Absolute-Valued Algebras Satisfying

$$ (xx^2)x = x(x^2x) $$

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Abstract

It is well known that every absolute-valued algebra of dimension \( \leq 4 \) satisfying \((xx^2)x = x(x^2x)\) is flexible and isomorphic to either \( \mathbb{R}, \mathbb{C}, \mathbb{C}^*, \mathbb{H} \) or \( \mathbb{H}^* \) [4]. Here we show that every eight-dimensional absolute-valued algebra containing four-dimensional sub-algebras and satisfying \((xx^2)x = x(x^2x)\) is also flexible and isomorphic to either \( \mathbb{O}, \mathbb{O}^* \) or \( \mathbb{P} \).

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1. Introduction

It is well known that every real third-power associative algebra satisfies the identity

\[(xx^2)x = x(x^2x)\]  \hspace{1cm} (1.1)

[4, Proposition 2.2] but the converse is false [3, Remarque 1.17]. Also, every absolute-valued algebra of dimension \( \leq 4 \) satisfying (1.1) is third-power associative (even flexible) and isomorphic to either \( \mathbb{R}, \mathbb{C}, \mathbb{C}^*, \mathbb{H} \) or \( \mathbb{H} \) [4, Theorem 3.10]. In addition, there are no known examples of real division algebras satisfying (1.1) which are not third-power associative. On the other hand it is well known that every finite-dimensional third-power associative absolute-valued algebra is flexible and isomorphic to either \( \mathbb{R}, \mathbb{C}, \mathbb{C}^*, \mathbb{H}, \mathbb{H}, \mathbb{O}, \mathbb{O}^* \) or \( \mathbb{P} \) ([6], [10, Theorem 2.7]). Thus, it is natural to ask whether an 8-dimensional absolute-valued algebra satisfying (1.1) is flexible.

In the present paper we will provide an affirmative answer through the additional assumption of the existence of four-dimensional subalgebras (Theorem 2). We show that every 8-dimensional absolute-valued algebra satisfying \((xx^2)x = x(x^2x)\) and containing 4-dimensional sub-algebras is flexible and isomorphic to either \( \mathbb{O}, \mathbb{O}^* \) or \( \mathbb{P} \).

2. Notations and preliminary results

For reasons of simplicity all algebras \( A \) will be considered over the field \( \mathbb{R} \) of real numbers. The algebra \( A \) is said to be third-power associative if it satisfies \( xx^2 = x^2x \) for all \( x \). It is said to be flexible if it satisfies \((xy)x = x(yx)\) for all \( x, y \). An element \( a \in A \) is said to be central if it commutes with all elements of \( A \).

\( A \) is said to be a division algebra if the operators \( L_x, R_x \) of left and right multiplication by \( x \) are bijective for all non-zero \( x \). It is said to be an absolute-valued algebra if the space \( A \) is endowed with a norm \( || \cdot || \) such that \( ||xy|| = ||x|| ||y|| \) for all \( x, y \). Clearly a finite-dimensional absolute-valued algebra is a division algebra. It is well known that every finite dimensional absolute-valued algebra \( A \) is isotopical to either \( \mathbb{R}, \mathbb{C} \) (complex numbers), \( \mathbb{H} \) (quaternions) or \( \mathbb{O} \) (octonions), so \( A \) has dimension 1, 2, 4 or 8 and the absolute value of \( A \) comes from an inner product [1, Theorem 2].

Let now \( A \) be one of absolute-valued algebras \( \mathbb{H}, \mathbb{O} \). Its standard involution \( \sigma_A : x \rightarrow \overline{x} \) is a linear isometry of the euclidian space \( A \) fixing 1. Let now \( f, g \)
be linear isometries of the euclidian space \( \mathbb{A} \), we denote by \( \mathbb{A}_{f,g} \) the absolute-valued algebra obtained by endowing the normed space \( \mathbb{A} \) with the product \( x \odot y = f(x)g(y) \). The absolute-valued algebra \( \mathbb{A}_{\sigma_{\mathbb{A}}, \sigma_{\mathbb{A}}} := \mathbb{A} \) contains a non-zero central idempotent \( 1 \). It is the only non-zero central idempotent of algebra \( \mathbb{A} \) [9, Theorem 3.6].

We have the following preliminary result:

**Proposition 1.** Let \( f, g : \mathbb{A} \rightarrow \mathbb{A} \) be two linear isometries fixing \( 1 \). Then

1. \( \mathbb{A}_{f,g} \) has unit-element if and only if \( f = g = I_{\mathbb{A}} \) (identity mapping).
2. \( \mathbb{A}_{f,g} \) is isomorphic to \( \mathbb{A} \) if and only if \( f = g = \sigma_{\mathbb{A}} \).

