A Note on Multiplicative (Generalized)-Derivations in Prime Rings

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Abstract

Let $R$ be a prime ring, $0 \neq a \in R$ and $I$ a nonzero ideal of $R$. A map $F : R \to R$ is called a multiplicative (generalized)-derivation if $F(xy) = F(x)y + xg(y)$ fulfilled for all $x, y \in R$ where $g : R \to R$ is any map(not necessarily derivation). Suppose that $F$ and $G$ are two multiplicative (generalized)-derivations. The main objective of the present paper is to study the following situations: (i) $a(G(xy)\pm [F(x), y] \pm xy) = 0$; (ii) $a(G(xy) + F(x)F(y) \pm xy) = 0$ for all $x, y \in I$.

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1 Introduction

Let $R$ be an associative ring. For $x, y \in R$, the symbol $[x, y]$ will denote the commutator $xy - yx$ and the symbol $x \circ y$ will denote the anticommutator $xy + yx$. We shall make use of the basic commutator identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$. Recall that a ring $R$ is prime if for $a, b \in R$, $aRb = (0)$ implies either $a = 0$ or $b = 0$ and is semiprime if for $a \in R$, $aRa = (0)$ implies $a = 0$. An additive map $d$ from $R$ to $R$ is called a derivation of $R$ if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. 
Let $F : R \to R$ be a map associated with another map $g : R \to R$ such that $F(xy) = F(x)y + xg(y)$ holds for all $x, y \in R$. If $F$ is additive and $g$ is a derivation of $R$, then $F$ is said to be a generalized derivation of $R$ that was introduced by Brešar [5]. In [13], Hvala gave the algebraic study of generalized derivations of prime rings. Following [6], a multiplicative derivation of $R$ is a map $D : R \to R$ which satisfies $D(xy) = D(x)y + xD(y)$ for all $x, y \in R$. Of course these maps are not additive. By our knowledge, the concept of multiplicative derivations appears for the first time in the work of Daif [6] and it was motivated by the work of Martindale [14]. Further, the complete description of those maps were given by Goldmann and Šemrl in [12]. The notion of multiplicative derivation was extended in [8] as follows: a map $F : R \to R$ is called a multiplicative generalized derivation if there exists a derivation $d$ of $R$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. Very recently, Dhara and Ali [10] gave a more precise definition of multiplicative (generalized)-derivation as follows: A mapping $F$ on $R$ is said to be a multiplicative (generalized)-derivation if there exists a map $g$ on $R$ such that $F(xy) = F(x)y + xg(y)$ for all $x, y \in R$, where $g$ is any map on $R$ (not necessarily additive). Hence, the concept of multiplicative (generalized)-derivation covers the concept of multiplicative derivation. Moreover, multiplicative (generalized)-derivation with $g = 0$ covers the notion of multiplicative centralizers (not necessarily additive).

During the past few years, some authors have been studying the commutativity in prime and semiprime rings admitting derivations or generalized derivations. For example, we refer the reader to ([1, 2, 3, 4, 7, 11, 9, 15, 16], where further references can be found). In [3], Ashraf and Rehman proved that if $R$ is a prime ring with a nonzero ideal $I$ of $R$ and $d$ is a derivation of $R$ such that either $d(xy) - xy \in Z$ for all $x, y \in I$ or $d(xy) + xy \in Z$ for all $x, y \in I$, then $R$ is commutative. Being inspired by this result, recently Ashraf et al. [2] have studied the situations replacing derivation $d$ with a generalized derivation $F$. More precisely, they proved that a prime ring $R$ must be commutative, if $R$ satisfies any one of the following conditions: (i) $F(xy) - xy \in Z$ for all $x, y \in I$, (ii) $F(xy) + xy \in Z$ for all $x, y \in I$, (iii) $F(xy) - yx \in Z$ for all $x, y \in I$, (iv) $F(xy) + yx \in Z$ for all $x, y \in I$, (v) $F(xy) - F(y)x - y \in Z$ for all $x, y \in I$, (vi) $F(xy) + xy \in Z$ for all $x, y \in I$; where $F$ is a generalized derivation of $R$ associated with a nonzero derivation $d$ and $I$ is a nonzero two-sided ideal of $R$. Recently, Dhara and Ali in [10] studied the above mentioned identities for multiplicative (generalized)-derivations in prime and semiprime rings.

