Centralizing with Generalized $(\sigma, \tau)$-Derivations on Semiprime Rings

C. Jaya Subba Reddy, K. Subbarayudu and S. Mallikarjuna Rao

Department of Mathematics, S. V. University
Tirupati, Andhra Pradesh, India

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Abstract

Let $R$ be a semiprime ring with characteristic not two, $I$ a nonzero ideal of $R$, and $\sigma, \tau$ are two an epimorphism of $R$. An additive mapping $f: R \to R$ is generalized $(\sigma, \tau)$-derivation on $R$ if there exists a $(\sigma, \tau)$-derivation $d: R \to R$ such that $f(xy) = f(x)\sigma(y) + \tau(x)d(y)$, holds for all $x, y \in R$ and $\tau(I)d(I) \neq 0$. In this paper, if $R$ satisfies any one of the following conditions: (i) $f([x,y]) = \pm(x\sigma y)_{\sigma,\tau}$; (ii) $f(x\sigma y) = \pm[x,y]_{\sigma,\tau}$; (iii) $[f(x),y]_{\sigma,\tau} = (f(x)\sigma y)_{\sigma,\tau}$; (iv) $[f(x),d(y)]_{\sigma,\tau} = 0$; (v) $[f(x),d(y)]_{\sigma,\tau} = y\sigma(x)$; (vi) $d(x)f(y) = \pm[x,y]_{\sigma,\tau}$; (vii) $f(xy) - x\sigma(y) \in C_{\sigma,\tau}$; (viii) $f(xy) + x\sigma(y) \in C_{\sigma,\tau}$, for all $x, y \in I$. Then we prove that $R$ contains a nonzero central ideal, that is $I \subseteq Z(R)$.

Keywords: Semiprime ring, Derivation, Generalized derivation, $(\sigma, \tau)$-derivation, Generalized $(\sigma, \tau)$-derivation

Introduction

In 2002, Ashraf and Rehman [4] established that if $R$ is a 2-torsion free prime ring and $L$ be a lie ideal of $R$ admits a derivation $d$ such that $d([x,y]) = [x,y]$, for all $x, y \in L$, then $L \subseteq Z(R)$. Further, N. Rehman [11] extended the mention results for generalized derivations. Recently, the several authors (see [1], [2], [3], [6], [7], [8], [9]) have extended the some results to generalized $(\sigma, \tau)$-derivation. After that Rehman et al [12] extended some results in the generalized $(\alpha, \beta)$-derivations in certain classes of rings. After that Basudeb Dhara and Atanu Pattanayak [5] extended the generalized $(\alpha, \beta)$-derivations into generalized $(\sigma, \tau)$-
derivations in semiprime rings, where $\sigma$ and $\tau$ are epimorphisms of $R$. Further Mahammad Anwar Chaudhary et al. [10] have discussed the some results on generalized $(\alpha, \beta)$-derivations in $*$-prime rings. Motivated by these results we extended some results on generalized $(\sigma, \tau)$-derivations in semiprime rings.

**Preliminaries**

Throughout this paper $R$ denote an associative ring with center $Z$. Recall that a ring $R$ is prime if $xRy = \{0\}$ implies $x = 0$ or $y = 0$. A ring $R$ is said to be semiprime if $xRx = \{0\}$ implies $x = 0$, for all $x \in R$. A ring $R$ is said to be characteristic not two if $2x = 0$ implies $x = 0$, for all $x \in R$. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol $(xoy)$ denotes the anti-commutator $xy + yx$. Let $S$ be a nonempty subset of $R$. A mapping $F: R \to R$ is called centralizing on $S$ if $[F(x), x] \in Z$ for all $x \in S$ and $F: R \to R$ is called commuting on $S$ if $[F(x), x] = 0$, for all $x \in S$. Let $\sigma, \tau$ be any two epimorphisms of $R$. For any $x, y \in R$, set $[x, y]_{\sigma, \tau} = x\sigma(y) - \tau(y)x$ and $(xoy)_{\sigma, \tau} = x\sigma(y) + \tau(y)x$. An additive mapping $d: R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$, holds for all $x, y \in R$. An additive mapping $d: R \to R$ is called a $(\sigma, \tau)$-derivation if $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$, holds for all $x, y \in R$. An additive mapping $f: R \to R$ is called a generalized derivation, if there exists a derivation $d: R \to R$ such that $f(xy) = f(x)y + xd(y)$, holds for all $x, y \in R$. An additive mapping $f: R \to R$ is said to be a generalized $(\sigma, \tau)$-derivation of $R$, if there exists a $(\sigma, \tau)$-derivation $d: R \to R$ such that $f(xy) = f(x)\sigma(y) + \tau(x)d(y)$, holds for all $x, y \in R$.