**Proof.** (1). If \( \mathbb{A}_{f,g} := (\mathbb{A}, \odot) \) contains unit-element \( e \) then \( e \) is the only non-zero idempotent. As \( 1 \) is a non-zero idempotent of \( \mathbb{A}_{f,g} \) we have: \( e = 1 \) and \( x = x \odot 1 = f(x)g(1) = f(x) \) for all \( x \in \mathbb{A} \). So \( f = I_{\mathbb{A}} = g \). The converse is clear.

(2). Let \( \Phi : \mathbb{A}_{f,g} \rightarrow \mathbb{A} \) be an isomorphism of algebras. We have:

\[
\Phi(f(x)g(y)) = \Phi(x) \Phi(y) \quad \text{for all} \quad x, y \in \mathbb{A}.
\]

On the other hand \( \Phi^{-1}(1) \) is the only one non-zero central idempotent in algebra \( \mathbb{A}_{f,g} \). So \( \Phi^{-1}(1) = 1 \), and \( f = g \) [9, Theorem 4.4]. Thus \( \Phi(1) = 1 \) and then \( \Phi \) commutes with \( \sigma_{\mathbb{A}} \). On the other hand, the equality (2.2) gives, for \( y = 1 : \Phi(f(x)) = \Phi(x) = \Phi(\bar{x}) \), that is \( f = \sigma_{\mathbb{A}} = g \). The converse is clear.

Let now \( \varphi, \psi, f, g \) be four linear isometries of the euclidian space \( \mathbb{H} \) with \( \varphi(1) = f(1) = 1 \). We denote by \( (\varphi, \psi) \) the linear isometry of the euclidian space \( \mathbb{H} \times \mathbb{H} \) defined by \( (\varphi, \psi)(x, y) = (\varphi(x), \psi(y)) \). We define a product on the vector space \( \mathbb{H} \times \mathbb{H} \) by:

\[
(x, y) \odot (u, v) = \left( \varphi(x), \psi(y) \right) \ast \left( f(u), g(v) \right)
\]

\( \ast \) means the usual Cayley-Dickson product. We obtain an absolute-valued algebra \( (\mathbb{H} \times \mathbb{H}, \odot) \) denoted by \( \mathbb{H} \times \mathbb{H}_{(\varphi, \psi), (f, g)} \). Such an algebra is said to be obtained from \( \mathbb{H}_{f, \varphi} \) par duplication [2, p. 211].

3. **When does \( \mathbb{H} \times \mathbb{H}_{(I_{\mathbb{H}}, f), (I_{\mathbb{H}}, g)} \) satisfy \( (xx^2)x = x(x^2x) \)?**

Let \( f, g \) be two linear isometries of the euclidian space \( \mathbb{H} \). We have the following two preliminary results:
Lemma 1. If the algebra $A = \mathbb{H} \times \mathbb{H}(I_{\mathbb{H}}, (I_{\mathbb{H}}, f), (I_{\mathbb{H}}, g)) := (\mathbb{H} \times \mathbb{H}, \odot)$ satisfies $(xx^2)x = x(x^2x)$ then the equality

\begin{equation}
(g(y)(f \circ g)(y) = (f \circ g)(y)f(y)
\end{equation}

holds for all $y \in \mathbb{H}$.

Proof. For $y \in \mathbb{H}$ we put $(0, y) := Y$ and we have

$Y \odot Y = (0, f(y)) \ast (0, g(y))$

$= (-g(y)f(y), 0)$

$(Y \odot Y) \odot Y = (-g(y)f(y), 0) \ast (0, g(y))$

$= -|g(y)|^2(0, f(y))$

$= -|y|^2(0, f(y))$

$Y \odot (Y \odot Y) = (0, f(y)) \ast (-g(y)f(y), 0)$

$= (0, -f(y)f(y)g(y))$

$= -|y|^2(0, g(y))$

$(Y \odot (Y \odot Y)) \odot Y = -|y|^2(0, (f \circ g)(y)) \ast (0, g(y))$

$= |y|^2(g(y)(f \circ g)(y), 0)$

$Y \odot ((Y \odot Y) \odot Y) = -|y|^2(0, f(y)) \ast (0, (g \circ f)(y))$

$= |y|^2((g \circ f)(y)f(y), 0)$.

So

$(Y \odot (Y \odot Y)) \odot Y - Y \odot ((Y \odot Y) \odot Y) = |y|^2(g(y)(f \circ g)(y) - (g \circ f)(y)f(y), 0)$.

