G-algebras, Lie Algebras, Hopf Algebras II

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Dedicated to Edward L. Green who introduced me to the world of Koszul algebras and non commutative Groebner basis.

Abstract

In a series of papers the author jointly with J. Mondragón studied in a systematic way the structure of homogeneous Groebner basis algebras, or G-algebras, and in two recent papers the author applied these results to the homogenized enveloping algebra of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. The results obtained in the study of this particular algebra inspired our previous paper on the Hopf algebra structure of the homogenized enveloping algebra of finite dimensional Lie algebras.

In this first section of this paper we continue further the study of the homogenization process, this time at the level of Ext and Tor. In the second part we study quantized homogeneous G-algebras $B_n$ and give the structure of their Yoneda algebra. Using this construction we find the structure of arbitrary quadratic G-algebras in terms of their structure of constants.

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1 Introduction

Let $k$ be a field and $k\langle X_1, X_2, \ldots, X_n \rangle$ the free algebra in $n$ generators. A quadratic Groebner basis algebra or a G-algebra is an algebra defined by gen-
generators and relations as \( A_n = \mathbb{k} \langle X_1, X_2, \ldots, X_n \rangle / \langle X_j X_i - c_{ij} X_i X_j - \sum_{k=1}^{n} b_{ij}^k X_k - a_{ij} \rangle \), for \( i < j \) and such that \( X_1, X_2, \ldots, X_n \) form a Poincare Birkhoff Witt basis. In case \( a_{ij} = 0, c_{ij} = 1 \) for \( i < j \), a \( G \)-algebra \( A_n \) is isomorphic to the enveloping algebra of a finite dimensional Lie algebra, and in the case \( a_{ij} = 1, c_{ij} = 1 \) for \( i < j \), and \( b_{ij}^k = 0 \) for all \( k \), \( A_n \) is the Weyl algebra \([5], [6]\).

A homogeneous \( G \)-algebra is an algebra given by generators and relations as \( B_n = \mathbb{k} \langle X_1, X_2, \ldots, X_n, Z \rangle / \langle X_j X_i - c_{ij} X_i X_j - \sum_{k=1}^{n} b_{ij}^k X_k Z - a_{ij} Z^2, X_j Z - Z X_i \rangle \) such that \( X_1, X_2, \ldots, X_n, Z \) form a Poincare Birkhoff Witt (or PBW) basis. We proved in \([15], [18]\) that an homogeneous algebra \( B_n \) has a PBW basis if and only if \( B_n/(Z-1)B_n \) has a PBW basis, moreover \( A_n \) is isomorphic to \( B_n/(Z-1)B_n \). We called to the relation between \( A_n \) and \( B_n \), the homogenization/deshomogenization process. The homogeneous version \( B_n \) of the \( G \)-algebra \( A_n \) has the advantage of being Koszul so we can use Koszul theory to relate the Koszul \( B_n \)-modules with the Koszul modules over the Yoneda algebra \( B_n^! \), this has been the approach we followed in \([15], [18], [19]\).

The paper consists of two parts. The first one is dedicated to the study of Ext and Tor for a homogenized \( G \)-algebra \( B_n \) and their relations with the corresponding Ext and Tor of the graded localization \( B_n Z \), and those of the deshomogenized \( G \)-algebra \( B_n/(Z-a)B_n \).

In the second part of the paper we look to the Yoneda algebra \( B_n^! \) of a homogenized \( G \)-algebra \( B_n \) with quantized relations. The algebra \( B_n^! \) has as a subalgebra the quantized exterior algebra \( C_n^! \), we prove that there is a graded derivation \( \partial : C_n^! \to C_n^! [1] \) with \( \partial^2 = 0 \), and that the tensor product \( \mathbb{k}[Z]/(Z)^2 \otimes C_n^! \), twisted with \( \partial \) is isomorphic to \( B_n^! \). Conversely, given a graded derivation \( \partial \) of \( C_n^! \) with \( \partial^2 = 0 \), the skew tensor product \( \mathbb{k}[Z]/(Z)^2 \otimes C_n^! \) is isomorphic to the Yoneda algebra \( B_n^! \) of a homogeneous \( G \)-algebra \( B_n \). Using this characterization we describe the structure of constants of an arbitrary quadratic \( G \)-algebra \( B_n \). We end the paper with the remark that the quantized polynomial algebra \( C_n \) and the quantized exterior algebra \( C_n^! \) have a Hopf structure under a skew tensor product that generalizes the graded tensor product.

### 2 Ext and Tor for homogeneous \( G \)-algebras and the des homogenization process

#### 2.1 The homogenization des homgenization of \( \mathbb{k}[Z] \)-modules

Through the paper \( \mathbb{k} \) will denote a field of characteristic zero. In this section we study in further detail the properties of the graded localization \( \mathbb{k}[Z]_Z \) of the
polynomial ring in one variable, considered in our previous papers. We start recollecting results from [15], [16], [18], [19] stating them in the form that we need for this paper.

We consider next the graded functors $\text{Ext}^*_{B_n}(-,?)$ and $\text{Tor}^*_{B_n}(-,?)$ and their relations with the functors $\text{Ext}^*_{B_{nZ}}(-,?)$ and $\text{Tor}^*_{B_{nZ}}(-,?)$ over the graded localization and the functors $\text{Ext}^*_{B_n/(z-a)B_n}(-,?)$ and $\text{Tor}^*_{B_n/(z-a)B_n}(-,?)$ over the des homogenized algebra $B_n/(z-a)B_n$.

The ring $k[Z]$ has the usual graduation. We denote by $k[Z]$ the localization $k[Z]_S$ with $S$ the multiplicative set $S=\{1, Z, Z^2, \ldots, Z^n, \ldots\}$. The ring $k[Z]$ is $Z$-graded with homogeneous elements of the degree $m$, $(k[Z]_m) = \{aZ^m \mid a \in k\} = kZ^m$. Hence the homogeneous elements of degree $m$ form a one dimensional $k$-vector space.

For $a \in k \{-0\}$ $(Z-a)$ is a maximal ideal of $k[Z]$, and $k[Z]/(Z-a)$ is a one dimensional $k$-vector space.

We have a composition of algebra maps: $k[Z] \xrightarrow{j} k[Z]_Z \xrightarrow{\pi} k[Z]/(Z-a)k[Z]$ with $\varphi = \pi j$ given by $\varphi(f) = f + (Z-a)k[Z]$. The map $\varphi$ is onto and has kernel $(Z-a)k[Z]$.

We have:

**Proposition 2.1.** There are ring isomorphisms: $k[Z]/(Z-a)k[Z] \cong k[Z]/(Z-a)k[Z] \cong k$, and for any $m \in Z$ $(k[Z]_m) = kZ^m \cong k$.

The proposition can be generalized for $Z$-graded $k[Z]$-module $M$, to do this we need first the following:

**Definition 2.2.** For a $Z$-graded $k[Z]$-module $M$ we define the $Z$-torsion part as:

$t_Z(M) = \{ m \in M \mid \text{there is a non negative integer } k \text{ such that } Z^k m = 0 \}$

and the usual torsion part

$t(M) = \{ m \in M \mid \text{there is } f \in k[Z] \text{ f \neq 0 and } fm = 0 \}$

Clearly $t_Z(M) \subseteq t(M)$.

