Detection Whether a Monoid of the Form 
\[ \mathbb{N}^n / \sim M \] is Affine or Not

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Abstract
In this study, we consider some types of monoids \( M \) such as \( R \) – trivial monoid, Brauer type monoid, Bicyclic monoid and matrix monoids called special linear semigroups, \( \text{SLS}(2, 2) \) and general linear semigroups, \( \text{GLS}(2, 2) \). We decide whether these monoids \( \mathbb{N}^n / \sim M \) are affine semigroups (cancellative, reduced and torsionfree) or not. Moreover Minkowski Farkas Lemma and related algorithms are an important tools.

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1 Introduction


Affine semigroups has some applications in other parts of algebra. For instance affine semigroups appear in algebraic geometry, commutative algebra, number theory. Affine semigroups also appear as Weierstrass semigroup of sets of points.

The aim of this paper is to investigate whether a monoid of the form \( \mathbb{N}^n / \sim M \) is affine or not. We show that \( R \) – trivial monoid, Brauer type monoid and
Bicyclic monoid are non-affine semigroups and matrix monoids are affine semigroups.

In order to state this result precisely, we need to introduce some concepts and notation.

**Definition 1** An affine semigroup is a finitely generated submonoid of \( \mathbb{N}^r \) for some positive integer \( r \) where \( \mathbb{N} \) denotes the set of non-negative integers. Hence any affine semigroup is cancellative \((a + b = a + c \Rightarrow b = c)\) reduced \((it\ is\ only\ unit\ is\ 0\ the\ identity\ element)\) and torsion − free \(( ka = kb \Rightarrow a = b \ for \ k \in \mathbb{Z}^+ )\).

**Definition 2** For a given positive integer \( n \) and \( 1 \leq i, j \leq n \) \((i \neq j)\),

1. \( R_{i \leftrightarrow j} \) is the matrix obtained from the identity matrix \( I_n \) interchanging the rows \( i \) and \( j \).
2. \( R_{i \leftarrow i} \) is the matrix obtained from the identity matrix by multiplying its \( i \) − \text{th} row by \(-1\).
3. \( R_{j \leftarrow j+z_i} \) is the matrix obtained from the identity matrix by adding to the \( j \) − \text{th} row the \( i \) − \text{th} row multiplied by \( z \in \mathbb{Z} \).

\( C_{i \leftrightarrow j}, C_{i \leftarrow i}, C_{j \leftarrow j+z_i} \) are defined similarly. \( R_{i \leftrightarrow j}, R_{i \leftarrow i}, R_{j \leftarrow j+z_i} \) are called row elementary matrices and \( C_{i \leftrightarrow j}, C_{i \leftarrow i}, C_{j \leftarrow j+z_i} \) are known as column elementary matrices.

**Definition 3** Let \( A \) be a matrix with \( s \) rows and \( t \) columns and with integer entries. Then \( A \) is equivalent to a matrix of the form

\[
\begin{pmatrix}
    d_1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
    0 & d_2 & \ldots & 0 & 0 & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & d_r & 0 & \ldots & 0 \\
    0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
\end{pmatrix}
\]

where \( r \leq \min\{s, t\} \), \( \{d_1, \ldots, d_r\} \subset \mathbb{N}/\{0\} \) and \( d_i \) divides \( d_{i+1} \) for all \( i \). The elements \( d_1, \ldots d_r \) are called invariant factors of \( A \).
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**Definition 4** The subgroup $M$ is homogeneous if its invariant factors are all equal to one.

The followings propositions and corollary have been given by Rosales and Garcia-Sanchez. You can see the proofs in [5].

**Proposition 5** Let $M$ be a subgroup of $\mathbb{Z}^n$ such that $(x_1, x_2, ..., x_n) \in M$ if and only if

\[
\begin{align*}
    a_{11}x_1 + ... + a_{1n}x_n &= 0 \pmod{d_1} \\
    &\vdots \\
    a_{r1}x_1 + ... + a_{rn}x_n &= 0 \pmod{d_r} \\
    a_{(r+1)1}x_1 + ... + a_{(r+1)n}x_n &= 0 \\
    &\vdots \\
    a_{(r+k)1}x_1 + ... + a_{(r+k)n}x_n &= 0
\end{align*}
\]

Then $\mathbb{N}^n/\sim M$ is isomorphic to the submonoid $S$ of $\mathbb{Z}_{d_1} \times ... \times \mathbb{Z}_{d_r} \times \mathbb{Z}^k$ generated by

\[
\{(a_{11})_{d_1}, ..., (a_{r1})_{d_r}, a_{(r+1)1}, ..., a_{(r+k)1}, ..., (a_{1n})_{d_1}, ..., (a_{rn})_{d_r}, a_{(r+1)n}, ..., a_{(r+k)n}\}
\]

where $[a]_d$ denotes the equivalence class of $a$ in $\mathbb{Z}_d$.