This concludes the result.
4. When does $\mathbb{H} \times \mathbb{H}_{(\sigma_\infty, f), (\sigma_\infty, g)}$ satisfy $(xx^2)x = x(x^2x)$?

Lemma 2. If the algebra $A = \mathbb{H} \times \mathbb{H}_{(\sigma_\infty, f), (\sigma_\infty, g)} := (\mathbb{H} \times \mathbb{H}, \circ)$ satisfies $(xx^2)x = x(x^2x)$ then the equality

$$g \left( g(y) f(y) g(y) \right) f(y) = \overline{g(y) f(y) g(y)} f(y)$$

holds for all $y \in \mathbb{H}$.

Proof. For $y \in \mathbb{H}$ we put $(0, y) := Y$ and we have

$$Y \circ Y = (-\overline{g(y) f(y)}, 0)$$

$$\left( Y \circ Y \right) \circ Y = (-\overline{g(y) f(y)}, 0) \ast (0, g(y))$$
$$= (-\overline{f(y)g(y)}, 0) \ast (0, g(y))$$
$$= (0, -g(y)f(y)g(y))$$

$$Y \circ (Y \circ Y) = (0, f(y)) \ast (-\overline{g(y) f(y)}, 0)$$
$$= (0, f(y)) \ast (-\overline{f(y)g(y)}, 0)$$
$$= (0, -f(y)g(y)f(y))$$

$$\left( Y \circ (Y \circ Y) \right) \circ Y = \left( 0, -\overline{f(y)g(y)f(y)} \right) \ast (0, g(y))$$
$$= \left( \overline{g(y)f(y)g(y)f(y)}, 0 \right)$$

$$Y \circ \left( (Y \circ Y) \circ Y \right) = (0, f(y)) \ast (0, -g(y)\overline{f(y)g(y)})$$
$$= \left( \overline{g(y)f(y)g(y)f(y)}, 0 \right).$$

The result is obtained from the equality of the two terms above.

Note 1. Let us identify the three-dimensional sphere $S^3$ with the set of norm-one quaternions and let $a, b, c, d$ be in $S^3$. The equality $x\overline{ac} \overline{d} = axb\overline{c}ax$ cannot hold in $\mathbb{H}$ otherwise both continuous spherical functions $f, g : S^3 \rightarrow S^3$ given by $f(x) = x\overline{ac} \overline{d}$ and $g(x) = axb\overline{c}ax$ must coincide and, therefore, their homotopic degree $\partial f, \partial g$ [4, p. 1531] must be equal. But $\partial f = 0$, while $\partial g = 2$. We deduce that the equality $x\overline{ac} \overline{d} = axb\overline{c}ax$ cannot hold in $\mathbb{H}$. Recall that the degree of such a function $f$ is $n - m$, where $n$ is the number of $x$ that appear in the polynomial expression of $f$ and $m$ the number of $\overline{f}$. To see this, we take into account that $\overline{d}, \overline{c}, \overline{ac} \overline{c} \in S^3$ and that $S^3$ is connected.
There are continuous mappings $\alpha, \beta : [0,1] \to S^3$ such that $\alpha(0) = \beta(0) = 1$ and $\alpha(1) = \overline{d}$, $\beta(1) = \overline{c} \overline{a} c$. Therefore, the mapping

$$H : S^3 \times [0,1] \to S^3 \quad (x,t) \mapsto H(x,t) = x\beta(t)\overline{c}a\overline{a}c\alpha(t)\overline{d}$$

becomes a homotopy from $f$ to the mapping $x \mapsto 1$. Thus $\partial f = 0$. Recall also that $\partial (x \mapsto x^n) = n$ for any integer $n$. ∎

We are going to see that the homotopical degree is a very effective tool in the calculations. We will denote by $T_{a,b}$ the operator $L_a R_b$ for $a, b \in H$.

**Proposition 2.** If one of the two algebras $H \times H_{(I_H,f),(I_H,g)}$, $H \times H_{(\sigma_H,f),(\sigma_H,g)}$ satisfies $(x, x^2, x) = 0$ then $f, g$ are proper isometries.

**Proof.** Taking into account [7, Theorem (Cayley) p. 215] it suffices to show that $(f, g)$ cannot be of the form:

$(T_{a,b}, T_{c,d} \circ \sigma_H)$, $(T_{a,b} \circ \sigma_H, T_{c,d})$, $(T_{a,b} \circ \sigma_H, T_{c,d} \circ \sigma_H)$ where $a, b, c, d \in S^3$.