**Lemma 2.3.** Let $M$ be a $Z$-graded $k[Z]$-module. Then $t_Z(M) = t(M)$. In particular $t(M)$ is $Z$-graded.

**Proof.** Let $m = m_k + m_{k-1} + \ldots + m_1$ be the decomposition in homogeneous components of $m \in t(M)$ and assume $\deg(m_i) \geq \deg(m_{i-1})$ and let $f(Z) = c_0 + c_1 Z + \ldots + c_\ell Z^\ell$ be a non zero polynomial of degree $\ell$ such that $f(Z)m = 0$. Then $f(Z)m$ has a homogeneous components $(f(Z)m)_n = \sum_{\deg(m_j) + i = n} c_i Z^i m_j = 0$. In particular, $c_\ell Z^\ell m_k = 0$ and $m_k \in t_Z(M) \subseteq t(M)$. Then $m - m_k \in t(M)$ and $m - m_k = m_{k-1} + \ldots + m_1$. It follows by induction on the number of homogeneous summands that each $m_i \in t_Z(M)$.
It follows that the module $M/t(M) = M/t_Z(M)$ is a graded torsion free $k[Z]$-module, hence; graded flat.

**Lemma 2.4.** Assume $M$ is a graded torsion $k[Z]$-module. Then for $a \in k-\{0\}$ $M/(Z-a)M = 0$.

**Proof.** Let $m \in M$ be non zero. Then there is a $k > 0$ such that $((Z-a)+a)^k m = Z^k m = 0$, hence there is a polynomial $h(Z)$ such that $a^k m + (Z-a)h(Z)m = 0$. Then $m \in (Z-a)M$.

If $M$ is torsion, then we know the localization $M_Z$ is zero. Therefore: the exact sequence: $0 \rightarrow t(M) \rightarrow M \rightarrow M/t(M) \rightarrow 0$, induces exact sequences:

$t(M)/(Z-a)t(M) \rightarrow M/(Z-a)M \rightarrow M/t(M)/(Z-a)(M/t(M)) \rightarrow 0$ and $0 \rightarrow t(M)Z \rightarrow MZ \rightarrow (M/t(M))Z \rightarrow 0$ with $t(M)/(Z-a)t(M) = t(M)Z = 0$.

We may assume $M$ is torsion free. We have a composition of maps: $M \rightarrow M_Z \rightarrow M/(Z-a)M_Z$ given by $m \rightarrow m/1+(Z-a)M$. If $m/1 = (Z-a)\sum_{\ell=-t}^{k} Z^\ell m_{\ell}$, then $Z^\ell m = (Z-a)m'$, with $m' \in M$.

As before, $Z^\ell m = ((Z-a)+a)^\ell m = (Z-a)h(Z)m + a^\ell m$. Therefore: $m = (Z-a)a^{-\ell}(m'-h(Z)m)$.

Hence the map $M \rightarrow M_Z/(Z-a)M_Z$ has kernel $(Z-a)M$.

Let $\sum_{\ell=-t}^{k} Z^\ell m_{\ell} + (Z-a)M_Z$ be an element of degree $k$. This is $\deg(m_{\ell}) + \ell = k$ with $\varphi((\sum_{\ell=-t}^{k} Z^\ell m_{\ell})) = 0$.

**Proposition 2.5.** Let $M$ be a graded $k[Z]$-module and $k \in k-\{0\}$. Then there are isomorphisms of $k$-vector spaces:

$$(M_Z)_k \cong M/(Z-a)M \cong M_Z/(Z-a)M_Z$$

**Proof.** In view of Lemma 2.4 we can assume $M$ is torsion free. Consider the map $\varphi: (M_Z)_k \rightarrow M_Z \rightarrow M_Z/(Z-a)M_Z \varphi = \pi j$.

Let $\sum_{\ell=-t}^{k} Z^\ell m_{\ell}$ be an element of degree $k$, this is $\deg(m_{\ell}) + \ell = k$ with $\varphi((\sum_{\ell=-t}^{k} Z^\ell m_{\ell})) = 0$. 

This is \( \sum_{\ell=-t}^{\ell=k} Z^\ell m_\ell = (\sum_{\ell=-t}^{\ell=t} Z^{\ell+t} m_\ell)/Z^r = (Z-a) m'/Z^r \), with \( m = \sum_{\ell=-t}^{\ell=k} Z^{\ell+t} m_\ell \in M \) an homogeneous element of degree \( k+t \).

\( m' \) has a decomposition \( m' = m'_s + m'_{s-1} + ... + m'_1 \), with \( \deg(m'_i) > \deg(m'_{i-1}) \).

Then \( Z^r m = (Z-a)Z^r m' = Z^{r+1} m'_s + Z^{r+1} m'_{s-1} + ... + Z^{r+1} m'_1 - aZ^s m'_s aZ^r m'_{s-1} - ... aZ^r m'_1. \)

Comparing degrees we get a contradiction. Therefore the map \( \varphi \) is injective.

If \( \sum_{\ell=-t}^{\ell=k} Z^\ell m_\ell + (Z-a)M_Z \) is an element of \( M_Z/(Z-a)M_Z \), then as above \( m/Z^r = \sum_{\ell=-t}^{\ell=k} Z^\ell m_\ell \) with \( m = \sum_{\ell=-t}^{\ell=k} Z^{\ell+t} m_\ell. \)

We may assume \( m \) homogeneous of degree \( \deg(m) = s+t. \)

When \( k \geq s \deg(Z^s m/Z^t) = k. \)

As above, \( (Z^s m)/Z^t = a^{s-k} m/Z^t + (Z-a)m'/Z^t \) and \( \varphi(a^{s-k} Z^s m/Z^t) = \sum_{\ell=-t}^{\ell=k} Z^\ell m_\ell + (Z-a)M_Z. \)

The case \( s<k \) is similar. \( \square \)

**Corollary 2.6.** Let \( 0 \to L \to M \to N \to 0 \) be an exact sequence of graded \( k[Z] \)-modules. Then the sequence:

\[
0 \to L/(Z-a)L \to M/(Z-a)M \to N/(Z-a)N \to 0
\]

is exact.

**Proof.** We have a commutative diagram with exact rows and such that the composition of the maps in the columns are isomorphisms:

\[
\begin{array}{cccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & (L_Z)_k & (M_Z)_k & (N_Z)_k \\
\downarrow & \downarrow & \downarrow \\
L_Z/(Z-a)L_Z & M_Z/(Z-a)M_Z & N_Z/(Z-a)N_Z & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0
\end{array}
\]

It follows the sequence \( 0 \to L_Z/(Z-a)L_Z \to M_Z/(Z-a)M_Z \to N_Z/(Z-a)N_Z \to 0 \) is exact.

We also have a commutative diagram with exact rows and the columns isomorphisms:
It follows the sequence \( 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \) is exact.

\[ L/(Z-a)L \rightarrow M/(Z-a)M \rightarrow N/(Z-a)N \rightarrow 0 \]

2.2 The homology and cohomology of homogeneous G-algebras

In this subsection we apply the results of the previous subsection to study the homology and cohomology of the homogeneous G-algebras \( B_n \) and their relations with the corresponding homology and cohomology of the algebras \( B_nZ \) and \( B_n/(Z-a)B_n \).