**Proposition 6** Let $M$ be a subgroup of $\mathbb{Z}^n$ such that $\{e_1, ..., e_n\} \cap M = \emptyset$. The monoid $\mathbb{N}^n/\sim M$ is reduced if and only if $M \cap \mathbb{N}^n = \{0\}$.

**Proposition 7** Let $M$ be a subgroup of $\mathbb{Z}^n$ such that $M \cap \mathbb{N}^n = \{0\}$. Then there exists a strongly positive element $\mathbf{a} = (a_1, a_2, ..., a_n) \in \mathbb{N}^n$ such that for every $(x_1, ..., x_n) \in M$ the equality

\[
a_1x_1 + ... + a_nx_n = 0
\]

holds.

**Proposition 8** Let $M$ be a subgroup of $\mathbb{Z}^n$. The following statements are equivalent.
(i) $M$ is homogeneous.

(ii) $\mathbb{N}^n/\sim M$ is isomorphic to a submonoid of $\mathbb{Z}^k$ for some positive integer $k$.

(iii) $\mathbb{N}^n/\sim M$ is torsion free.

**Proposition 9** Let $S$ be a monoid. Then $S$ is a group if and only if $S$ is cancellative.

**Corollary 10** Let $M$ be a subgroup of $\mathbb{Z}^n$ with rank $n$ and with invariant factors $d_1, \ldots, d_n$.

1. $\mathbb{N}^n/\sim M$ is a finite group.
2. $M$ contains a strongly positive element.
3. $\mathbb{N}^n/\sim M$ is isomorphic to $\mathbb{Z}_{d_1} \times \ldots \times \mathbb{Z}_{d_n}$.

**Theorem 11** (Grillet) Let $S$ be a finitely generated monoid. The monoid is cancellative, torsion free and reduced if and only if is isomorphic to a submonoid of $\mathbb{N}^k$ for some positive integer $k$.

**Proof.** See [5].

**Theorem 12** (Minkowski – Farkas' Lemma) Let $A$ and $b$ be $m \times n$ and $m \times 1$ matrices with rational entries respectively. Let $x$ and $y$ be $n \times 1$ and $1 \times m$ matrices of unknowns respectively. The following statements are equivalent.

(i) The system

$$
Ax = b \\
x \geq 0
$$

has a solution over the rationals.

(ii) The system

$$
yA \geq 0 \\
yb < 0
$$

has no solution over the rationals.

**Proof.** See [5].

**Remark 13** We use the idea involved in the proof of Minkowski – Farkas' Lemma to give a way for finding a nonnegative element of subspace $V$ of $\mathbb{Q}^n$ whose first coordinate is greater than zero (if this element exists).
Now, we present some algorithms called \textit{algorithm FP} and \textit{algorithm SP} to determine whether a subspace of $\mathbb{Q}^n$ has non-trivial nonnegative (strongly) elements.

\textbf{Algorithm 14 (algorithm FP):} The algorithm is used for computing non-negative element of a subspace $V$ of $\mathbb{Q}^n$ whose first coordinate is greater than zero. (if it exists). The input of the algorithm is a matrix $A$ whose rows form a basis of a subspace $V$ of $\mathbb{Q}^n$. The output is a nonnegative element in $V$ whose first coordinate is greater than zero. If this element does not exists then the algorithm return false.

(1) If $n=1$ take $y=(0, ..., 0, a_{i1}, 0, ..., 0)$, where $i = \min \{k \mid a_{k1} \neq 0\}$ and return $x = ya = yA_1$. Observe that if $A_1$ has all entries equal to zero, the problem has no solution. Return false.

(2) If $n>1$, solve the problem for the matrix $(A_1, A_2, ..., A_{n-1})$. If there is no solution for $(A_1, A_2, ..., A_{n-1})$, then there is no solution for $A$. Otherwise, let $z$ be an element in $\mathbb{Q}^r$ such that $zA_1 > 0$ and $zA_i \geq 0$ for all $i \in \{2, ..., n-1\}$.

(a) If $zA_n \geq 0$, then $zA$ is nonnegative element in $V$ whose first coordinate is greater than zero. Return $zA$.