Indeed, the equality (3.3) gives:

$$\begin{cases}
y\overline{c}a\overline{d}y\overline{d} = ayb\overline{c}ay \quad \text{for all } y \in H & \text{if } (f, g) = (T_{a,b}, T_{c,d} \circ \sigma_H) \\
y\overline{d}a\overline{c}y\overline{c} = b\overline{a}c\overline{a}y \quad \text{for all } y \in H & \text{if } (f, g) = (T_{a,b} \circ \sigma_H, T_{c,d}) \\
y\overline{c}a\overline{d}y\overline{c} = a\overline{g}b\overline{c}ay \quad \text{for all } y \in H & \text{if } (f, g) = (T_{a,b} \circ \sigma_H, T_{c,d} \circ \sigma_H)
\end{cases}$$

This cannot hold by virtue of the homotopic degrees. This shows the property in algebra $H \times H_{(I_H,f),(I_H,g)}$ and it is the same for $H \times H_{(\sigma_H,f),(\sigma_H,g)}$. ∎

5. **When does $H \times H_{(I_H,T_{a,b}),(I_H,T_{c,d})}$ satisfy $(xx^2)x = x(x^2x)$?**

**Note 2.** There are other homotopic invariants for continuous functions $f : S^3 \to S^3$ namely the notion of **bidegree** finer than that of the degree. According to the notations in Note 1 the bidegree of $f$, denoted $\text{bideg}(f)$ is the pair $(n, -m)$ [2, p. 8]. We have:

$$\text{bideg}(fg) = \text{bideg}(f) + \text{bideg}(g).$$

We need the following preliminary results:

**Lemma 3.** Let $a, b, c, d \in S^3$ and let $A = H \times H_{(I_H,T_{a,b}),(I_H,T_{c,d})}$. If $A$ satisfies $(xx^2)x = x(x^2x)$ for all elements in $\{0\} \times H$ then $a, c \in \{1, -1\}$ and there exists $\varepsilon \in \{1, -1\}$ such that $(c,d) = (\varepsilon a, \overline{c}b)$. 

Proof. The equality (3.3) gives:

\[ \overline{y} \cdot c \cdot ayd = \overline{b} \cdot \overline{y} \cdot \overline{x} \cdot ay \quad \text{for all } y \in \mathbb{H} \]

The equality (5.5) is replaced by the following one if we introduce the scalar \(|y|^2 = y \overline{y}:

\[ \overline{y} \cdot c \cdot a(y \overline{y})cyd = \overline{b} \cdot \overline{y} \cdot a(y \overline{y}) \cdot \overline{c} \cdot ay \quad \text{for all } y \in \mathbb{H} \]

Multiplying on the left the two members of the equality (5.6) by \(y \cdot ay \cdot b\) we get:

\[ y \cdot ay \cdot b \cdot \overline{y} \cdot c \cdot a(y \overline{y})cyd = |y|^4 y \cdot \overline{y} \cdot a(y \overline{y})cyd \quad \text{for all } y \in \mathbb{H} \]

The same equality with a new arrangement of parentheses is:

\[ (y \cdot ay \cdot b)(y \cdot \overline{c} \cdot ay)(y \cdot cyd) = |y|^4 y \cdot \overline{c} \cdot ay \quad \text{for all } y \in \mathbb{H}. \]

Let

\[ f(y) = \overline{y} \cdot ay \cdot b, \quad y \in S^3 \]
\[ g(y) = y \cdot \overline{c} \cdot ay, \quad y \in S^3 \]
\[ h(y) = y \cdot cyd, \quad y \in S^3. \]

The equality (5.7) restricted to \(S^3\) becomes \(f(y)g(y)h(y) = g(y)\). Passing to bidegree we get: \(\text{bideg}(f) = \text{bideg}(h) = (0,0)\). So \(a, c\) are scalars and there exists \(\varepsilon \in \{1, -1\}\) such that \(c = \varepsilon a\). The equality (5.5) then gives \(d = \overline{\varepsilon b}\).

Note also that

\[ \mathbb{H} \times \mathbb{H}(I_{\mathbb{H}}, T_a, b), (I_{\mathbb{H}}, T_c, d) = \mathbb{H} \times \mathbb{H}(I_{\mathbb{H}}, T_a, b), (I_{\mathbb{H}}, T_c, \overline{a}) \]
\[ = \mathbb{H} \times \mathbb{H}(I_{\mathbb{H}}, T_{-a}, -b), (I_{\mathbb{H}}, T_{-a}, \overline{-b}). \]

This allows to take \(a = c = 1\) and \(d = \overline{b}\).