Throughout this paper, we shall make use of the some basic commutator identities:

$$[x, yz] = y[x, z] + [x, y]z, [xy, z] = [x, z]y + x[y, z],$$

$$(x o(yz)) = (xoy)z - y[x, z] = y(xoz) + [x, y]z,$$

$$[xy, z]_{\sigma, \tau} = x[y, z]_{\sigma, \tau} + [x, \tau(z)]y = x[y, \sigma(z)] + [x, z]_{\sigma, \tau}y,$$

$$[x, yz]_{\sigma, \tau} = \tau(y)[x, z]_{\sigma, \tau} + [x, y]_{\sigma, \tau}\sigma(z),$$

$$(xo(yz))_{\sigma, \tau} = (xoy)_{\sigma, \tau}\sigma(z) - \tau(y)[x, z]_{\sigma, \tau} = \tau(y)(xoz)_{\sigma, \tau} + [x, y]_{\sigma, \tau}\sigma(z),$$

$$((xy)oz)_{\sigma, \tau} = x(yoz)_{\sigma, \tau} - [x, \tau(z)]y = (xoz)_{\sigma, \tau}y + x[y, \sigma(z)].$$

**Theorem 1**: Let $R$ be a semiprime ring with characteristic not two, $I$ a nonzero ideal of $R$, $\sigma$ and $\tau$ two epimorphisms of $R$ and $f$ a generalized $(\sigma, \tau)$-derivation associated with a $(\sigma, \tau)$-derivation $d$ of $R$ such that $\tau(I)d(I) \neq 0$. If $f([x, y]) = \pm (xoy)_{\sigma, \tau}$, for all $x, y \in I$, then $R$ contains a non zero central ideal.
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**Proof:** First we assume that \(f([x, y]) = (x o y)_{\sigma, \tau}\), for all \(x, y \in I\).\(^1\)

We replace \(y\) by \(2y x\) in equation (1) and using \(\text{char}R \neq 2\), we get
\[
f([x, y x]) = (x o (y x))_{\sigma, \tau}
\]
\[
f([x, y]x) = (x o (y x))_{\sigma, \tau}
\]
\[
f([x, y]) \sigma(x) + \tau([x, y]) d(x) = (x o y)_{\sigma, \tau} \sigma(x) - \tau(y)[x, x]_{\sigma, \tau}, \text{ for all } x, y \in I.\(^2\)
\]

Using equation (1) in equation (2), we get
\[
\tau([x, y]) d(x) = -\tau(y)[x, x]_{\sigma, \tau}, \text{ for all } x, y \in I.\(^3\)
\]

We replace \(y\) by \(2wy, w \in R\) in equation (3) and using \(\text{char}R \neq 2\), we get
\[
\tau([x, w]y + w[y, x]) d(x) = -\tau(w) \tau(y)[x, x]_{\sigma, \tau}
\]
\[
\tau([x, w]) \tau(y) d(x) + \tau(w) \tau([x, y]) d(x) = -\tau(w) \tau(y)[x, x]_{\sigma, \tau}, \text{ for all } x, y \in I \text{ and } w \in R.\(^4\)
\]

Using equation (3) in equation (4), we get
\[
\tau([x, w]) \tau(y) d(x) = 0, \text{ for all } x, y \in I \text{ and } w \in R.\(^5\)
\]

We replace \(y\) by \(sy, s \in R\) in the above equation, we get
\[
\tau([x, w]) \tau(s) \tau(y) d(x) = 0
\]
\[
\tau([x, w]) R \tau(y) d(x) = 0, \text{ for all } x, y \in I \text{ and } w \in R.
\]

Since \(\tau\) is an epimorphism of \(R\), we can write \(([R, \tau(x)]) R \tau(l) d(x) = 0, \text{ for all } x \in I.\)

Since \(R\) is semiprime, it must contain a family \(\Omega = \{P_\alpha / \alpha \in \Lambda\}\) of prime ideals
such that \(\cap P_\alpha = \{0\}\). If \(P\) is typical member of \(\Omega\) and \(x \in I\), it follows that
\([R, \tau(x)] \subseteq P\) or \(\tau(I) d(x) \subseteq P\). Construct two additive subgroups \(T_1 = \{x \in I/ [R, \tau(x)] \subseteq P\}\) and \(T_2 = \{x \in I/ \tau(I) d(x) \subseteq P\}\). Then \(T_1 \cup T_2 = I\). Since
a group cannot be a union of two its proper subgroups, either \(T_1 = I\) or \(T_2 = I\).