A homogenous G-algebra is an algebra \( B_n \) defined by generators and relations as \( B_n = k < X_1, X_2, \ldots, X_n, Z > / \langle X_jX_i - c_{ij}X_iX_j - \sum b_{ij}X_iZ - a_{ij}Z, X_iZ - ZX_i \rangle \) and such that \( X_1, X_2, \ldots, X_n, Z \) form a PBW basis. We will assume the reader is familiar with the results of [18], [19].

By definition \( Z \) is in the center of \( B_n \) and \( B_n \) is a \( k \)-algebra. The (graded) localization \( B_nZ \) is the (graded) tensor product \( B_n \otimes k[Z] \), and \( k[Z] \) is isomorphic to the Laurent polynomials \( k[Z, Z^{-1}] \). For any right (graded) \( B_n \)-module \( M \) the localization (graded) is \( MZ = M \otimes_{B_n} B_nZ \cong M \otimes_{k[Z]} k[Z, Z^{-1}] \).

We know homogeneous G-algebras are noetherian of finite global dimension. The next proposition is a slight generalization of the commutative case, we refer to [26] for the proof.

**Proposition 2.7.** Let \( B_n \) be a homogeneous G-algebra, \( M \) a finitely generated (graded) right module, \( N \) an arbitrary (graded) right module. Then for \( m \geq 0 \) there is a natural isomorphism of (graded) \( k[Z] \)-modules

\[
\text{Ext}^m_{B_n}(M, N)_Z = \text{Ext}^m_{B_n}(M, N) \otimes_{k[Z]} k[Z, Z^{-1}] \cong \text{Ext}^m_{B_nZ}(M_Z, N_Z)
\]

**Corollary 2.8.** For any (graded) right \( B_n \)-module \( M \) there is an isomorphism of graded rings: \( (\oplus \text{Ext}^m_{B_n}(M, M))_Z \cong (\oplus \text{Ext}^m_{B_nZ}(M_Z, M_Z))_Z \).

We see next that there is a similar proposition for the deshomogenized algebra.

**Proposition 2.9.** Let \( B_n \) be a homogeneous G-algebra, \( M \) a finitely generated graded right module, \( N \) an arbitrary graded right module. Then for \( m \geq 0 \), \( a \in k - \{ 0 \} \) there is a natural isomorphism of \( k \)-vector spaces:

\[
\text{Ext}^m_{B_n}(M, N)/(Z-a)\text{Ext}^m_{B_n}(M, N) \cong \text{Ext}^m_{B_n/(Z-a)B_n}(M/(Z-a)M, N/(Z-a)N).
\]
Proof. Taking a graded projective presentation of $M$, $P_1 \to P_0 \to M \to 0$ we obtain an exact sequence of $\mathbb{Z}$-graded $k[Z]$-modules

$$0 \to \text{Hom}_{B_n}(M,N) \to \text{Hom}_{B_n}(P_0,N)$$

$$\text{Hom}_{B_n}(P_1,N)$$

By Corollary 2.6 we obtain an exact sequence of $k[Z]$-modules

$$0 \to \text{Hom}_{B_n}(M,N)/(Z-a)\text{Hom}_{B_n}(M,N) \to \text{Hom}_{B_n}(P_0,N)/(Z-a)\text{Hom}_{B_n}(P_0,N)$$

$$\text{Hom}_{B_n}(P_1,N)/(Z-a)\text{Hom}_{B_n}(P_1,N)$$

For each finitely generated graded projective $P$ there are natural isomorphisms of $k$-vector spaces:

$$\text{Hom}_{B_n}(P,N)/(Z-a)\text{Hom}_{B_n}(P,N) \cong \text{Hom}_{B_n}(P,N) \otimes_{k[Z]} k[Z]/(Z-a)$$

$$\text{Hom}_{B_n}(P,B_n) \otimes_{B_n} N \otimes_{k[Z]} k[Z]/(Z-a)N \cong \text{Hom}_{B_n}(P,B_n) \otimes_{B_n} N/(Z-a)N$$

Hence the exact sequence (*) is isomorphic to the exact sequence

$$0 \to \text{Hom}_{B_n}(M,N)/(Z-a)\text{Hom}_{B_n}(M,N)$$

For $m \geq 0$ Ext$^m_{B_n}(M,N)/(Z-a)\text{Ext}^m_{B_n}(M,M) \cong \text{Ext}^m_{B_n/(Z-a)B_n}(M/(Z-a)M,N/(Z-a)N)$.

Corollary 2.10. For a finitely generated graded $B_n$-module $M$ there is an isomorphism of graded algebras

$$\bigoplus_{m \geq 0} \text{Ext}^m_{B_n}(M,M)/(Z-a) \bigoplus_{m \geq 0} \text{Ext}^m_{B_n}(M,M) \cong \bigoplus_{m \geq 0} \text{Ext}^m_{B_n/(Z-a)B_n}(M/(Z-a)M,M/(Z-a)M).$$

We have analogous results for the homology.

Proposition 2.11. i) Let $M$ be a (graded) right $B_n$-module and $N$ a left (graded) $B_n$-module. Then for $m \geq 0$ there is a natural isomorphism of (graded) $k[Z]$-modules

$$\text{Tor}^m_{B_n}(M,N) \cong \text{Tor}^m_{B_n}(M,N)$$
ii) Let $M$ be a finitely generated graded right $B_n$-module and $N$ a graded left $B_n$-module. Then for $m \geq 0$ there is a natural isomorphism of $k$-vector spaces

$$\text{Tor}^B_m(M,N)/(Z-a)\text{Tor}^B_m(M,N) \cong \text{Tor}^B_m/(Z-a)B_n(M/(Z-a)M,N)/(Z-a)N$$

**Proof.** We give the proof for $m=0$ and leave the general case to the reader.

i) $(M \otimes_{B_n} N)_Z = (M \otimes_{B_n} N) \otimes_{k[Z]} k[Z,Z^{-1}] \cong M \otimes_{B_n} N_Z \cong M \otimes_{B_n} B_n \otimes_{B_n \otimes B_n} N_Z \cong M_Z \otimes_{B_n \otimes B_n} N_Z$.

ii) $(M \otimes_{B_n} N)/(Z-a)(M \otimes_{B_n} N) \cong (M \otimes_{B_n} N) \otimes_{k[Z]} k[Z]/(Z-a)k[Z] \cong M \otimes_{B_n} k N/(Z-a)N \cong M \otimes_{B_n} B_n/(Z-a)B_n \otimes_{B_n \otimes B_n} N/(Z-a)N$.

For the general case use Corollary 2.6. $\square$

### 3 The structure of quadratic $G$-algebras

In this section we give the structure of constants of the quadratic $G$-algebras, [3],[13] generalizing our results from [17]. We divide the section in three subsections:

In the first one we study the quantized exterior algebra and its derivations.

In the second subsection we give the construction of a family of homogeneous $G$-algebras. From this construction we get the general structure of $G$-algebras.

In the last subsection we study the Hopf structure of the quantized polynomial ring and the quantized exterior algebras.

#### 3.1 The quantized exterior algebra and its derivations

The quantized exterior algebra is the algebra $C_q^n = k \langle X_1X_2,...,X_n,\rangle/\langle X_i^2, X_iX_j + c_{ij}X_jX_i, c_{ij} \in k \{0\}, i<j \rangle$. We are interested in graded derivations $\partial:C_q^n \to C_q^n[1]$ with $\partial^2=0$. This means $\partial$ is $k$-linear, $\partial(1)=0$ and if $w_1$, $w_2$ are homogeneous elements of lengths $\ell(w_1)$, $\ell(w_2)$, then $\partial(w_1w_2)=\partial(w_1)w_2 + (-1)^{\ell(w_1)}w_1\partial(w_2)$.