Lemma 4. Let \(b\) be in \(S^3\) and let \(A_b = \mathbb{H} \times \mathbb{H}(I_{\mathbb{H}}, R_a, R_b) := (\mathbb{H} \times \mathbb{H}, \circ)\). If \(A_b\) satisfies \((xx^2)x = x(x^2x)\) then \(b = 1\) and \(A_b\) is isomorphic to \(\mathbb{O}\).

Proof. Let \(b + \overline{b} := \lambda \in \mathbb{R}\) and \((1, 1) := X\). We have:

\[ X \circ X = (1, b) * (1, \overline{b}) = (1 - b^2, \lambda). \]
\[(X \circ X) \circ X = (1 - b^2, \lambda b) \ast (1, \tilde{b}) \]
\[= (1 - b^2 - \lambda b^2, \lambda b + \tilde{b}(1 - b^2)) \]
\[= (1 - (1 + \lambda)b^2, (\lambda - 1)b + \tilde{b}). \]

\[X \circ (X \circ X) = (1, b) \ast (1 - b^2, \lambda \tilde{b}) \]
\[= (1 - (1 + \lambda)b^2, b + (\lambda - 1)\tilde{b}). \]

\[X \circ (X \circ (X \circ X)) = (1, b) \ast (1 - (1 + \lambda)b^2, \lambda - 1 + \tilde{b}^2) \]
\[= (1 - (1 + \lambda)b^2 - (\lambda - 1 + b^2)b, b - (1 + \lambda)b + \lambda - 1 + \tilde{b}^2) \]
\[= (1 - (1 + \lambda)b^2 - (\lambda - 1 + b^2)b, 2b - 2). \]

\[\left( X \circ (X \circ X) \right) \circ X = (1 - (1 + \lambda)b^2, b^2 + \lambda - 1) \ast (1, \tilde{b}) \]
\[= (1 - (1 + \lambda)b^2 - b(b^2 + \lambda - 1), b^2 + \lambda - 1 + \tilde{b}(1 - (1 + \lambda)b^2)) \]
\[= (1 - (1 + \lambda)b^2 - b(b^2 + \lambda - 1), 2\tilde{b} - 2). \]

We have \(X \circ (X \circ X) \circ X = 2(0, \tilde{b} - b).\) It vanishes only if \(b\) is a scalar equal to \(\pm 1.\) Now the algebra \(A_{-1}\) contains a non-zero central idempotent \((1, 0)\) and

\[\left( (i, 1) \circ (1, 0) \right) \circ (i, 1) - (i, 1) \circ \left( (1, 0) \circ (i, 1) \right) \neq (0, 0). \]

Thereby \(A_{-1}\) cannot satisfy \((xx^2)x = x(x^2x)\). Otherwise \(A_{-1}\) would be flexible [9, Theorem 2.3] which is absurd. So if \(A_b\) satisfies \((xx^2)x = x(x^2x)\) then \(b \neq -1.\)

We can now state the following result:

**Theorem 1.** Let \(A = \mathbb{H} \times \mathbb{H}_{(I_\mathbb{H}, f), (I_\mathbb{H}, g)}\). The following four statements are equivalent:

1. \(A\) satisfies \((xx^2)x = x(x^2x)\).
2. \(f, g\) are proper and coincide with the identity mapping \(I_\mathbb{H}\).
3. \(A\) is flexible.
4. \(A\) is isomorphic to \(\mathbb{O}\).
Proof. The implication $(1) \Rightarrow (2)$ is a consequence of Proposition 2 and Lemmas 3, 4. The implications $(2) \Rightarrow (4)$, $(4) \Rightarrow (3)$, $(3) \Rightarrow (1)$ are clear.

6. When does $H \times H_{(\sigma, T_{a, b}), (\sigma, T_{c, d})}$ satisfy $(xx^2)x = x(x^2x)$?

We need the following preliminary results:

Lemma 5. Let $a, b, c, d$ be in $S^3$ and assume that $A = H \times H_{(\sigma, T_{a, b}), (\sigma, T_{c, d})}$ satisfies $(xx^2)x = x(x^2x)$. Then $b, d, a^3, c^3 \in \{1, -1\}$ and there exists $\varepsilon \in \{1, -1\}$ such that $c = \varepsilon a$.