That is, either \([R, \tau(I)] \subseteq P\) or \(\tau(I) d(I) \subseteq P\).
Thus both the cases together implies \([R, \tau(I)]\tau(I)d(I) \subseteq P\) for any \(P \in \Omega\).

Therefore, \([R, \tau(I)]\tau(I)d(I) \subseteq \bigcap_{\alpha \in A} P_\alpha = 0\),

That is, \([R, \tau(I)]\tau(I)d(I) = 0\). \hspace{1cm} \text{(6)}

Thus \([R, \tau(RIR)]\tau(RI)d(I) = 0\)

\([R, R\tau(I)R]\tau(I)d(I) = 0\)

And so \([R, R\tau(I)d(I)R]\tau(I)d(I)R = 0\).

Therefore \([R, K]R\tau(I)d(I)R = 0\), where \(K = R\tau(I)d(I)R\) is nonzero ideal of \(R\), since \(\tau(I)d(I) \neq 0\).

Then \([R, K]R[K, R] = 0\). Since \(R\) is semiprime, it follows that \([R, K] = 0\).

That is, \(K \subseteq Z(R)\).

Similarly, we can obtain the same conclusion when \(F([x, y]) = -(xoy)_{\sigma, \tau}\), for all \(x, y \in I\).

Thus the proof is completed.

**Theorem 2:** Let \(R\) be a semiprime ring with characteristic not two, \(I\) a nonzero ideal of \(R\), \(\sigma\) and \(\tau\) two epimorphisms of \(R\) and \(f\) a generalized \((\sigma, \tau)\)-derivation associated with a \((\sigma, \tau)\)-derivation \(d\) of \(R\) such that \(\tau(I)d(I) \neq 0\). If \(f(xoy) = \pm [x, y]_{\sigma, \tau}\), for all \(x, y \in I\), then \(R\) contains a non zero central ideal.

**Proof:** By the hypothesis we have \(f(xoy) = [x, y]_{\sigma, \tau}\), for all \(x, y \in I\). \hspace{1cm} \text{(7)}

We replace \(y\) by \(2yx\) in equation (7) and using char\(R \neq 2\), we get

\[f(xo(yx)) = [x, yx]_{\sigma, \tau}\]

\[f((xoy)x) = [x, yx]_{\sigma, \tau}\]

\[f(xoy)\sigma(x) + \tau(xoy)d(x) = [x, y]_{\sigma, \tau}\sigma(x) + \tau(y)[x, x]_{\sigma, \tau},\] for all \(x, y \in I\). \hspace{1cm} \text{(8)}

Using equation (7) in equation (8), we get

\[\tau(xoy)d(x) = \tau(y)[x, x]_{\sigma, \tau},\] for all \(x, y \in I\). \hspace{1cm} \text{(9)}
We replace $y$ by $2wy$, $w \in R$ in equation (9) and using $\text{char} R \neq 2$, we get

\[
(\tau(w(xoy) + [x,w]y))d(x) = \tau(w)\tau(y)[x,x]_{\sigma,\tau}
\]

\[
\tau(w)\tau(xoy)d(x) + \tau[x,w]\tau(y)d(x) = \tau(w)\tau(y)[x,x]_{\sigma,\tau}, \text{ for all } x,y \in I \text{ and } w \in R.
\] (10)

Using equation (9) in equation (10), we get

\[
\tau([x,w])\tau(y)d(x) = 0, \text{ for all } x,y \in I \text{ and } w \in R.
\] (11)

The equation (11) is same as equation (5) in theorem 1. Thus, by same argument of theorem 1, we can conclude the result here. Similarly, we can obtain the same conclusion when $f(xoy) = -[x,y]_{\sigma,\tau}$, for all $x,y \in I$. Thus the proof is completed.