In the present paper, our main object is to discuss the commutativity of prime rings involving multiplicative (generalized)-derivations with annihilating conditions. More precisely, we study the following identities: (i) $a(G(xy) + F(x)y \pm yx) = 0$; (ii) $a(G(xy) + F(x)y \pm yx) = 0$; for all $x, y \in I$, where $I$ is a nonzero ideal of $R$ and $F, G$ are multiplicative (generalized)-derivations of prime ring $R$. 

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2 Main Results

Now we begin with our first theorem:

**Theorem 2.1.** Let $R$ be a prime ring, and $I$ be a nonzero ideal of $R$. Suppose that $G$ and $F$ are two multiplicative (generalized)-derivations of $R$ associated with mappings $g$ and $d$, respectively. If $a \neq 0 \in R$ such that $a(G(xy) \pm [F(x), y] \pm yx) = 0$ for all $x, y \in I$, then one of the following holds:

(i) $G(r) = \pm r$ for all $r \in R$ and $R$ is commutative;
(ii) $g(I) = 0$, $[d(x), x] = 0$ and $[F(x), x] = 0$ for all $x \in I$.

**Proof.** By the assumption, we have

$$a(G(xy) + [F(x), y] + yx) = 0 \text{ for all } x, y \in I. \quad (1)$$

Substituting $yz$ for $y$ in (1), we get

$$a(G(xy)z + xyg(z) + [F(x), y]z + y[F(x), z] + yzx) = 0 \quad (2)$$

Application of (1) yields

$$a(xy g(z) + y[F(x), z] + y[z, x]) = 0 \text{ for all } x, y, z \in I. \quad (3)$$

Replacing $y$ by $ay$ in (3), we obtain

$$a(xay g(z) + ay[F(x), z] + ay[z, x]) = 0 \text{ for all } x, y, z \in I. \quad (4)$$

Left multiplication by $a$ to (3) and then subtracting from (4), we find that

$$a[x, a]yg(z) = 0 \text{ for all } x, y, z \in I. \quad (5)$$

This implies that $a[x, a]Rg(z) = (0)$ for all $x, y, z \in I$. Since $R$ is prime, so $I$ is too. Thus, the last relation forces that $a[x, a] = 0$ for all $x \in I$ or $g(z) = 0$ for all $z \in I$. First, we consider the case $a[x, a] = 0$ for all $x \in I$. Since $a \neq 0$, it follows that $a \in Z(R)$. It is well known that the center of prime ring is free from zero divisor. Hence, the relation (3) reduces to

$$xy g(z) + y[F(x), z] + y[z, x] = 0 \text{ for all } x, y, z \in I. \quad (6)$$

Replace $y$ by $ty$ in (6), to get

$$xty g(z) + ty[F(x), z] + ty[z, x] = 0 \text{ for all } x, y, z \in I, t \in R. \quad (7)$$
Left multiplication by \( t \) to relation (6) gives

\[
txyg(z) + ty[F(x), z] + ty[z, x] = 0 \quad \text{for all } x, y, z \in I, t \in R.
\]

On combining the last two expressions, we obtain \([x, t]yg(z) = 0\) for all \( x, y, z \in I, t \in R \). Since \( R \) is Prime, so we are force conclude that either \( R \) is commutative or \( g(I) = 0 \). If \( R \) is commutative, then relation (6) reduces to \( xyg(z) = 0 \) for all \( x, y, z \in I \). Again the primeness of \( R \) yields that \( g(I) = 0 \). Thus, till now we proved that \( R \) is commutativeand \( g(I) = 0 \). By using these facts, hypothesis yields that \((G(x) + x)y = 0\) for all \( x, y \in I \). Now, substituting \( rx \) for \( x \), we find that \((G(r) + r)xy = 0\) for all \( r \in R \) and \( x, y \in I \). This implies further that \((G(r) + r)Iy = (0)\) for all \( r \in R \) and \( y \in I \). The primeness of \( R \) and hence of \( I \) forces that \( G(r) = -r \) for all \( r \in R \).