Proof. The equality (4.4) gives:

\begin{equation}
\bar{d}, \bar{y}, \bar{c}, ayb, \bar{d}, \bar{y}, \bar{x}a^2 ay = \bar{y}, \bar{c}, a^2 yb, \bar{d}, \bar{y}, \bar{c}, ayb.
\end{equation}

Multiplying on the left the two members of the equality (6.8) by $b, y, \bar{a}^2, cy$ we get:

\begin{equation}
\bar{b}, \bar{y}, \bar{a}^2 cy(\bar{d}, \bar{y}, \bar{c}ayb)\bar{d}, \bar{y}, \bar{x}a^2 ay = |y|^4 \bar{d}, \bar{y}, \bar{c}, ayb \quad \text{for all} \quad y \in H.
\end{equation}

Let now

\begin{align*}
f(y) &= \bar{b}, \bar{y}, \bar{a}^2 cy, y \in S^3 \\
g(y) &= \bar{d}, \bar{y}, \bar{c}ayb, y \in S^3 \\
h(y) &= \bar{d}, \bar{y}, \bar{x}a^2 ay, y \in S^3.
\end{align*}

The equality (6.9) restricted to $S^3$ becomes $f(y)g(y)h(y) = g(y)$ for all $y \in H$. Passing to bidegree we get: $\text{bideg}(f) = \text{bideg}(h) = (0, 0)$ that is $\bar{a}^2 c$, $\bar{x}a \in \{1, -1\}$. So there exists $\varepsilon \in \{1, -1\}$ such that $\bar{x}a = \varepsilon \bar{a}^2 c \in \{1, -1\}$. Thus

\begin{align*}
\bar{x}a &= \varepsilon \bar{a}^2 c \\
&= \varepsilon \bar{a}^2.
\end{align*}

This gives, after simplifications: $\bar{x} = \varepsilon a$ and finally $c = \varepsilon a$. Through the equality $\bar{x}a = \varepsilon \bar{a}^2 c$ we get $c^3 = \varepsilon a^3 = (\varepsilon a)^3 \in \{1, -1\}$. The equality (6.8) becomes

\begin{equation}
\bar{d}, \bar{y}a^2 yb\bar{d} = \varepsilon b, \bar{d}, \bar{y}a^2 yb : y \in H.
\end{equation}
By introducing the scalar $|y|^2$ in both members of the equality (6.10) and multiplying on the right by $y$, we get 

$$d.ya^2yb(y)y = |y|^2b.d.ya^2yb \overline{y}$$

and finally

$$(6.11) \quad (d.ya^2yb)(y\overline{d.y}) = \varepsilon|y|^2b(d.ya^2yb) : y \in \mathbb{H}.$$ 

Let now $(\varphi(y), \psi(y)) = (d.ya^2yb, y\overline{d.y})$, $y \in S^3$. The equality (6.11) restricted to $S^3$ becomes $\varphi(y)\psi(y) = \varepsilon b\varphi(y)$ for all $y \in S^3$. Passing to bidegree we get: 

$bideg(\psi) = (0, 0)$. So $d$ is a scalar. The equality (6.8) shows that $b = \varepsilon d$ is also a scalar. Thus

$$H \times H(\sigma_{H}, T_{a,b}), (\sigma_{H}, T_{c,d}) = H \times H(\sigma_{H}, T_{a,b}), (\sigma_{H}, T_{c, a, b}) = H \times H(\sigma_{H}, T_{a,b}), (\sigma_{H}, T_{c, -a, b}) = H \times H(\sigma_{H}, T_{a,b}), (\sigma_{H}, T_{c, -a, b}).$$

This allows to take $b = d = 1$, that is $A = H \times H(\sigma_{H}, L_{a}), (\sigma_{H}, L_{a'})$ where $a' = \pm a$. 

**Lemma 6.** The algebra $B = H \times H(\sigma_{H}, L_{a}), (\sigma_{H}, L_{a'})$ does not satisfy $(xx^2)x = x(x^2x)$. 

**Proof.** It is easily verified that $B$ is not a flexible algebra and contains a non-zero central idempotent $(1, 0)$. It cannot satisfy $(xx^2)x = x(x^2x)$. 

**Lemma 7.** If $a \in \{1, -1\}$ then the following affirmations are equivalent:

1. The algebra $A = H \times H(\sigma_{H}, L_{a}), (\sigma_{H}, L_{a'})$ satisfies $(xx^2)x = x(x^2x)$.
2. $c = a = -1$.
3. $A$ is isomorphic to $O^*$. 

**Proof.** Clearly (1) $\Rightarrow$ (2) is an immediate consequence of Lemmas 5, 6. 

(2) $\Rightarrow$ (3) ? The involution $\sigma_{O} de O$ coincides in $H \times H$ with $(\sigma_{H}, -I_{H})$. So 

$$A = H \times H(\sigma_{H}, -I_{H}), (\sigma_{H}, -I_{H}) = O_{\sigma_{O}, \sigma_{O}} = O^*.$$ 

The implication (3) $\Rightarrow$ (1) is clear. 