**Theorem 3:** Let $R$ be a semiprime ring with characteristic not two, $I$ a nonzero ideal of $R$, $\sigma$ and $\tau$ two epimorphisms of $R$ and $f$ a generalized $(\sigma,\tau)$-derivation associated with a $(\sigma,\tau)$-derivation $d$ of $R$ such that $\tau(I),\tau(I)d(I) \neq 0$. If $[f(x),y]_{\sigma,\tau} = (f(x)oy)_{\sigma,\tau}$, for all $x,y \in I$, then $R$ contains a non zero central ideal.

**Proof:** We have $[f(x),y]_{\sigma,\tau} = (f(x)oy)_{\sigma,\tau}$, for all $x,y \in I$. (12)

We replace $y$ by $2yw$, $w \in R$ in equation (12) and using $\text{char} R \neq 2$, we get

\[
[f(x),yw]_{\sigma,\tau} = (f(x)o(yw))_{\sigma,\tau}, \text{ for all } x,y \in I \text{ and } w \in R.
\]

\[
[f(x),y]_{\sigma,\tau}\sigma(w) + \tau(y)[f(x),w]_{\sigma,\tau} = (f(x)oy)_{\sigma,\tau}\sigma(w) - \tau(y)[f(x),w]_{\sigma,\tau}, \text{ for all } x,y \in I \text{ and } w \in R.
\]

Using equation (12) in the above equation, we get

\[
2\tau(y)[f(x),w]_{\sigma,\tau} = 0
\]

Since $\text{char} R \neq 2$, we have $\tau(y)[f(x),w]_{\sigma,\tau} = 0$, for all $x,y \in I$ and $w \in R$. (13)

Hence we conclude that $[f(x),w]_{\sigma,\tau} = 0$, for all $x \in I$ and $w \in R$. (14)
We substitute $2xy$ for $x$ in equation (14) and using $\text{char}R \neq 2$, we get
\[ [f(xy), w]_{\sigma, \tau} = 0 \]
\[ f(x)[\sigma(y), \sigma(w)] + [f(x), w]_{\sigma, \tau} + \tau(x)(d(y), w)_{\sigma, \tau} + [\tau(x), \tau(w)]d(y) = 0, \text{ for all } x, y \in I \text{ and } w \in R. \]

Using equation (14) in the above equation, we get
\[ f(x)[\sigma(y), \sigma(w)] + \tau(x)(d(y), w)_{\sigma, \tau} + [\tau(x), \tau(w)]d(y) = 0, \text{ for all } x, y \in I \text{ and } w \in R. \]  
(15)

We replace $w$ by $y$ in the equation (15), we get
\[ \tau(x)[d(y), y]_{\sigma, \tau} + \tau([x, y])d(y) = 0, \text{ for all } x, y \in I. \]  
(16)

Taking $2wx$, $w \in R$ instead of $x$ in the equation (16) and using $\text{char}R \neq 2$, we obtain
\[ \tau(wx)[d(y), y]_{\sigma, \tau} + \tau([wx, y])d(y) = 0 \]
\[ \tau(w)[\tau(x)[d(y), y]_{\sigma, \tau} + \tau([x, y])d(y)] + \tau([w, y])\tau(x)d(y) = 0, \text{ for all } x, y \in I \text{ and } w \in R. \]  
(17)

Using equation (16) in equation (17), we get
\[ \tau([w, y])\tau(x)d(y) = 0, \text{ for all } x, y \in I \text{ and } w \in R. \]  
(18)

We replace $x$ by $y$ and $y$ by $x$ in equation (18), we get
\[ \tau([w, x])\tau(y)d(x) = 0, \text{ for all } x, y \in I \text{ and } w \in R. \]  
(19)

The equation (19) is same as equation (5) in theorem 1. Thus, by same argument of theorem 1, we can conclude the result here. Thus the proof is completed.

**Theorem 4**: Let $R$ be a semiprime ring with characteristic not two, $I$ a nonzero ideal of $R$, $\sigma$ and $\tau$ two epimorphisms of $R$ and $f$ a generalized $(\sigma, \tau)$-derivation associated with a $(\sigma, \tau)$-derivation $d$ of $R$ such that $\tau(I)d(I) \neq 0$. If $[f(x), d(y)]_{\sigma, \tau} = 0$, for all $x, y \in I$, then $R$ contains a non zero central ideal.
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**Proof:** We have \([f(x), d(y)]_{\sigma, \tau} = 0\), for all \(x, y \in I\).  