We know by [18] $C_q^n$ has a PBW basis, ordering the variables $X_1 < X_2 < ... < X_n$ the basis of $C_q^n$ consists of the words $X_{i_1}X_{i_2}...X_{i_m}$ with $X_{i_k} \neq X_{i_j}$ for $j \neq k$ and $X_{i_1} < X_{i_2} < ... < X_{i_m}$.

Since $\partial$ is of degree one for any $1 \leq k \leq n$ $\partial(X_k) = -\sum_{i<j}b_{ij}^k X_jX_i$.

We have in $C_q^n$ the relation $X_iX_j + c_{ij}X_jX_i = 0$, hence $c_{ij}^{-1}X_iX_j + X_iX_j = 0$ and we define $c_{ji} = c_{ij}^{-1}$.

We have:

$$\partial(X_k) = -1/2 \sum_{i<j}b_{ij}^k X_jX_i - 1/2 \sum_{i<j}b_{ij}^k X_jX_i = -1/2 \sum_{i<j}b_{ij}^k X_jX_i - 1/2 \sum_{i<j}b_{ij}^k (-c_{ji})X_iX_j.$$

For $i < j$ we define $b_{ij}^k = b_{ij}^k (-c_{ji})$ and $b_{ii}^k = 0$.

The derivation becomes $\partial(X_k) = -1/2 \sum_{i<j}b_{ij}^k X_jX_i$. 

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For any word $X_{i_1}X_{i_2}\ldots X_{i_m}$ we have by definition
$$\partial(X_{i_1}X_{i_2}\ldots X_{i_m})=\partial(X_{i_1}X_{i_2}\ldots X_{i_{m-1}})X_{i_m}+(-1)^{m-1}X_{i_1}X_{i_2}\ldots X_{i_{m-1}}\partial(X_{i_m}).$$

It follows by induction $\partial(X_{i_1}X_{i_2}\ldots X_{i_m})=\sum_{r=1}^{m}(-1)^{r-1}X_{i_1}\ldots X_{i_{r-1}}\partial(X_{i_r})X_{i_{r+1}}\ldots X_{i_m}.$

With this definition follows by induction $\partial^2=0$ if and only if for $1\leq k\leq n,$ $\partial^2(X_k)=0$.

We have proved:

i) It is enough to define $\partial$ for each $X_k$.

ii) $\partial^2=0$ if and only if for $1\leq k\leq n,$ $\partial^2(X_k)=0$.

We want to find conditions on the structure of constants $\{c_{ij}\}$ and $\{b_{ij}^k\}$ in order to have $\partial^2=0$. We will make some calculations.

$$\partial(X_k)=-\frac{1}{2}\sum_{i,j}b_{ij}^kX_jX_i$$
$$\partial(X_j)=-\frac{1}{2}\sum_{r,s}b_{rs}^jX_sX_r$$
$$\partial(X_i)=-\frac{1}{2}\sum_{u,v}b_{uv}^iX_uX_v$$
$$\partial^2(X_k)=-\frac{1}{2}\sum_{i,j}b_{ij}^k(\partial(X_j)X_jX_j\partial(X_i))=$$
$$\frac{1}{4}(\sum_{i,j}b_{ij}^k(\sum_{r,s}b_{rs}^jX_sX_rX_i+\sum_{u,v}b_{uv}^iX_jX_u))=$$
$$\frac{1}{4}(\sum_{i,j}b_{ij}^k)X_sX_rX_i+\sum_{j,u,v}(\sum_{i}b_{ij}^k)X_jX_u)$$
$$\partial^2(X_k)=0 \text{ if and only if }$$
$$^*) \sum_{i,j}(\sum_{r,s}b_{ij}^k)X_sX_rX_i=\sum_{j,u,v}(\sum_{i}b_{ij}^k)X_jX_u.$$

We assume $s<r<i$ and $j<v<u$ and consider the permutations:

$X_sX_rX_i$, $X_iX_sX_r$, $X_rX_iX_s$, $X_jX_uX_v$, $X_vX_uX_j$, $X_uX_jX_v$

We have equalities:

$X_rX_iX_s=c_{is} c_{rs} X_sX_rX_i$ and $X_iX_rX_s=c_{is} c_{ir} X_sX_rX_i$
$X_rX_sX_j=c_{uj} c_{ij} X_jX_uX_v$, $X_uX_jX_v=c_{uj} c_{uv} X_jX_uX_v$
$X_sX_rX_i=c_{iv} c_{ij} X_jX_iX_v$ and $X_nX_nX_n=c_{iv} c_{iv} X_iX_iX_i$

Substituting in equation $^*)$ we get the following coefficients of $X_sX_rX_i$:

$^*) (\sum_{j}b_{ij}^k)+\sum_{j}b_{ij}^k c_{is} c_{ir} X_sX_rX_i$ and the coefficients of $X_jX_uX_v$:

$^*) (\sum_{j}b_{ij}^k)+\sum_{j}b_{ij}^k c_{ij} c_{uv} X_jX_uX_v$

The index $j$ is just the summation variable in the first equation and $i$ the summation variable in the second expression, to avoid confusion we change $i$ for $\alpha$ in the first expression and $j$ for $\beta$ in the second to have:

$**)$ $\sum_{\alpha}b_{\alpha i}^{\alpha} + (\sum_{\alpha}b_{\alpha i}^{\beta} c_{is} c_{ir}) X_sX_rX_i$

and $***)$ $\sum_{\beta}b_{\beta j}^{\beta} c_{uj} c_{uv} X_jX_uX_v$

From equation $^{***)}$ we have the same coefficients when $s=j$, $r=v$ and $i=u$. Using the fact $b_{su}^{\alpha} c_{us}=-b_{us}^{\alpha}$ $^{**)$ becomes:
Let $C_n$ be the quantized exterior algebra $C_n = \mathbb{k}[X_1, X_2, \ldots, X_n]/\langle X_i^2, X_iX_j + c_{ij}X_jX_i, c_{ij} \in \mathbb{k}\{-0\}, i<j\rangle$ that has graded derivation $\partial: C_n \to C_n[1]$ given by $\partial(X_k) = -\sum_{i<j} b_{ij}X_kX_i$. When $\partial^2 = 0$, it is easy to check that the product is associative and $(\mathbb{k}[Z]/(Z^2)) \otimes_\partial C_n$ is a graded $\mathbb{k}$-algebra.
To prove next theorem we need the following:

**Definition 3.2.** Given a finite alphabet $S=\{X_1, X_2, \ldots, X_n\}$ ordered as $X_1 < X_2 < \ldots < X_n$ and $W$ the words in the alphabet $S$, with the degree lexicographic order, $I$ a two sided ideal of $\mathbb{k} < W>$, and $f: \mathbb{k} < W> \to \mathbb{k} < W>/I$ the natural projection. Given monomials $v, w$ we write $v \subseteq w$ if there are monomials $w_1, w_2$, such that $w_1 w_2 = w$. We say that a non empty subset $M$ of $W$ is an order ideal of monomials, and write $o.i.m.$, if given $u \in M$ and $v \subset u$, then $v \in M$.