(b) If $zA_n < 0$, let $A_i = A_i - ((zA_i)/(zA_n))A_n$ and solve the problem for $(A_1, A_2, ..., A_{n-1})$.

(i) If there is no solution for $(A_1, A_2, ..., A_{n-1})$, then the original problem has no solution. Return false.

(ii) If $z$ is a solution for $(A_1, A_2, ..., A_{n-1})$, define $y = ((zA_n)/(zA_n))z$ and return $x = yA$.

We apply the algorithm to the matrix $(A_iA_{i+1}...A_nA_1...A_{i-1})$ to find a nonnegative element with its $i$-th coordinate greater than zero. (if it exists). If there exists such an element for every $i$, adding them together we get a strongly positive element belonging to $V$. We stop when we find a \textit{strongly positive element} or Algorithm fails. Therefore the algorithm could be described as follows.

\textbf{Algorithm 15 (Algorithm SP):} The input of the algorithm is a matrix $A$ whose rows form a basis of a subspace $V$ of $\mathbb{Q}^n$. The output is a strongly positive element $x$ in $V$ if this element exists and false otherwise.

(1) Apply algorithm FP to the matrix $A$. If the algorithm fails, then return false. Otherwise, if $y$ is the output of FP, let $x = yA$.

(2) If all the coordinates of $x$ are greater than zero, then return $x$. Otherwise, find the first zero coordinate of $x$. Let $i$ be the position of this coordinate. Apply FP to the matrix $(A_i...A_nA_1...A_{i-1})$. If the output of FP is false, then return false. If the output is $z$, then redefine $x$ as $x + (zA_1...zA_n)$, and repeat this step.
Remark 16  We want to know whether $\mathbb{N}^n/\sim M$ is an affine semigroup. For this reason, we must determine whether it is cancellative, torsion free and reduced.

(1) $\mathbb{N}^n/\sim M$ is cancellative.

(2) We may assume that $\{e_1,...e_n\} \cap M = \emptyset$. Compute the invariant factors and defining equations of $M$. If $M$ is not homogeneous then $\mathbb{N}^n/\sim M$ is not torsion free and therefore it cannot be isomorphic to a submonoid of $\mathbb{N}^k$. Otherwise, let

$$
\begin{align*}
    a_{11}x_1 + ... + a_{1n}x_n &= 0 \\
    & \vdots \\
    & \vdots \\
    a_{r1}x_1 + ... + a_{rn}x_n &= 0
\end{align*}
$$

be the defining equations of $M$.

(3) If $\mathbb{N}^n/\sim M$ is reduced, then $M$ admit an equation of the form

$$
a_1x_1 + ... + a_nx_n = 0, \quad a_i > 0 \text{ for all } i.
$$

This holds if and only if the subspace

$$
V = L_Q(\{(a_{11},...,a_{1n}),...(a_{r1},...,a_{rn})\})
$$

of $Q^n$ has a strongly positive element.

2 R-Trivial Monoid

$R$–trivial monoid has been studied by Bergi C., Bergeron, N., Bhargava, S., and Saliola F. [9] in 2011. In this section, we show that $R$–trivial monoid is non–affine semigroup.

Theorem 17 Let $S$ be the monoid generated by the matrices $\rho_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $\rho_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then, $R$–trivial monoid, $S = \{1, \rho_1, \rho_2, \rho_1\rho_2, \rho_2\rho_1\}$ is not an affine semigroup.
**Proof.** Following the above algorithms, we are looking for a nonnegative element in $S$ whose first coordinate is greater than zero. This is equivalent to finding $y \in \mathbb{N}^3$ such that $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $yA_1 > 0$, $yA_2 \geq 0$, $yA_3 \geq 0$.