(20)

We substitute \(2xd(y)\) for \(x\) in the equation \((20)\) and using \(char R \neq 2\), we obtain \([f(xd(y), d(y))]_{\sigma, \tau} = 0\), for all \(x, y \in I\)

\[ [f(x), d(y)]_{\sigma, \tau} \sigma(d(y)) + \tau(x)[d(d(y)), d(y)]_{\sigma, \tau} + [\tau(x), \tau(d(y))]d(d(y)) = 0, \text{ for all } x, y \in I. \]  

(21)

Using equation \((20)\) in the equation \((21)\), we get

\[ \tau(x)[d(d(y)), d(y)]_{\sigma, \tau} + [\tau(x), \tau(d(y))]d(d(y)) = 0, \text{ for all } x, y \in I. \]  

(22)

We replace \(x\) by \(2wx\), \(w \in R\) in equation \((22)\) and using \(char R \neq 2\), we get

\[ \tau(wx)[d(d(y)), d(y)]_{\sigma, \tau} + [\tau(wx), \tau(d(y))]d(d(y)) = 0 \]

\[ \tau(w)\{\tau(x)[d(d(y)), d(y)]_{\sigma, \tau} + \tau([x, d(y)])d(d(y))\} + \tau([w, d(y)])\tau(x)d(d(y)) = 0, \text{ for all } x, y \in I \text{ and } w \in R. \]  

(23)

Using equation \((22)\) in the equation \((23)\), we get

\[ \tau([w, d(y)])\tau(x)d(d(y)) = 0, \text{ for all } x, y \in I \text{ and } w \in R. \]  

(24)

We replace \(x\) by \(y\) and \(d(y)\) by \(x\) in equation \((24)\), we get

\[ \tau([w, x])\tau(y)d(x) = 0, \text{ for all } x \in I \text{ and } y, w \in R. \]  

(25)

The equation \((25)\) is same as equation \((5)\) in theorem 1. Thus, by same argument of theorem 1, we can conclude the result here. Thus the proof is completed.

**Theorem 5:** Let \(R\) be a semiprime ring with characteristic not two, \(I\) a nonzero ideal of \(R\), \(\sigma\) and \(\tau\) two epimorphisms of \(R\) and \(f\) a generalized \((\sigma, \tau)\)-derivation associated with a \((\sigma, \tau)\)-derivation \(d\) of \(R\) such that \(\tau(I)d(I) \neq 0\). If \([f(x), d(y)]_{\sigma, \tau} = y\sigma(x)\), for all \(x, y \in I\), then \(R\) contains a non zero central ideal.

**Proof:** By the hypothesis, \([f(x), d(y)]_{\sigma, \tau} = y\sigma(x)\), for all \(x, y \in I\).  

(26)
We replace $x$ by $2xw$, $w \in R$ in equation (26) and using $\text{char}R \neq 2$, we get
\[
[f(xw), d(y)]_{\sigma, \tau} = y \sigma(xw), \text{ for all } x, y \in I \text{ and } w \in R.
\]
\[
[f(x), d(y)]_{\sigma, \tau} \sigma(w) + f(x)[\sigma(w), \sigma(d(y))] + \tau(x)[d(w), d(y)]_{\sigma, \tau} +
[\tau(x), \tau(d(y))]d(w) = y \sigma(x) \sigma(w), \text{ for all } x, y \in I \text{ and } w \in R.
\] (27)

Using equation (26) in equation (27), we get
\[
f(x)[\sigma(w), \sigma(d(y))] + \tau(x)[d(w), d(y)]_{\sigma, \tau} + [\tau(x), \tau(d(y))]d(w) = 0, \text{ for all } x, y \in I \text{ and } w \in R.
\] (28)

Again we replace $w$ by $d(y)$ in equation (28), we get
\[
\tau(x)[d(d(y)), d(y)]_{\sigma, \tau} + [\tau(x), \tau(d(y))]d(d(y)) = 0, \text{ for all } x, y \in I.
\] (29)

Taking $2wx$, $w \in R$ instead of $x$ in equation (29) and using $\text{char}R \neq 2$, we get
\[
\tau(wx)[d(d(y)), d(y)]_{\sigma, \tau} + [\tau(wx), \tau(d(y))]d(d(y)) = 0
\]
\[
\tau(w)[\tau(x)[d(d(y)), d(y)]_{\sigma, \tau} + [\tau(x), \tau(d(y))]d(d(y))] +
[\tau(w), \tau(d(y))]\tau(x)d(d(y)) = 0, \text{ for all } x, y \in I \text{ and } w \in R.
\] (30)

Using equation (29) in equation (30), we get
\[
\tau([w, d(y)])\tau(x)d(d(y)) = 0, \text{ for all } x \in I \text{ and } y, w \in R.
\] (31)

By further application of similar arguments as used in the end of the proof of theorem 4, we get the required result.