We need a series of computational preliminary results:

**Lemma 8.** If $a \neq \pm 1$ then the algebra $A = H \times H(\sigma_{H}, L_{a}), (\sigma_{H}, L_{a'})$ contains no non-zero central elements.
Proof. Let \((u, v)\) be a central element of algebra \(A := (\mathbb{H} \times \mathbb{H}, \circ)\) and let \(y \in \mathbb{H}\). We have:

\[(u, v) \circ (y, 0) = (y, 0) \circ (u, v) \iff (\overline{u} \overline{y}, avy) = (\overline{y} \overline{u}, \overline{a}v \overline{y}).\]

The latter equality, valid for all \(y\), shows that \(u\) is a scalar. For \(y = 1\) it gives \(v = 0\) taking into account that \(a \neq \pm 1\). On the other hand, the element \((1, 0) \in A\) is not central. So \((u, v) = (0, 0)\).

Lemma 9. If \(a^3 = -1 \neq a\) then the algebra \(A = \mathbb{H} \times \mathbb{H}(\sigma_{\mathbb{H}}, L_0),(\sigma_{\mathbb{H}}, L_\pi) := (\mathbb{H} \times \mathbb{H}, \circ)\) is flexible and isomorphic to \(\mathbb{P}\).

Proof. Note that the algebra \(A\) contain no central idempotent. Thus it cannot be isomorphic neither to \(\mathbb{O}\) nor to \(\mathbb{O}^*\). Let now \(u, v, y, z \in \mathbb{H}\) and let \(U = (u, v), Y = (y, z)\) we have:

\[U \circ Y = (\overline{u}, av) * (\overline{y}, \overline{a}z) = (\overline{u} \overline{y} - \overline{a}^2v, avy + \overline{a}z \overline{u}).\]

\[(U \circ Y) \circ U = \left(\overline{u} \overline{y} - \overline{a}^2v, avy + \overline{a}z \overline{u}\right) * (\overline{u}, \overline{a}v) = (yu - \overline{u} \overline{a}^2z, a^2vy + zu) * (\overline{u}, \overline{a}v) = \left(\left(yu - \overline{a}^2z\right)\overline{u} - \overline{a}v(a^2vy + zu), (a^2vy + zu)u + \overline{a}v(yu - \overline{a}^2z)\right) = \left(\left(yu + avaz\right)\overline{u} - \overline{a}v(a^2vy + zu), (a^2vy + zu)u + \overline{a}v(yu + vaza)\right) = \left(\left(|u|^2 + |v|^2\right)y, (|u|^2 + |v|^2)z\right) = |U|^2Y.\]

\[U \circ (Y \circ U) = (u, v) \circ (\overline{y} \overline{u} - \overline{a}^2z, zu + \overline{av} \overline{y}) = (\overline{u}, av) * \left(\overline{y} \overline{u} - \overline{a}^2z, \overline{a}zu + \overline{av} \overline{y}\right) = (\overline{u}, av) * (uy - z a^2v, zu + a^2vy) = \left(\overline{u}(uy - z a^2v) - zu + a^2vy, av, av. uy - z a^2v + (zu + a^2vy)u\right) = \left(\overline{u}(uy + z av) - (\overline{u} z + yav^2), av, av(\overline{u} \overline{u} - \overline{a}^2z) + (zu - avy)u\right) = \left(\left(|u|^2 + |v|^2\right)y, (|u|^2 + |v|^2)z\right) = (U \circ Y) \circ U.\]

So \(A\) is flexible and isomorphic to \(\mathbb{P}\).
Lemma 10. If $a \neq \pm 1$ and the algebra $A = \mathbb{H} \times \mathbb{H}(\sigma_{\mathbb{H}, L_a}, (\sigma_{\mathbb{H}, L_a})$ satisfies $(xx^2)x = x(x^2x)$ then $a^3 = -1$.

Proof. Assume that $a^3 = 1$ and note that the trace, $a + \bar{a}$, of $a$ equals $-1$ and that $a^2 + a + 1 = 0$. Let $V = (1,1)$ we have:

\[
V \odot V = (1, a) \ast (1, \bar{a})
\]
\[
= (1 - a^2, a + \bar{a})
\]
\[
= (1 - a^2, -1).
\]

\[
(V \odot V) \odot V = (1 - \bar{a}^2, -a) \ast (1, \bar{a})
\]
\[
= (1 - \bar{a}^2 + a^2, -a + \bar{a} - 1)
\]
\[
= (-2a, 2a^2).
\]

\[
V \odot (V \odot V) = (1, a) \ast (1 - \bar{a}^2, -\bar{a})
\]
\[
= (1 - \bar{a}^2 + a^2, a - 1 - \bar{a})
\]
\[
= (-2a, 2a).
\]

\[
(V \odot (V \odot V)) \odot V = 2(-\bar{a}, a^2) \ast (1, \bar{a})
\]
\[
= 2(-\bar{a} - 1, a^2 - \bar{a}^2)
\]
\[
= 2(a, -1 - 2a).
\]

\[
V \odot ((V \odot V) \odot V) = 2(1, a) \ast (-\bar{a}, a)
\]
\[
= 2(-\bar{a} - 1, -a^2 + a)
\]
\[
= 2(a, 1 + 2a)
\]
\[
\neq 2(a, -1 - 2a).
\]