Following the algorithm FP, we must solve the problem for $A_1$. Since $a_{11} \neq 0$, we get that $z = (1, 0, 0, 0, 0, 0)$ verifies that $zA_1 = 1 > 0$, $zA_2 = 0 \geq 0$, $zA_3 = 0 \geq 0$. So, $z$ is a solution for $A$. Return $zA = y = (1, 0, 0)$ according to the algorithm FP (a). Therefore, $zA = y$ is a nonnegative element of $\mathbb{N}^3$ such that its first coordinate is greater than zero. Now, we find a strongly positive element whose all coordinates are greater than zero. Algorithm SP (2) says that the first zero of $y$ is found since $zA = y = (1, 0, 0)$ is not a strongly positive element. Then we apply algorithm FP for $(A_2A_3A_1)$. This is equivalent to finding $y_1 \in \mathbb{N}^3$ such that $y_1A_1 > 0$, $y_1A_2 \geq 0$, $y_1A_3 \geq 0$. $z = (0, 0, 0, 0, 1, 0)$ verifies that $zA_1 = 1 > 0$, $zA_2 = 0 \geq 0$, $zA_3 = 0 \geq 0$. Hence, $z$ is a solution for $(A_2A_3A_1)$. Return is $zA = y = (1, 0, 0)$. Here $S$ has no strongly positive elements since the second coordinate of $y$ is not greater than zero. So $S = \mathbb{N}^3/\tilde{M}$ is not reduced. As a result $S$ is not an affine semigroup with proposition 6, proposition 7 and (3) of remark 16. Alternatively, we can see the non-affine property of monoid as below. Let us show that $R$ - trivial monoid is not torsion free by using the definition. Alternatively, $R$ - trivial monoid is $S = \{1, \rho_1, \rho_2, \rho_1\rho_2, \rho_2\rho_1\}$. For $\rho_1, \rho_2 \in S$,

$$k\rho_1 = k \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = k\rho_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq \rho_1 = \rho_2.$$  

Hence, $R$ - trivial monoid, $S$ is not torsion-free and therefore $S$ is non-affine by Grillet theorem.

3 Brauer Type Monoid

The Brauer Type Monoid has been studied by G. Kudryavtseva and V. Mazorchuk [10] in 2005. In this section, we show that the Brauer Type Monoid is $B_3$ is not an affine semigroup.

**Theorem 18** The Brauer Type Monoid is $B_n = \{s_i\} \cup \{\pi_i\}$ where

$$\pi_k = (1 \ k) \quad k = 2, 3, \ldots n$$

$$s_i = (i \ i + 1) \in S_n, \quad i = 1, 2, \ldots, n - 1.$$
\[(i\ j) = (1\ i)(1\ j)(1\ i)\]. Particularly if we take \(n = 3\), then the monoid,

\[B_3 = \{(1\ 2), (2\ 3), (1\ 3)\} = \{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{pmatrix}\}\]

is not an affine semigroup.

**Proof.** Firstly, we write

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 2 & 3 \\
2 & 1 & 3 \\
1 & 2 & 3 \\
1 & 3 & 2 \\
1 & 2 & 3 \\
3 & 2 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 6 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\]

then applying elementary row and column operations to the middle and rightmost matrix, we obtain,

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 6 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}
\]

From here, we get the equations

\[
x_1 = 0 \pmod{1} \\
x_1 + x_2 = 0 \pmod{2} \\
x_1 + x_2 + x_3 = 0 \pmod{6}
\]

The first equation is redundant. The invariant factors of \(B_3\) are \(d_1 = 1, d_2 = 2, d_3 = 6\).

\[
\mathbb{Z}_2 = \langle [1]_2, [1]_2 \rangle = \langle [1]_2 \rangle \\
\mathbb{Z}_6 = \langle [1]_6, [1]_6, [1]_6 \rangle
\]

and therefore the monoid \(B_3 = \mathbb{N}^3/\tilde{M}\) is isomorphic to the submonoid \(S = \langle [1]_2, [1]_6 \rangle\) of \(\mathbb{Z}_2 \times \mathbb{Z}_6\). As a result, according to the (2) of remark 16, \(\mathbb{N}^3/\tilde{M}\) is not torsion-free since \(\mathbb{N}^3/\sim M\) is not isomorphic to the submonoid of \(\mathbb{N}^3\). Hence \(B_3\) is not an affine semigroup.
4 Bicyclic Monoid

*Bicyclic monoid* is very known monoid type in semigroup theory. Here, we say that *Bicyclic monoid* is *non-affine semigroup*.

**Theorem 19** \( B = \langle (1, 0), (0, 1) \rangle \) is a subgroup of \( GL(2, \mathbb{N}) \). Bicyclic monoid \( B \) is *non-affine semigroup*.