**Theorem 6:** Let $R$ be a semiprime ring with characteristic not two, $I$ a nonzero ideal of $R$, $\sigma$ and $\tau$ two epimorphisms of $R$ and $f$ a generalized $(\sigma, \tau)$-derivation associated with a $(\sigma, \tau)$-derivation $d$ of $R$ such that $\tau(I)d(I) \neq 0$. If $d(x)f(y) = \pm[x, y]_{\sigma, \tau}$, for all $x, y \in I$, then $R$ contains a non zero central ideal.

**Proof:** First we consider the case $d(x)f(y) = [x, y]_{\sigma, \tau}$, for all $x, y \in I$. (32)

We replace $y$ by $2yw$, $w \in R$ in equation (32) and using the fact that $\text{char}R \neq 2$, we have
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\[ d(x)f(yw) = [x, yw]_{\sigma,\tau}, \text{ for all } x, y \in I. \]

\[ d(x)f(y)\sigma(w) + d(x)\tau(y)d(w) = [x, y]_{\sigma,\tau}\sigma(w) + \tau(y)[x, w]_{\sigma,\tau}, \text{ for all } x, y \in I \text{ and } w \in R. \] (33)

Using equation (32) in equation (33), we get

\[ d(x)\tau(y)d(w) = \tau(y)[x, w]_{\sigma,\tau}, \text{ for all } x, y \in I \text{ and } w \in R. \] (34)

We replace \(2ry, r \in R\) for \(y\) in equation (34) and using \(\text{char}R \neq 2\), we get

\[ d(x)\tau(r)\tau(y)d(w) = \tau(r)\tau(y)[x, w]_{\sigma,\tau}, \text{ for all } x, y \in I \text{ and } w, r \in R. \] (35)

Using equation (34) in equation (35), we get

\[ d(x)\tau(r)\tau(y)d(w) = \tau(r)d(x)\tau(y)d(w) \]

\[ [\tau(r), d(x)]\tau(y)d(w) = 0, \text{ for all } x, y \in I \text{ and } w, r \in R. \] (36)

We replace \(d(x)\) by \(\tau(s)d(x), s \in R\) in the equation (36), we get

\[ [\tau(r), \tau(s)d(x)]\tau(y)d(w) = 0 \]

\[ \tau(s)[\tau(r), d(x)]\tau(y)d(w) + [\tau(r), \tau(s)]d(x)\tau(y)d(w) = 0, \text{ for all } x, y \in I \text{ and } s, r, w \in R. \] (37)

Using equation (36) in the equation (37), we get

\[ [\tau(r), \tau(s)]d(x)\tau(y)d(w) = 0 \]

We replace \(d(x)\) by \(\tau(pq)d(x), p, q \in R\) in the above equation, we get

\[ [\tau(r), \tau(s)]\tau(p)\tau(q)d(x)\tau(y)d(w) = 0 \]

\[ [\tau(r), \tau(s)]R\tau(q)d(x)\tau(y)d(w) = 0, \text{ for all } x, y \in I \text{ and } q, r, s, w \in R. \]

Since \(\tau\) is an epimorphism of \(R\), we can write

\[ [R, \tau(s)]R\tau(R)d(I)\tau(I)d(w) = 0, \text{ for all } w \in R. \] (38)

Since \(R\) is semiprime, it must contain a family \(\Omega = \{P_\alpha/\alpha \in \Lambda\}\) of prime ideals such that \(\cap P_\alpha = \{0\}\). If \(P\) is typical member of \(\Omega\) and \(w \in R\), it follows that \([R, \tau\]

(s) \subseteq P \text{ or } \tau(R)d(I)\tau(I)d(w) \subseteq P. \text{ Construct two additive subgroups } T_1 = \{s \in R/ [R, \tau(s)] \subseteq P\} \text{ and } T_2 = \{w \in R/ \tau(R)d(I)\tau(I)d(w) \subseteq P\}. \text{ Then } T_1 \cup T_2 = R. \text{ Since a group cannot be a union of two its proper subgroups, either } T_1 = R \text{ or } T_2 = R.