Our main tool in this section is the following:

**Lemma 3.3.** [1] Let $M$ be the subset of $W$ defined as $M = \{x \in W | f(x) \notin \text{span}_f(y) \text{ if } y < x\}$. Then $M$ is an $o.i.m.$ and the elements $f(w)$, with $w \in M$ form a $\mathbb{k}$-vector space basis of $\mathbb{k} < W>/I$.

Observe that $M$ is precisely the set of monomial which are non tips of elements of $I$.

**Definition 3.4.** $V_M$ is the set of obstructions. This is: $V_M = \{w \in W | w \notin M \text{ but } v \subseteq w, v \neq w \text{ implies } v \in M\}$.

Then:
- The 1-chain consist of the element 1.
- The 0-chains are the letters in the alphabet.
- The 1-chains are the elements of $V_M$.

Define by induction the $n$-chains:

A monomial $\mu = X_{i_1}X_{i_2}\ldots X_{i_t}$ is a $n$-prechain if there exist natural numbers $a_j, b_j$ with $1 \leq j \leq n$, satisfying:

- $1 = a_1 < a_2 < a_3 < \ldots < a_n < b_n - 1, b_n = t$
- $X_{a_{i_1}} \ldots X_{a_{i_j}} \in V_M.$

A $n$-prechain $X_{i_1}X_{i_2}\ldots X_{i_t}$ is a $n$-chain, if $X_{i_1}X_{i_2}\ldots X_{i_s}$ is not an $m$-prechain for $s < b_m$ and $m \leq n$.

**Theorem 3.5.** Let $C_n^l = \mathbb{k} < X_1X_2, \ldots X_n, X_iX_j + c_{ij}X_jX_i, c_{ij} \in \mathbb{k}-\{0\}, i < j>$ be a quantized exterior algebra and $\partial: C_n^l \to C_n^l [1]$ $\partial(X_k) = - \sum_{i < j} b_{ij}^k X_jX_i$ graded derivation with $\partial^2 = 0$. Then $i)$ the $\partial$-skew tensor product $\mathbb{k}[Z]/(Z^2) \otimes_\partial C_n^l$ is isomorphic to the algebra $B_n^l = \mathbb{k} < X_1X_2, \ldots X_n, X_kZ + ZX_k + \sum_{i < j} b_{ij}^k X_jX_i, c_{ij} \in \mathbb{k}-\{0\}, i < j, X_kX_j - c_{ij}X_jX_k, c_{ij} \in \mathbb{k}-\{0\}, i < j, X_kZ - ZX_k >$.

- $ii)$ The algebra $B_n^l$ is Koszul selfinjective.
- $iii)$ The Yoneda algebra $B_n = \mathbb{k} < X_1X_2, \ldots X_n, X_jX_i - c_{ij}X_iX_j, c_{ij} \in \mathbb{k}-\{0\}, i < j, X_iX_j - c_{ij}X_jX_i, c_{ij} \in \mathbb{k}-\{0\}, i < j, X_iZ - ZX_i >$ is Artin-Schelter regular and it has a PBW basis.
Proof. i) By the universal property of free algebras there is a map \( \varphi: k \to X_1X_2, \ldots, X_n \) \( \to \mathbb{k}[Z]/(Z^2) \otimes \mathcal{C}_n^i \) given by \( \varphi(Z) = Z \otimes 1 \), \( \varphi(X_i) = 1 \otimes X_i \). Then \( \varphi(Z^2) = Z^2 \otimes 1 \) = 0 and \( \varphi(X_i^2) = 1 \otimes X_i^2 = 0 \), \( \varphi(X_iX_j + c_{ij}X_iX_j) = 1 \otimes (X_iX_j + c_{ij}X_iX_j) = 0 \).

We also have \( \varphi(X_iZ + ZX_k + \sum b_{ij}X_jX_i) = (Z \otimes 1)(1 \otimes X_k) + (1 \otimes X_k)(Z \otimes 1) + \sum_{i<j} b_{ij}(1 \otimes X_j)(1 \otimes X_i) = Z \otimes X_k - Z \otimes X_k - 1 \sum_{i<j} b_{ij}X_jX_i + \sum_{i<j} b_{ij} \otimes X_jX_i = 0 \).

Hence \( \varphi \) induces an onto map \( \overline{\varphi}: B_n^i \to \mathbb{k}[Z]/(Z^2) \otimes \mathcal{C}_n^i \). The algebra \( B_n^i \) is generated by words of the form \( X_iX_{i_2} \ldots X_{i_k} \) and \( ZX_{i_1}X_{i_2} \ldots X_{i_k} \) with \( X_{i_r} \neq X_{i_s} \) and \( \mathbb{k}[Z]/(Z^2) \otimes \mathcal{C}_n^i \) has as \( \mathbb{k} \)-basis the elements \( 1 \otimes X_{i_1}X_{i_2} \ldots X_{i_k}, Z \otimes X_{i_1}X_{i_2} \ldots X_{i_k} \).

It follows \( \dim_k(B_n^i) = \dim_k(\mathbb{k}[Z]/(Z^2) \otimes \mathcal{C}_n^i) \).

Therefore: \( \dim_k(B_n^i) = \dim_k(\mathbb{k}[Z]/(Z^2) \otimes \mathcal{C}_n^i) \), and \( \overline{\varphi} \) is an isomorphism.

ii) The algebra \( \mathbb{k}[Z]/(Z^2) \otimes \mathcal{C}_n^i \) has PBW basis the elements \( 1 \otimes X_{i_1}X_{i_2} \ldots X_{i_k}, Z \otimes X_{i_1}X_{i_2} \ldots X_{i_k} \), and if \( \mathcal{M} = \{ \rho_{ij}, \rho_k, X_i^2, Z^2 \} \) is a Groebner basis and by definition \( \text{tip}(\rho_{ij}, \rho_k, X_i^2, Z^2) = \text{tip}I \) where \( I \) is the ideal generated by \( \{ \rho_{ij}, \rho_k, X_i^2, Z^2 \} \).

To achieve this we first use Anick’s resolution to find a graded projective resolution of the simple left \( B_n^i \)-module \( k \).

We give the variables the following order: \( Z > X_1 > X_2 \ldots X_n \) and we give to the words \( W \) in \( Z, X_1, X_2 \ldots X_n \) the degree lexicographic order.

Since \( B_n^i \) has a PBW basis, it follows by [7], [8], [11], [14] that \( X_i^2, Z^2, \rho_{ij} = X_iX_j + c_{ij}X_iX_j, c_{ij} \in k - \{ 0 \} \), \( i < j \), \( \rho_k = kZ + ZX_k + \sum b_{ij}X_jX_i \) are a Groebner basis and by definition \( \text{tip}(\rho_{ij}, \rho_k, X_i^2, Z^2) = \text{tip}I \) where \( I \) is the ideal generated by \( \{ \rho_{ij}, \rho_k, X_i^2, Z^2 \} \).

The set \( M = \{ w \in W | \text{is non tip of I} \} \) is according to Anick [1] an ordered monomial ideal.