**Proof.** Bicyclic monoid, \( B \) is not reduced with proposition 6 since \( GL(2, \mathbb{N}) \) has not a subgroup such that \( B \cap \{e_1, e_2\} = \emptyset \). Thus, \( B \) is *non-affine* by Grillet theorem. Also, we may see the non-affine property of bicyclic monoid by using algorithm FP. We are looking a nonnegative element whose first coordinate is greater than zero. This is equivalent to finding \( y \in \mathbb{Q}^2 \) such that

\[
A = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}, \quad yA_1 > 0, \quad yA_2 \geq 0
\]

We solve problem for \((A_1, A_2)\). Take \( z = (1, 0) \). Then \( zA_1 = 1 > 0 \), \( y = zA = (1, 0) \). So we did not find an element whose second coordinate greater than zero. Thus bicyclic monoid has no strongly positive element hence \( \mathbb{N}^2/\sim M = B \) is not reduced so is *non-affine*. ■

5 Matrix Monoids

For a prime number \( p \) and a natural number \( n \), \( GF(p^n) \) denotes the finite field with \( p^n \) elements. It is well known that the multiplicative group \( GF(p^n) - \{0\} \) is cyclic and we will denote by \( \xi \) a generator. We write \( GF(d, p^n) \) for the group of all non-singular \( d \times d \) matrices over \( GF(p^n) \), and \( SL(d, p^n) \) for the group of all matrices from \( GF(d, p^n) \) having determinant 1. \( SL(d, p^n) \) is a normal subgroup of \( GF(d, p^n) \).

Semigroup analogues of \( GF(d, p^n) \) and \( SL(d, p^n) \) are the semigroup of all \( d \times d \) matrices over \( GF(p^n) \) and the semigroup of all \( d \times d \) matrices of determinant 0 or 1. We call these semigroups the *general linear semigroup* and the *special linear semigroup* respectively, and denote them by a \( GLS(d, p^n) \) and \( SLS(d, p^n) \) respectively. Both these semigroups are regular and possess a zero and an identity.

These semigroups have been examined by Nikola Ruskuc in [8].

In the following sections, we prove that matrix monoids are affine semigroups.

5.1 Special Linear Semigroups, SLS(2,p)

**Theorem 20** Let \( SLS(2, p) \) is generated by the matrices
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\[ A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} \xi & 0 \\ 0 & 0 \end{pmatrix} \]

where \( \xi \) is generator for the group \( GF(p) - \{0\} \) or in other words, a primitive root of 1 modulo \( p \). Then, the special linear semigroup, \( SLS(2, p) \) is an affine semigroup provided that \( p = 2, \xi = 3 \).

**Proof.** \( 3^{2-1} \equiv 1(\text{mod } 2) \) and so we take \( S = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \).

\[
\text{Generator}(SLS(2, 2)) = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \right\}
\]

Following the algorithms. We are looking for a nonnegative element in \( SLS(2, 2) \) whose first coordinate is greater than zero. This is equivalent to finding \( y \in \mathbb{N}^2 \) such that \( A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 3 & 0 \\ 0 & 0 \end{pmatrix} \), \( yA_1 > 0, yA_2 \geq 0 \). Following the algorithm FP, we must solve the problem for \( A_1 \). Since \( a_{11} \neq 0 \), we get that \( z = (1, 0, 0, 0, 0, 0) \) verifies that \( zA_1 = 1 > 0, zA_2 = 0 \geq 0 \). So, \( z \) is a solution for \( A \). Return \( zA = y = (1, 0) \) according to the algorithm FP(a). Therefore, \( zA = y \) is a nonnegative element of \( \mathbb{N}^2 \) such that its first coordinate is greater than zero. Now, we find a strongly positive element whose all coordinates are greater than zero. Algorithm SP(2) says that the first zero of \( y \) is found since \( zA = y = (1, 0) \) is not a strongly positive element. Then we apply algorithm FP for \( A_2A_1 \). This is equivalent to finding \( y_1 \in \mathbb{N}^2 \) such that \( y_1A_10, y_1A_2 \geq 0 \). \( z = (0, 1, 0, 0, 0, 0) \) verifies that \( zA_1 = 1 > 0, zA_2 = 1 > 0 \). Hence, \( z \) is a solution for \( (A_2A_1) \). Return is \( zA = y = (1, 1) \). Here \( SLS(2, 2) \) has strongly positive elements since the second coordinate of \( y \) is greater than zero. So \( S = \mathbb{N}^2 / SLS(2, 2) \) is reduced. Now we compute invariants factors of \( A \). Firstly we write,

\[
\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 3 & 0 \\ 0 & 0 \end{pmatrix}\]

then applying elementary row and column operations to the middle and right-
most matrix, we obtain,
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0 \\
-1 & 1 \\
0 & 0
\end{pmatrix}
\]

From here, we get the equations. The invariant factors of \(A\) are \(d_1 = 1\), \(d_2 = 1\) so \(SLS(2, 2)\) is homogeneous by definition 1 and hence \(\mathbb{N}^n/\sim SLS(2, 2)\) is torsion-free with proposition 8. Moreover, since \(SLS(2, 2)\) has strongly positive elements and \(\mathbb{N}^n/\sim SLS(2, 2) \cong \mathbb{Z} \times \mathbb{Z}\), \(\mathbb{N}^n/\sim SLS(2, 2)\) is a finite group according to corollary 10 and so \(\mathbb{N}^n/\sim SLS(2, 2)\) is cancellative by proposition 9. As a result, \(\mathbb{N}^n/\sim SLS(2, 2)\) is an affine semigroup by Grillet theorem.