That is, either \([R, \tau(R)] \subseteq P\) or \(\tau(R)d(I)\tau(I)d(R) \subseteq P\).

Thus both the cases together implies \([R, \tau(I)]\tau(I)d(R)\tau(I)d(R) \subseteq P\) for any \(P \in \Omega\).

Therefore, \([R, \tau(I)]\tau(I)d(R)\tau(I)d(R) \subseteq \cap_{\alpha \in \Lambda} P_\alpha = 0\).

It follows that \([R, \tau(I)]\tau(I)d(R)\tau(RI)d(R) = 0\).

\([R, \tau(I)]\tau(I)d(R)R\tau(I)d(R) = 0\) and hence we can write

\([R, \tau(I)]\tau(I)d(R)R[R, \tau(I)]\tau(I)d(R) = 0\).

Since \(R\) is semiprime, it follows that \([R, \tau(I)]\tau(I)d(R) = 0\).

In particular \([R, \tau(I)]\tau(I)d(I) = 0\). \hspace{1cm} (39)

The equation (39) is same as equation (6) in theorem 1. Thus, by same argument of theorem 1, we can conclude the result here.

Similarly, we can obtain the same conclusion when \(d(x)f(y) = -[x, y]_{\sigma, \tau}\), for all \(x, y \in I\).

Thus the proof is completed.

**Theorem 7:** Let \(R\) be a semiprime ring with characteristic not two, \(I\) a nonzero ideal of \(R\), \(\sigma\) and \(\tau\) two epimorphisms of \(R\) and \(f\) a generalized \((\sigma, \tau)\)-derivation associated with a \((\sigma, \tau)\)-derivation \(d\) of \(R\) such that \(\tau(I)d(I) \neq 0\). If \(f(xy) - x\sigma(y) \in C_{\sigma, \tau}\), for all \(x, y \in I\), then \(R\) contains a non zero central ideal.

**Proof:** By the hypothesis, we have \(f(xy) - x\sigma(y) \in C_{\sigma, \tau}\), for all \(x, y \in I\). \hspace{1cm} (40)

This can be rewritten as \(f(x)\sigma(y) + \tau(x)d(y) - x\sigma(y) \in C_{\sigma, \tau}\), for all \(x, y \in I\). \hspace{1cm} (41)

We replace \(y\) by \(2yw, w \in R\) in equation (41) and using \(\text{char} R \neq 2\), we get
Centralizing with generalized \((σ,τ)\)-derivations on semiprime rings

\[
f(x)σ(yw) + τ(x)d(yw) - xσ(yw) ∈ C_{σ,τ}, \text{ for all } x, y ∈ I \text{ and } w ∈ R.
\]
\[
(f(x)σ(y) + τ(x)d(y) - xσ(y))σ(w) + τ(x)τ(y)d(w) ∈ C_{σ,τ}, \text{ for all } x, y ∈ I \text{ and } w ∈ R.
\]

(42)

Commuting this term with \(w\) and using equation (41) in equation (42), we obtain

\[
τ(xy)d(w) ∈ C_{σ,τ}
\]

Therefore \([τ(xy)d(w), w]_{σ,τ} = 0, \text{ for all } x, y ∈ I \text{ and } w ∈ R\). (43)

\[
τ(xy)[d(w), w]_{σ,τ} + [τ(xy), τ(w)]d(w) = 0, \text{ for all } x, y ∈ I \text{ and } w ∈ R.
\]

(44)

We replace \(x\) by \(2sx, s ∈ R\) in equation (44) and using \(char R ≠ 2\), we have

\[
τ(sxy)[d(w), w]_{σ,τ} + [τ(sxy), τ(w)]d(w) = 0, \text{ for all } x, y ∈ I \text{ and } w, s ∈ R
\]

\[
τ(s)[τ(xy)[d(w), w]_{σ,τ} + [τ(xy), τ(w)]d(w)] + [τ(s), τ(w)]τ(xy)d(w) = 0,
\]

for all \(x, y ∈ I \text{ and } w, s ∈ R\). (45)

Using equation (44) in equation (45), we get

\[
[τ(s), τ(w)]τ(xy)d(w) = 0, \text{ for all } x, y ∈ I \text{ and } w, s ∈ R.
\]

