Torsion Groups of Elliptic Curves with Everywhere Good Reduction over Quadratic Fields

Takaaki Kagawa

Department of Mathematical Sciences
Ritsumeikan University
1-1-1, Nojihigashi, Kusatsu, Shiga, 525–8577, Japan

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Abstract

This paper concerns elliptic curves defined over quadratic fields with everywhere good reduction. The curves are classified over the structure of the torsion subgroup. There are only 5 torsion types and 22 curves if the torsion subgroups are not isomorphic to \( \{0\} \), \( \mathbb{Z}/2\mathbb{Z} \) and \( \mathbb{Z}/3\mathbb{Z} \).

Mathematics Subject Classification: 11G05

Keywords: Elliptic curves with everywhere good reduction, Torsion subgroups

1 Introduction

We first recall terminology “admissible elliptic curve” defined over a number field \( K \). If an elliptic curve defined over \( K \) with everywhere good reduction has a \( K \)-rational 2-division point, then it is said to be admissible. Comalada [1] studies the admissible curves and proves many important results. On the other hand, Zhao [19] gives terminology 3-admissible. An elliptic curve defined over a number field \( K \) is called 3-admissible if it has everywhere good reduction over \( K \) and has a \( K \)-rational 3-division point. He proves that there are infinitely many 3-admissible curves in the case where \( K \) is real quadratic.

Here we introduce a new notion. Let \( E \) be an elliptic curve over a number field \( K \). For a finite group \( G \), \( E \) is called \( G \)-admissible if it has everywhere
good reduction over $K$ and $E(K)_\text{tors} \cong G$. (Note that Comalada uses the word “g-admissible”. But his “g” means “allowing global minimal equation”. So our terminology is not the cause of confusion.)

When $G \cong \mathbb{Z}/2\mathbb{Z}$ or $G \cong \mathbb{Z}/3\mathbb{Z}$, the number of $G$-admissible curves over real quadratic field is infinite, as is referred above. In this paper, we determine all $G$-admissible curves over quadratic fields when $G \not\cong \{0\}, \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$.

**Remark.** For an entire collection of the torsion group structures of elliptic curves over quadratic fields, see [7].

## 2 Result

We introduce our main theorem.

**Theorem 2.1.** Let $K$ be a quadratic field. Let $G$ be a finite group not isomorphic to $\{0\}, \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$, and let $E$ be a $G$-admissible elliptic curve defined over $K$. Then $K$ must be either $\mathbb{Q}(\sqrt{6}), \mathbb{Q}(\sqrt{7}), \mathbb{Q}(\sqrt{26}), \mathbb{Q}(\sqrt{37}), \mathbb{Q}(\sqrt{41})$ or $\mathbb{Q}(\sqrt{65})$, and $E$ is isomorphic to a curve in Table 1 through Table 5.

We explain the contents of the tables.

1. The code, what the author uses in his thesis [4], is of the form $mXi$, where $m$ indicates that the curve is defined over $\mathbb{Q}(\sqrt{m})$, $X = A, B$ denotes a $K$-isogeny class (but in our tables, except for $m = 65$, only 1 isogeny class exists over an each field), and $i$ is the ordinal number of the curve over $\mathbb{Q}(\sqrt{m})$. (When $m = 6, 17, 41, 65$, the codes in Comalada’s paper [1] are also given.) A curve with ’ such as 7A2’ is its conjugate without ’ such as 7A2. Note that, for example, 26A1’ is not on the tables, since 26A1 and 26A1’ are isomorphic over $\mathbb{Q}(\sqrt{26})$. For the same reason, 37A1’, 65B1’, 65B2’ and 65A2’ are not on the tables.

2. The coefficients $a_1, a_2, a_3, a_4$ and $a_6$ are the ones of a defining equation of each curve.

3. The discriminant of $E$ is denoted by $\Delta$. In all cases, we denote by $\varepsilon$ the fundamental unit greater than 1. Note that 65A4, 65A4’, 65B1 and 65B2 have no global minimal models and so $\Delta$ cannot be units.

4. The $j$-invariant of $E$ is denoted by $j$. 

Table 1: $\mathbb{Z}/4\mathbb{Z}$-admissible
Elliptic curves

3 Proof of Theorem

The following lemma is crucial to prove Theorem.

Lemma 3.1. If an elliptic curve over a number field \( K \) has everywhere good reduction over \( K \), then its \( j \)-invariant is an integer of \( K \).

Proof. Let \( E \) be an elliptic curve defined over \( K \) and let \( j \) be the \( j \)-invariant of \( E \). Suppose that \( E \) has good reduction at a prime ideal \( \mathfrak{p} \) of \( K \). Then we can choose a model of \( E \) with \( \mathfrak{p} \)-integer coefficients and thus \( \mathfrak{p} \)-unit discriminant \( \Delta \). Hence \( j = c_4^3/\Delta \) (here \( c_4 \) is the usual invariant) is a \( \mathfrak{p} \)-integer of \( K \).

Therefore all \( G \)-admissible curves appear in the tables in [11] already when \( G \not\cong \{0\}, \mathbb{Z}/2\mathbb{Z} \) and \( \mathbb{Z}/3\mathbb{Z} \). Hence, to prove Theorem, it is enough to select all \( G \)-admissible curves from the tables. But they are too large to check throughly. To reduce the amount of consideration, we use following two lemmas:

Lemma 3.2. Let \( j \) be the \( j \)-invariant of an elliptic curve with everywhere good reduction over a quadratic field \( F \).