Now we assume that \( g(I) = 0 \). Then relation (3) reduces to \( ay([F(x), z] + [z, x]) = 0 \) for all \( x, y, z \in I \). Since \( a \neq 0 \) and \( R \) is prime, we find that

\[
[F(x), z] + [z, x] = 0 \quad \text{for all } x, y, z \in I. \tag{8}
\]

Substituting \( xz \) for \( x \) in (8) and using it, we obtain \([xd(z), z] = 0\) for all \( x, z \in I \). This further implies \([d(z), z] = 0\) for all \( z \in I \). In particular, for \( z = x \) in (8), we get \([F(x), x] = 0\) for all \( x \in I \).

Similarly, we can prove for the case \( a(G(xy) \pm [F(x), y] - yx) = 0\) for all \( x, y \in I \). This proves the theorem completely.

The following corollary is an immediate consequence of the above theorem.

**Corollary 2.2.** Let \( R \) be a prime ring, and \( I \) be a non zero ideal of \( R \). Suppose that \( G \) is a multiplicative (generalized)-derivation associated with the \( a \) mapping \( g \) of \( R \). If \( a \neq 0 \in R \) such that \( a(G(xy) \pm yx) = 0 \) for all \( x, y \in I \), then \( G(r) = r \) for all \( r \in R \) and \( R \) is commutative.

**Theorem 2.3.** Let \( R \) be a prime ring, and \( I \) be a non zero ideal of \( R \). Suppose that \( G \) and \( F \) are two multiplicative (generalized)-derivations associated with the mappings \( g \) and \( d \) of \( R \). If \( a \neq 0 \in R \) such that \( a(G(xy) + F(x)F(y) \pm yx) = 0 \) for all \( x, y \in I \), then \( F(xy) = F(x)y, G(xy) = G(x)y \) for all \( x, y \in R \) and \( R \) is commutative.

**Proof.** First we assume that

\[
a(G(xy) + F(x)F(y) + yx) = 0 \quad \text{for all } x, y \in I. \tag{9}
\]

Replacing \( y \) by \( yz \) in (9), we get

\[
a((G(xy) + F(x)F(y) + yx)z + xyg(z) + F(x)yd(z) + y[z, x]) = 0 \quad \text{for all } x, y, z \in I. \tag{10}
\]
In view of relation (9), (10) reduces to

\[ a(xyg(z) + F(x)yd(z) + y[z, x]) = 0 \text{ for all } x, y, z \in I. \]  

(11)

Substituting \( ry \) for \( y \) in (11), we get

\[ a(xryg(z) + F(x)ryd(z) + ry[z, x]) = 0 \text{ for all } x, y, z \in I, r \in R. \]  

(12)

Replace \( x \) by \( xr \) in (11) to get

\[ a(xryg(z) + F(x)ryd(z) + xd(r)yd(z) + y[z, xr]) = 0 \text{ for all } x, y, z \in I, r \in R. \]  

(13)

From relations (12) and (13), we obtain

\[ a(xd(r)yd(z) + y[z, xr] - ry[z, x]) = 0 \text{ for all } x, y, z \in I, r \in R. \]  

(14)

Substituting \( ay \) for \( y \) in (14), we find that

\[ a(xd(r)ayd(z) + ay[z, xr] - ray[z, x]) = 0 \text{ for all } x, y, z \in I, r \in R. \]  

(15)

Left multiplication by \( a \) to (14) and then subtracting from (15), we obtain

\[ a([xd(r), a]yd(z) + [a, r]y[z, x]) = 0 \text{ for all } x, y, z \in I, r \in R. \]  

(16)

Again replacing \( x \) by \( ax \) in relation (16), we find

\[ a(a[xd(r), a]yd(z) + [a, r]y[z, ax]) = 0 \text{ for all } x, y, z \in I, r \in R. \]  

(17)

Multiplying to (16) by \( a \) from left and then subtracting from (17), we get

\[ a([a, r]y[z, ax] - a[a, r]y[z, x]) = 0 \text{ for all } x, y, z \in I, r \in R. \]  

(18)

Taking \( z = x \) in relation (18), we obtain

\[ a[a, r]y[x, a]x = 0 \text{ for all } x, y \in I, r \in R. \]  

(19)

Taking \( sr \) instead of \( r \) in (19), we get
\[ as[a, r]y[x, a]x + a[a, s]ry[x, a]x = 0 \text{ for all } x, y \in I, r, s \in R. \] 

(20)