We replace \(x\) by \(rx, r ∈ R\) in the above equation, we get

\[
[τ(s), τ(w)]τ(r)τ(xy)d(w) = 0, \text{ for all } x, y ∈ I \text{ and } w, r, s ∈ R.
\]

\[
[τ(s), τ(w)]Rτ(x)τ(y)d(w) = 0
\]

Since \(τ\) is an epimorphism of \(R\), the above relation implies that

\[
[R, τ(w)]Rτ(I)τ(I)d(w) = 0, \text{ for all } w ∈ R.
\]

(46)

Since \(R\) is semiprime, it must contain a family \(Ω = \{P_α/α ∈ Λ\}\) of prime ideals such that \(∩ P_α = \{0\}\). If \(P\) is typical member of \(Ω\) and \(w ∈ R\), it follows that

\[
[R, τ(w)] ⊆ P \text{ or } τ(I)τ(I)d(w) ⊆ P.
\]
Construct two additive subgroups $T_1 = \{ w \in R / [R, \tau(w)] \subseteq P \}$ and $T_2 = \{ w \in R / \tau(I) \tau(I) d(w) \subseteq P \}$. Then $T_1 \cup T_2 = R$. Since a group cannot be a union of two its proper subgroups, either $T_1 = R$ or $T_2 = R$.

That is, either $[R, \tau(R)] \subseteq P$ or $\tau(I) \tau(I) d(R) \subseteq P$.

Thus both the cases together implies $[R, \tau(I)] \tau(I) d(R) \subseteq P$ for any $P \in \Omega$.

Therefore, $[R, \tau(I)] \tau(I) d(R) \subseteq \cap_{\alpha \in A} P_{\alpha} = 0$

That is, $[R, \tau(I)] \tau(I) d(R) = 0$

In particular $[R, \tau(I)] \tau(I) d(I) = 0$. \hfill (47)

The equation (47) is same as equation (6) in theorem 1. Thus, by same argument of theorem 1, we can conclude the result here. Thus the proof is completed.

**Theorem 8:** Let $R$ be a semiprime ring with characteristic not two, $I$ a nonzero ideal of $R$, $\sigma$ and $\tau$ two epimorphisms of $R$ and $f: R \to R$ is a generalized $(\sigma, \tau)$-derivation associated with a $(\sigma, \tau)$-derivation $d$ of $R$ such that $\tau(I) d(I) \neq 0$. If $f(xy) + x \sigma(y) \in C_{\sigma, \tau}$ for all $x, y \in I$, then $R$ contains a non zero central ideal.

**Proof:** If $f$ is a generalized $(\sigma, \tau)$-derivation satisfying the property $f(xy) + x \sigma(y) \in C_{\sigma, \tau}$ for all $x, y \in I$.

Then $(-f)$ satisfies the condition $(-f)(xy) - x \sigma(y) \in C_{\sigma, \tau}$, for all $x, y \in I$.

Hence we have $I \subseteq Z(R)$ by theorem 7.

**Corollary 9:** Let $R$ be a prime ring with characteristic not two, $I$ a nonzero ideal of $R$, and $\sigma, \tau$ are two an epimorphism of $R$. An additive mapping $f: R \to R$ is a generalized $(\sigma, \tau)$-derivation associated with a nonzero $(\sigma, \tau)$-derivation $d: R \to R$ such that $f(xy) = f(x) \sigma(y) + \tau(x) d(y)$, holds for all $x, y \in R$. If $R$ satisfies any one of the following conditions i.e., (i) $f([x, y]) = \pm (x o y)_{\sigma, \tau}$; (ii) $f(x o y) = \pm [x, y]_{\sigma, \tau}$; (iii) $[f(x), y]_{\sigma, \tau} = (f(x) o y)_{\sigma, \tau}$; (iv) $[f(x), d(y)]_{\sigma, \tau} = 0$; (v) $[f(x), d(y)]_{\sigma, \tau} = y \sigma(x)$; (vi) $d(x) f(y) = \pm [x, y]_{\sigma, \tau}$; (vii) $f(xy) - x \sigma(y) \in C_{\sigma, \tau}$; (viii) $f(xy) + x \sigma(y) \in C_{\sigma, \tau}$, for all $x, y \in I$, then $R$ contains a nonzero central ideal, That is $I \subseteq Z(R)$.
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References


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