1. The principal ideal \( (j) \) is the cube of some ideal.
2. If \( j \in \mathbb{Z} \), then \( j \) is the cube of some rational integer.
Proof. (1) is essentially proved in the proof of Lemma 3.1.

(2) [15], Theorem 1.

Lemma 3.3. Let \( K \) be an imaginary quadratic field. If the class number of \( K \) is prime to 6, then there is no elliptic curve defined over \( K \) having everywhere good reduction.

Proof. [2], Corollary 3.3, [14], Theorem 5 and [16], Theorem 1.9.

Over real quadratic fields, many nonexistence results and determination results exist. See Ishii [2], Kagawa [3], [5], [6], Kida [8], [9], Kida–Kagawa [10], Pinch [12], Takeshi [17] and Yokoyama–Shimasaki [18]. Using their results together with Lemmas 3.2 and 3.3, we can ignore many quadratic fields.

[I] We show only eight \( \mathbb{Z}/4\mathbb{Z} \)-admissible curves exist. There are many curves in Table 2 in [11] whose torsion subgroups contain \( \mathbb{Z}/4\mathbb{Z} \). Most curves cannot have everywhere good reduction by Lemma 3.2. For over many fields, for example \( \mathbb{Q}(\sqrt{17}) \), it is shown that there are no curves with everywhere good reduction by [10], Corollary 1 and [5], Theorem 2. And over \( \mathbb{Q}(\sqrt{33}) \), in [4], [6], all elliptic curves with everywhere good reduction are determined and none of them has a \( \mathbb{Q}(\sqrt{33}) \)-rational point of order 4. For the remaining curves, we compute the conductors and check whether they are trivial or not using the software Sage [13].

[II] Next, we determine \( \mathbb{Z}/5\mathbb{Z} \)-admissible curves. In Table 8 of [11], the elliptic curves which have integral j-invariant and defined over quadratic fields with torsion subgroup contain \( \mathbb{Z}/5\mathbb{Z} \) are classified. We consider the curves over \( \mathbb{Q}(\sqrt{26}) \) and \( \mathbb{Q}(\sqrt{37}) \) only, because there are no elliptic curve over \( K = \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{10}) \) and \( \mathbb{Q}(\sqrt{13}) \) with everywhere good reduction over \( K \) by Lemma 3.3, Example of [2] and Theorem 2 of [5]. Moreover, over \( K = \mathbb{Q}(\sqrt{65}) \), there are no \( \mathbb{Z}/5\mathbb{Z} \)-admissible curves. (See [4].) Finally, other than 26A1 and 37A1, they have shown not to have everywhere good reduction using Sage. The remaining curves 26A1 and 37A1 have everywhere good reduction. Note that 26A1 (resp. to 37A1) is isomorphic over \( \mathbb{Q}(\sqrt{26}) \) (resp. \( \mathbb{Q}(\sqrt{37}) \)) to 26A1′ (resp. 37A1′).

[III] The case of \( \mathbb{Z}/6\mathbb{Z} \)-admissible curves determined in a similar way.

[IV] In [11], there are eight curves with \( E(K)_{\text{tors}} \cong \mathbb{Z}/8\mathbb{Z} \) and integral j-invariant. They are defined over \( \mathbb{Q}(\sqrt{5}) \) and \( \mathbb{Q}(\sqrt{17}) \). But there are no elliptic curves with everywhere good reduction over these fields. (See [2], [5], [10] and [12].) Thus there are no \( \mathbb{Z}/8\mathbb{Z} \)-admissible curves.
[V] Nonexistence of $\mathbb{Z}/7\mathbb{Z}$- and $\mathbb{Z}/10\mathbb{Z}$-admissible curves follows from Lemma 3.3.

[VI] There are no $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$-admissible curves because in Table 7 in [11], the elliptic curves with integral $j$-invariants defined over with torsion group isomorphic $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ are defined $\mathbb{Q}(\sqrt{-3})$. But by Lemma 3.3, all these curves do not have everywhere good reduction.

[VII] Finally, we determine $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$- and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$-admissible curves and show nonexistence of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$-admissible curves.

When an elliptic curve $E$ over a quadratic field $K$ having four 2-division $K$-rational points and integral $j$-invariant, the quadratic twists of $E$ have the same property as $E$. Moreover, Table 1 in [11], where such curves listed modulo twists, is huge. (9 pages long!) But the following lemma allows us to skip it:

**Lemma 3.4.** Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic field (real or imaginary). Let $E$ be an elliptic curve defined over $K$ with everywhere good reduction satisfying $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \subset E(K)_{\text{tors}}$. Then we must have $d = 7, 41$ or 65 and $E$ is isomorphic to either $7A2$, $7A2'$, $7A3$, $7A3'$, $41A2$, $41A2'$, $65A2$ or $65A2'$.

**Proof.** [1], Theorem 2. \hfill \Box

Computing the torsion subgroups by Sage, we can know that only 65A2 has a torsion point of order 4, that is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$-admissible and the others $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$-admissible curves do not exist, because in Table 6 of [11], there are four curves with torsion subgroups isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$. They are defined over $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{3})$. But over these fields, there are no elliptic curve with everywhere good reduction. (Lemma 3.3, Theorem 2 of [5] and Theorem 2 of [8].) Therefore there are no elliptic $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$-admissible curves over quadratic fields.

Theorem is now proved.

**Acknowledgements.** The author would like to thank Nao Takeshi and Shun’ichi Yokoyama for the several useful discussions. In particular he thanks about the information about the curve 26A1.

**References**


https://doi.org/10.2140/pjm.1990.144.237


Elliptic curves


Received: September 15, 2016; Published: October 15, 2016