So $a^3 = -1$. 

Corollary 1. Let $a,b,c,d$ be in $S^3$ and let $A = \mathbb{H} \times \mathbb{H}(\sigma_{\mathbb{H}, T_a, b}, (\sigma_{\mathbb{H}, T_c, d})$. The following affirmations are equivalent:

(1) $A$ satisfies $(xx^2)x = x(x^2x)$.

(2) $b$ et $d$ are scalars which can be chosen equal to 1. Moreover, either $a = c = -1$ or $a^3 = -1 \neq a = \bar{c}$.

(3) $A$ is flexible.

(4) $A$ is isomorphic to eithet $\mathbb{O}$ or $\mathbb{P}$.

Proof. The implication $(1) \Rightarrow (2)$ is a consequence of Lemmas 5, 7, 9, 10. The implication $(2) \Rightarrow (4)$ is a consequence of Lemmas 7, 9, 10. The
implication $(4) \Rightarrow (3)$ is contained in ([6], [10, Theorem 2.7]). The implication $(3) \Rightarrow (1)$ is clear.

7. The main result

**Proposition 3.** Let $A$ be an eight-dimensional absolute-valued algebra which contain a four-dimensional sub-algebra $B$ and satisfies $(xx^2)x = x(x^2x)$. Then $A$ is obtained from $B$ by duplication. In addition, the subalgebra $B$ is flexible and isomorphic to either $\mathbb{H}$ or $\hat{\mathbb{H}}$. Concretely, there are three linear isometries $f, \varphi, \psi : \mathbb{H} \to \mathbb{H}$ with $f \in \{I_{\mathbb{H}}, \sigma_{\mathbb{H}}\}$ such that:

$$A = \mathbb{H} \times \mathbb{H}_{(f, \varphi), (f, \psi)} \quad \text{and} \quad B = \mathbb{H}_{f, f}.$$ 

**Proof.** Let $e \in B$ be a non-zero idempotent. Then algebra $A_{R_e^{-1}, L_e^{-1}}$ has unit element $e$ and is isomorphic to $\mathbb{O}$. Moreover, $B$ is both $R_e$ and $L_e$ invariant. So $A$ is obtained from $B$ par duplication [2, Theorem 6.4] and there are four linear isometries $f, f', \varphi, \psi : \mathbb{H} \to \mathbb{H}$ with $f(1) = f'(1) = 1$ such that $A = \mathbb{H} \times \mathbb{H}_{(f, \varphi), (f', \psi)}$ and $B = \mathbb{H}_{f, f}$. If, in addition, $A$ satisfies $(xx^2)x = x(x^2x)$ then $B$ is flexible and isomorphic to either $\mathbb{H}$ or $\hat{\mathbb{H}}$. The Proposition 1 shows that $f = f' \in \{I_{\mathbb{H}}, \sigma_{\mathbb{H}}\}$.  

**Theorem 2.** Let $A$ be an 8-dimensional absolute-valued algebra satisfying $(xx^2)x = x(x^2x)$. Then the following four affirmations are equivalent:

1. $A$ has a four-dimensional subalgebra,
2. $A$ is third-power associative,
3. $A$ is flexible,
4. $A$ is isomorphic to either $\mathbb{O}, \hat{\mathbb{O}}$ or $\mathbb{P}$.

**Proof.** The implication $(1) \Rightarrow (3)$ follows from [2, Theorem 6.4], Propositions 2, 3, Theorem 1 and Corollary 1. The equivalences $(3) \iff (2) \iff (4)$ are contained in ([6], [10, Theorem 2.7]). The implication $(4) \Rightarrow (1)$ is clear.

It is shown in [5] that every third-power associative absolute-valued algebra whose norm comes from an inner product is finite-dimensional. It may be conjectured that every absolute-valued algebra whose norm comes from an inner product and satisfying both identities $(xx^2)x = x(x^2x), \ (x^2)^2x^2 = x^2(x^2)^2$ is finite-dimensional.
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References


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