If \( X_{i_1}X_{i_2} \ldots X_{i_{j-1}}X_{i_k} \) is a word in \( M \) then \( X_{i_1}X_{i_{j+1}} \) is in \( M \) this means it is not a tip of any of the relations \( \{ \rho_{ij}, \rho_k, X_i^2, Z^2 \} \). It follows \( X_{i_j} < X_{i_{j+1}} \) and if \( Z \) appears in the word then the word is \( X_{i_1}X_{i_2} \ldots X_{i_{j-1}}X_{i_k} Z \) with \( X_{i_j} < X_{i_{j+1}}, 1 \leq j \leq k-1 \).

Since we have \( \text{tip}(\rho_{ij}, \rho_k, X_i^2, Z^2) = \text{tip}I \) for \( X_{i_j} < X_{i_{j+1}}, 1 \leq j \leq k-1 \) any word either of the form \( X_{i_1}X_{i_2} \ldots X_{i_{j-1}}X_{i_k} Z \) or \( X_{i_1}X_{i_2} \ldots X_{i_{j-1}}X_{i_k} Z \) in \( M \).

We look at the obstructions.

It is clear \( V_M = \{ X_i^2, Z^2, ZX_i, X_iX_j, i < j \} \). Observe that the number of elements of \( V_M \) is equal to the number of homogeneous polynomials of degree two in \( Z, X_1, X_2 \ldots X_n \).

In our case the chains are easy to compute:
\( C_{-1} = \{ 1 \} \).

The zero chains are \( C_0 = \{ Z, X_i \} \).

The one chains are \( C_1 = V_M \).
The two chains \(C_2=\{X_i^3, i<j, X_i^2X_j, X_iX_j^2, Z^2X_i, ZX_i^2, X_iX_jX_k \mid i<j<k \}\) 
In general the \(m\)-chains are 
\[C_m=\{X_{i_1}X_{i_2}...X_{i_{m+1}} \mid i_1 \leq i_2 \leq ... \leq i_{m+1}\}\] 
\[\cup \{Z^kX_{i_1}X_{i_2}...X_{i_{m+1-k}} \mid i_1 \leq i_2 \leq ... \leq i_{m+1-k}\}\]. 

Let \(<C_m>\) be the \(k\)-vector space generated by the \(m\)-chains.

By [1], [2], we have Anick’s resolution

\[...<C_m>\otimes_k B_n^1 \rightarrow <C_{m-1} \otimes_k B_n^1 \rightarrow ... <C_1> \otimes_k B_n^1 \rightarrow <C_0> \otimes_k B_n^1 \rightarrow B_n^1 \rightarrow \mathbb{k} \rightarrow 0\]

The resolution is graded and linear. By Koszul theory, [9], [10] if \(J\) denotes the graded radical of \(B_n\), then there is an isomorphism of \(k\)-vector spaces 
\[<C_m> \cong \text{Hom}_k(J^{m+1}/J^{m+2}, \mathbb{k}).\]

It follows \(\dim_k(J^{m+1}/J^{m+2})\) is the number of homogeneous polynomials in \(n+1\) variables of degree \(m+1\).

If we can prove that the elements 
\[X_{i_1}X_{i_2}...X_{i_{m+1}}, Z^kX_{i_1}X_{i_2}...X_{i_{m+1-k}}\]
with \(i_1 \leq i_2 \leq ... \leq i_{m+1}\), \(i_1 \leq i_2 \leq ... \leq i_{m+1-k}\), respectively, generate \(J^{m+1}/J^{m+2}\) then they would be a basis.

We look now to the algebra \(B_n\) and give to the variables the order 
\[X_n > X_{n-1} > ... > X_2 > X_1 > Z\] (the order opposite to the one considered before).

Let \(W\) be the words in \(Z, X_1, X_2,...X_n\) in the degree lexicographic order.

We consider the relations 
\[\rho_{ij} = X_jX_i - \text{c}_{ij}X_iX_j - \sum_k b_{ij}^k X_k Z, \quad j > i \quad \text{and} \quad \rho_i = X_i Z - ZX_i, \quad \Gamma = <\rho_{ij}, \rho_i>\] the ideal generated by the relations. Then \(\text{tip}\{\rho_{ij}, \rho_i\} \subseteq \text{tip}\Gamma\).

We will prove the equality.

Let \(M'\) be the ordered monomial ideal 
\[M' = \{w \in W \mid \text{w is non tip of} \ \Gamma\}\] 
hence if \(w \in M\) then it is not a tip of \(\{\rho_{ij}, \rho_i\}\) and \(w\) is either of the form \(X_{i_1}X_{i_2}...X_{i_{m+1}}\) with \(i_1 \leq i_2 \leq ... \leq i_{m+1}\) or 
\[Z^kX_{i_1}X_{i_2}...X_{i_{m+1-k}}, \quad \text{with} \quad i_1 \leq i_2 \leq ... \leq i_{m+1-k}\].

By Anick [1] \(M'\) is a basis of \(B_n\). It follows the set of elements \(X_{i_1}X_{i_2}...X_{i_{m+1}}\), 
with \(i_1 \leq i_2 \leq ... \leq i_{m+1}\) and \(Z^kX_{i_1}X_{i_2}...X_{i_{m+1-k}}\) with \(i_1 \leq i_2 \leq ... \leq i_{m+1-k}\) generate \(J^{m+1}/J^{m+2}\). By the above argument they are a basis. We have proved \(B_n\) has PBW basis \(Z^kX_1^{\alpha_1}X_2^{\alpha_2}...X_n^{\alpha_n}\) with \(k \geq 0, \ \alpha_i \geq 0\).

As a consequence of this we have \(\text{tip}\{\rho_{ij}, \rho_i\} = \text{tip}\Gamma\).

We can prove now the converse.

**Theorem 3.6.** Let \(B_n=\mathbb{k} <X_1X_2,...,X_n,Z> / <X_jX_i-c_{ij}X_iX_j - \sum_k b_{ij}^k X_k Z, \ c_{ij} \in \mathbb{k}-\{0\} i<j, X_k Z - ZX_k>\) be a homogenized \(G\)-algebra and \(B_n^1=\mathbb{k} <X_1X_2,...,X_n,Z> / <X_1^2, Z^2, X_iX_j + c_{ij}X_jX_i, \ c_{ij} \in \mathbb{k}-\{0\} i<j, X_k Z + ZX_k + \sum_i b_{ij}^k X_jX_i>\) its Koszul dual. Then \(\partial(X_k) = \sum_{i<j} b_{ij}^k X_jX_i\) is a graded derivation \(\partial: C_n^1 \rightarrow C_n^1[1]\) of the quantized exterior algebra \(C_n^1=\mathbb{k} <X_1X_2,...,X_n> / <X_1^2, X_iX_j + c_{ij}X_jX_i, \ c_{ij} \in \mathbb{k}-\{0\} i<j>\), with \(\partial^2=0\). Furthermore, if \(\mathbb{k}[Z]/(Z^2)\otimes \partial C_n^1\) is the \(\partial\)-skew tensor product, then there is an isomorphism of graded \(k\)-algebras \(\mathbb{k}[Z]/(Z^2)\otimes \partial C_n^1 \cong B_n\).
Proof. We have in $B^1_n$ the following equality $X_kZ+ZX_k=-\sum_{i<j} b^k_{ij} X_i X_j$, as before we rewrite $\partial(X_k)$ as $\partial(X_k)=-1/2\sum_{i<j} b^k_{ij} X_j X_i-1/2\sum_{i<j} b^k_{ij} X_j X_i=-1/2\sum_{i<j} b^k_{ij} X_j X_i-1/2\sum_{i<j} (-c_{ij})^{-1} X_i X_j$.