Application of (19) yields

\[ as[a, r]y[x, a]x = 0 \text{ for all } x, y \in I, r, s \in R. \] 

(21)

This implies that \( as[a, r]I[x, a]x = (0) \) for all \( x \in I, r, s \in R \). The primeness of \( R \) and hence of \( I \) yields that \( [x, a]x = 0 \) for all \( x \in I \). Replacing \( x \) by \( x + y \) in the last expression, we obtain

\[ [x, a]y + [y, a]x = 0 \text{ for all } x, y \in I. \] 

(22)

Substituting \( yt \) for \( y \) in (22), we obtain \([x, a]yt + [y, a]tx + y[t, a]x = 0\). Right multiplying (22) by \( t \) and then substituting from last expression, we find that \([y, a][t, x] + y[t, a]x = 0\) for all \( x, y \in I, t \in R \). Replacing \( y \) by \( sy \), we obtain \( s[y, a][t, x] + [s, a]y[t, x] + sy[t, a]x = 0 \) for all \( x, y \in I, t, s \in R \). In view of the fact that \([y, a][t, x] + y[t, a]x = 0\), last relation yields \([s, a]y[t, x] = 0\) for all \( x, y \in I, s, t \in R \). In particular, for \( t = a \) and \( s = x \), we conclude that \([x, a][a, x] = 0\) for all \( x, y \in I \). The primeness of \( R \) forces that \([I, a] = 0\), which implies that \( a \in Z(R) \). Since the center of prime ring is free from zero divisor, so the relation (14) reduces to

\[ xd(r)yd(z) + y[z, xr] - ry[z, x] = 0 \text{ for all } x, y, z \in I, r \in R. \] 

(23)

In particular, taking \( r = z = x \) in (23), we obtain \( xd(x)yd(x) = 0 \) for all \( x, y \in I \). Again, the primeness of \( R \) yields that \( xd(x) = 0 \) for all \( x \in I \). Taking \( r = x \) in (23) and using the fact that \( xd(x) = 0 \), we get \( y[z, x^2] = xy[z, x] = 0 \) for all \( x, y, z \in I \). Substituting \( zy \) in place of \( y \), we have \( zy[z, x^2] - xzy[z, x] = 0 \) for all \( x, y, z \in I \). Now using the fact that \( y[z, x^2] = xy[z, x] \), we obtain \( z, x)[y[z, x] = 0 \) for all \( x, y, z \in I \). By the primeness of \( R \), we get \( [I, I] = 0 \), which implies that \( R \) is commutative. Hence, (23) implies that \( xd(r)yd(z) = 0 \) for all \( x, y, z \in I, r \in R \). Again by the primeness of \( R \), it follows that \( d(I) = 0 \). Therefore till now we have proved that \( R \) is commutative, \( a \in Z(R) \) and \( d(I) = 0 \). Using these facts in expression (12), we conclude that \( xygg(z) = 0 \) for all \( x, y, z \in I \) and \( r \in R \). Thus, the primeness of \( R \) gives \( g(I) = 0 \). Then our assumption yields \((G(x) + x)y + F(x)F(y) = 0\) for all \( x, y \in I \). Taking \( y = yr \) for \( r \in R \) in the last relation, we get \((G(x) + x)yr + F(x)F(y)r + F(x)yd(r) = 0\) for all \( x, y \in I \). Thus, we find that \( F(x)yd(r) = 0 \). Replace \( x \) by \( xs \) in last expression, we get \( xd(s)yd(r) = 0 \) for all \( x, y \in I, s \in R \). The primeness of \( R \) yields \( d(R) = 0 \). Similarly, we can prove that \( g(R) = 0 \).
By the similar arguments, same conclusion holds for the case \( a(G(xy) + F(x)F(y) - yx) = 0 \) for all \( x, y \in I \). This completes the proof of the theorem.

**Corollary 2.4.** Let \( R \) be a prime ring, and \( I \) be a nonzero ideal of \( R \). Suppose that \( G \) is a multiplicative (generalized)-derivation associated with a mapping \( g \) on \( R \). If \( a \neq 0 \in R \) such that \( a(G(x)G(y) \pm yx) = 0 \) for all \( x, y \in I \), then \( G(xy) = G(x)y \) for all \( x, y \in R \) and \( R \) is commutative.

**References**


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