5.2 General Linear Semigroups, SLS(2,p)

**Theorem 21** Let \(GLS(2, p)\) is generated by the matrices

\[
A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 0 & \xi \end{pmatrix}, \quad S = \begin{pmatrix} \xi & 0 \\ 0 & 0 \end{pmatrix}
\]

where \(\xi\) is generator for the group \(GF(p) - \{0\}\) or in other words, a primitive root of 1 modulo \(p\). Then, the general linear semigroup, \(SLS(2, p)\) is an affine semigroup provided that \(p = 2, \xi = 3\).

**Proof.** \(3^{2-1} \equiv 1(\text{mod} 2)\) and so we take \(S = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}\).

\[\text{Generator}(GLS(2, 2)) = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \right\}\]

Following the algorithms, we are looking for a nonnegative element in \(GLS(2, 2)\) whose first coordinate is greater than zero. This is equivalent to finding \(y \in \mathbb{N}^2\) such that \(A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 3 & 0 \\ 0 & 0 \end{pmatrix}\), \(yA_1 > 0, yA_2 \geq 0\). Following the algorithm FP, we must solve the problem for \(A_1\). Since \(a_{11} \neq 0\), we get that \(z = (1, 0, 0, 0, 0, 0)\) verifies that \(zA_1 = 10, zA_2 = 0 \geq 0\). So, \(z\) is a solution.
for \( A \). Return \( zA = y = (1, 0) \) according to the algorithm \( \text{FP}(a) \). Therefore, \( zA = y \) is a nonnegative element of \( \mathbb{N}^2 \) such that its first coordinate is greater than zero. Now, we find a strongly positive element whose all coordinates are greater than zero. Algorithm \( \text{SP}(2) \) says that the first zero of \( y \) is found since \( zA = y = (1, 0) \) is not a strongly positive element. Then we apply algorithm \( \text{FP} \) for \( (A_2A_1) \). This is equivalent to finding \( y_1 \in \mathbb{N}^2 \) such that \( y_1A_1 > 0 \), \( y_1A_2 \geq 0 \). \( z = (0, 1, 0, 0, 0, 0, 0) \) verifies that \( zA_1 = 1 > 0 \), \( zA_2 = 1 \geq 0 \). Hence, \( z \) is a solution for \( (A_2A_1) \). Return is \( zA = y = (1, 1) \). Here \( \text{SLS}(2, 2) \) has strongly positive elements since the second coordinate of \( y \) is greater than zero. So \( S = \mathbb{N}^2/\sim \text{GLS}(2, 2) \) is reduced. Now we compute invariant factors of \( A \). Firstly we write,

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
1 & 1 \\
1 & 1 \\
0 & 1 \\
1 & 0 \\
0 & 3 \\
3 & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}
\]

then applying elementary row and column operations to the middle and right-most matrix, we obtain,

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
1 & 1 \\
0 & 1 \\
1 & 0 \\
0 & 3 \\
0 & 0 \\
3 & 0 \\
0 & 0
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}
\]

From here, we get the equations

\[
x_1 - x_2 &= 0 \text{(mod 1)} \\
x_2 &= 0 \text{(mod 1)}
\]

Since all the invariant factors of \( A \) is 1, \( A \) is homogeneous with proposition 8. So \( A \) is torsion-free. Moreover, since \( \text{GLS}(2, 2) \) has strongly positive elements and \( \mathbb{N}^n/\sim \text{GLS}(2, 2) \simeq \mathbb{Z} \times \mathbb{Z} \), \( \mathbb{N}^n/\sim \text{SLS}(2, 2) \) is a finite group according to corollary 10 and so \( \mathbb{N}^n/\sim \text{GLS}(2, 2) \) is cancellative by proposition 9. As a result, \( \mathbb{N}^n/\sim \text{GLS}(2, 2) \) is an affine semigroup by Grillet theorem. ■
References


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