We let $c_{ji}$ be $c_{ji}^{-1}$ for $i\neq j$ and $c_{ii}=1$ for $i<j$ we define $b^k_{ji}=-c_{ji} b^k_{ij}$ and $b^k_{ii}=0$.

Then $\partial(X_k)=-1/2\sum_{i,j} b^k_{ij} X_j X_i$.

Hence, we have in $B^1_n$ the following equalities:

$X_kZ+ZX_k=-1/2\sum_{i,j} b^k_{ij} X_j X_i$

$ZX_kZ=-1/2\sum_{i,j} b^k_{ij} X_j X_i Z=-1/2\sum_{i,j} b^k_{ij} ZX_j X_i$

$X_jZ+ZX_j=-1/2\sum_{r,s} b^r_{is} X_s X_r$

Then $X_jZX_i+ZX_j X_i=-1/2\sum_{i,j} b^r_{is} X_s X_r X_i-1/2\sum_{i,j} b^k_{ij} ZX_j X_i-1/2\sum_{i,j} b^k_{ij} Z X_j X_i=1/4 \sum_{i,j} b^k_{ij} b^r_{is} X_s X_r X_i=1/4 \sum_{i,j} (\sum b^k_{ij} b^r_{is}) X_s X_r X_i$.

In the other hand, $X_i Z+ZX_i=-1/2\sum_{\ell,m} b_{\ell m}^i X_m X_\ell$

and $X_j X_i Z+X_i Z X_j =-1/2\sum_{\ell,m} b_{\ell m}^i X_j X_m X_\ell$

Therefore: $-1/2\sum_{i,j} b^k_{ij} X_j X_i Z+1/2\sum_{i,j} b^k_{ij} X_j Z X_i=1/4 \sum_{i,j} \sum_{\ell,m} b^k_{ij} b_{\ell m}^i X_j X_m X_\ell$

$=1/4 \sum_{i,j} \sum_{\ell,m} (b^k_{ij} b_{\ell m}^i) X_j X_m X_\ell$

From these equalities we have:

$\sum_{i,r,s,j} (b^k_{ij} b^r_{is}) X_s X_r X_i =\sum_{j,\ell,m} (b^k_{ij} b_{\ell m}^i) X_j X_m X_\ell$

But we saw above that this equality hold if and only if $\partial^2=0$, which in turn imposes the conditions given in Theorem 3.1 on the structure of constants \{\(c_{ij}\)\} and \{\(b^k_{ij}\)\).

As a corollary we have:

**Theorem 3.7.** The quadratic algebra $B_n=\k <X_1X_2...X_n,Z>/\langle X_jX_i-c_{ij}X_iX_j-\sum b^k_{ij} X_k Z, c_{ij} \in \k-\{0\} \ i<j, X_k Z-ZX_k >$ is a homogenized G-algebra if and only if the structure of constants \{\(c_{ij}\)\} and \{\(b^k_{ij}\)\} satisfies the equation:

$\sum_j (b^k_{ij} b^r_{us} (1+c_{ju} c_{us} c_{uv})+b^k_{ij} b^s_{uv} (c_{us} c_{us} +c_{js})+\sum b^k_{ij} b^r_{us} (c_{us} +c_{uv} c_{vj}))=0$.

Since the algebra $B_n=\k <X_1X_2...X_n,Z>/\langle X_jX_i-c_{ij}X_iX_j-\sum b^k_{ij} X_k Z, c_{ij} \in \k-\{0\} \ i<j, X_k Z-ZX_k >$ has a PBW basis if and only if $B_n/(Z-a)B_n$ has a PBW basis we have:
Theorem 3.8. The quadratic algebra $A_n=\mathbb{k} \langle X_1 X_2, \ldots, X_n \rangle/\langle X_j X_i - c_{ij} X_i X_j - \sum_k b^k_{ij} X_k, \ c_{ij} \in \mathbb{k}-\{0\} \rangle$ has a PBW basis if and only if the structure of constants $\{c_{ij}\}$ and $\{b^k_{ij}\}$ satisfy the equation:

$$\sum_j (b^k_{ij} b^j_{vs} (1+c_{ju} c_{us} c_{uv}) + b^k_{sj} b^j_{uv} (c_{us} c_{vs} + c_{js}) + \sum_j b^k_{ju} b^j_{us} (c_{vs} + c_{uv} c_{vj})) = 0.$$ 

We can consider now the general case of a quadratic algebra $A_n=\mathbb{k} \langle X_1 X_2, \ldots, X_n \rangle/\langle X_j X_i - c_{ij} X_i X_j - \sum_k b^k_{ij} X_k - a_{ij}, \ b^k_{ij}, a_{ij} \in \mathbb{k}, c_{ij} \in \mathbb{k}-\{0\} \rangle$. We proved in [17] that $A_n$ has a PBW basis if and only if its partial homogenization $A_n(Z)=\mathbb{k} \langle X_1 X_2, \ldots, X_n, Z \rangle/\langle X_j X_i - c_{ij} X_i X_j - \sum_k b^k_{ij} X_k - a_{ij} Z, \ b^k_{ij}, a_{ij} \in \mathbb{k}, c_{ij} \in \mathbb{k}-\{0\}, X_i Z - Z X_i \rangle$ has a PBW basis.

From this we can prove:

Theorem 3.9. A quadratic algebra $A_n=\mathbb{k} \langle X_1 X_2, \ldots, X_n \rangle/\langle X_j X_i - c_{ij} X_i X_j - \sum_k b^k_{ij} X_k - a_{ij}, \ b^k_{ij}, a_{ij} \in \mathbb{k}, c_{ij} \in \mathbb{k}-\{0\} \rangle$ has a PBW basis if and only if the structure of constants $\{c_{ij}\}$, $\{a_{ij}\}$ and $\{b^k_{ij}\}$ satisfy the equations:

$$\sum_j (b^k_{ij} b^j_{vs} (1+c_{ju} c_{us} c_{uv}) + b^k_{sj} b^j_{uv} (c_{us} c_{vs} + c_{js}) + \sum_j b^k_{ju} b^j_{us} (c_{vs} + c_{uv} c_{vj})) = 0.$$ 

$$\sum_j (a_{uj} b^j_{vs} (1+c_{ju} c_{us} c_{uv}) + a_{sj} b^j_{uv} (c_{us} c_{vs} + c_{js}) + \sum_j a_{ju} b^j_{us} (c_{vs} + c_{uv} c_{vj})) = 0.$$ 

Proof. The algebra $A_n$ has a PBW basis if and only if its partial homogenization $A_n(Z)$ has a PBW basis. Making $Z=X_{n+1}$ and $b^{n+1}_{ij}=a_{ij}$, $c_{n+1}=1$, $b^{k}_{in}=0$, the algebra $A_n(Z)$ becomes $A_n(Z)=\mathbb{k} \langle X_1 X_2, \ldots, X_n, X_{n+1} \rangle/\langle X_j X_i - c_{ij} X_i X_j - \sum_k b^k_{ij} X_k \rangle$ and the equations follow from the previous theorem.

We finally get:

Theorem 3.10. The quadratic algebra $B_n=\mathbb{k} \langle X_1 X_2, \ldots, X_n, Z \rangle/\langle X_j X_i - c_{ij} X_i X_j - \sum_k b^k_{ij} X_k Z - a_{ij} Z^2, \ c_{ij} \in \mathbb{k}-\{0\}, i \neq j, X_k Z - Z X_k \rangle$ is a homogenized G-algebra if and only if the structure of constants $\{c_{ij}\}$, $\{a_{ij}\}$ and $\{b^k_{ij}\}$ satisfies the equations:

$$\sum_j (b^k_{ij} b^j_{vs} (1+c_{ju} c_{us} c_{uv}) + b^k_{sj} b^j_{uv} (c_{us} c_{vs} + c_{js}) + \sum_j b^k_{ju} b^j_{us} (c_{vs} + c_{uv} c_{vj})) = 0.$$ 

$$\sum_j (a_{uj} b^j_{vs} (1+c_{ju} c_{us} c_{uv}) + a_{sj} b^j_{uv} (c_{us} c_{vs} + c_{js}) + \sum_j a_{ju} b^j_{us} (c_{vs} + c_{uv} c_{vj})) = 0.$$ 

Observe that if we have $c_{ij}=1$ for all $i \neq j$ then the equations of Theorems 3.9 and 3.10 are just the equations we had obtained in [17].
3.3 The Hopf structure of the quantized polynomial algebra and the quantized exterior algebra

We know the polynomial algebra is a Hopf algebra and the exterior algebra with the graded tensor product is also a Hopf algebra [4], [20]. In this section we want to prove that the quantized versions of the polynomial algebra and the exterior algebra have a Hopf structure with a skew tensor product that will be defined below, thus generalizing the situation of the usual exterior algebra.

We use the same notation as above the quantized polynomial algebra is $\mathbb{C}_n = k <x_1, x_2, \ldots, x_n>/ <x_i x_j - c_{ij} X_i X_j, c_{ij} \in k \{-0\}>$ and the quantized exterior algebra is $\mathbb{C}_n^! = \mathbb{C}_n^! = k <x_1^2, x_i x_j + c_{ij} X_i X_j, c_{ij} \in k \{-0\}>$. Since both cases are similar we will work the details for the quantized polynomial algebra and leave to the reader the quantized exterior algebra.

Let $n \leq m$ be positive integers. Then there is a natural inclusion of the algebra $C_n$ in $C_m$, when $\{c_{ij} | 1 \leq i,j \leq n\} \subseteq \{c_{ij} | 1 \leq i,j \leq m\}$.

We give to the vector space tensor product $C_n \otimes_k C_m$ an algebra product as follows:

Let $w$ be the word $w = x_{i_1} x_{i_2} \ldots x_{i_k}$ and write $\rho_r(w) = i_r$ the index of $X_{i_r}$ in $w$. Thus $w = x_{\rho(1)} x_{\rho(2)} \ldots x_{\rho(k)}$.

Let $u,v$ be homogeneous elements of $C_n$ and $C_m$, respectively, then $(u \otimes 1)(1 \otimes v) = u \otimes v$ and $(1 \otimes v)(u \otimes 1) = c_{\rho_r(u) \rho_s(v)} u \otimes v$, where for $i<j$ $c_{ij} = c_{ij}^{-1}$ and $c_{ii} = 1$.

Using that for $v$, $w$ homogenous of lengths $k$ and $\ell$ respectively we have for $1 \leq r \leq k$ $\rho_r(vw) = \rho_r(v)$ and for $k + 1 \leq r \leq \ell + k$ $\rho_{r-k}(vw) = \rho_{r-k}(w)$. It is easy to prove that the product is associative. Hence: $C_n \otimes_q C_m$ with the twisted tensor product that we just defined is a $k$-algebra, in particular, $C_n \otimes_q C_n$ is a $k$-algebra, that we denote by $C_n \otimes_q C_n$ to distinguish it from the usual tensor product.

By the universal property of free algebras there is a morphism of $k$-algebras $\Delta: \mathbb{C}_n \rightarrow \mathbb{C}_n \otimes_q \mathbb{C}_n$ given by $\Delta(X_j) = X_j \otimes 1 + 1 \otimes X_j$. Then $\Delta(X_j x_i - c_{ij} X_i X_j) = (X_j \otimes 1 + 1 \otimes X_j)(X_i \otimes 1 + 1 \otimes X_i) - c_{ij}(X_i \otimes 1 + 1 \otimes X_i)(X_j \otimes 1 + 1 \otimes X_j) = X_i X_j \otimes 1 + X_j \otimes X_i + c_{ij} X_i \otimes X_j + X_j X_i - c_{ij}(X_i X_j \otimes 1 + X_i \otimes X_j + c_{ij} X_j \otimes X_i + 1 \otimes X_i X_j) = 0$.

Therefore $\Delta$ induces a ring homomorphism $\Delta: C_n \rightarrow C_n \otimes_q C_n$ given by $\Delta(X_j) = X_j \otimes 1 + 1 \otimes X_j$ and $\Delta(1) = 1 \otimes 1$.

Assume $n \leq m$ and $\{c_{ij} | 1 \leq i,j \leq n\} \subseteq \{c_{ij} | 1 \leq i,j \leq m\}$, in order to describe $C_n \otimes_q C_n$ by generators and relations we define a ring homomorphism of graded algebras.

$\phi: \mathbb{C}_n \otimes_q \mathbb{C}_n \rightarrow C_n \otimes_q C_m$ by $\phi(1) = 1 \otimes 1$, for $1 \leq i \leq n$ $\phi(X_i) = X_i \otimes 1$ and for $1 \leq i \leq m$ $\phi(X_i) = 1 \otimes X_i$

Then for $1 \leq i,j \leq n$, $\phi(X_i X_j - c_{ij} X_i X_j) = (X_i X_j - c_{ij} X_i X_j) \otimes 1 = 0$

and for $1 \leq i,j \leq m$, $\phi(X_{i+n} X_{i+n} - c_{ij} X_{i+n} X_{i+n}) = 1 \otimes (X_i X_j - c_{ij} X_i X_j) = 0$
For $i \neq j$ \( \phi(X_{j+n}X_i-c_{ij}X_jX_{j+n})=(1 \otimes X_j)(X_i \otimes 1)-c_{ij} (X_i \otimes X_j)(1 \otimes X_j)=c_{ij}X_i \otimes X_j-c_{ij}X_i \otimes X_j=0 \)
and $\phi(X_{i+n}X_iX_{i+n})=(1 \otimes X_i)(X_i \otimes 1)-c_{ii} (X_i \otimes X_i)=X_i \otimes X_iX_i \otimes X_i=0$.

Let $(\hat{c}_{ij})$ be the matrix $\hat{c}_{ij}=c_{ij}$ for $1 \leq i,j \leq n$ and for $1 \leq i,j \leq m$ $\hat{c}_{i+n,j+n}=c_{ij}$.

For $i \neq j$ $\hat{c}_{ij+n}=c_{ij}$ and $\hat{c}_{ii+n}=c_{ii}=1$.

Then there is a surjective ring homomorphism of graded rings.

\( \phi: C_{n+m}=\mathbb{k} \langle X_1X_2,...,X_n,...,X_{n+m} \rangle / \langle X_jX_i-\hat{c}_{ij}X_iX_j \rangle \rightarrow C_n \otimes_q C_m \).

Since $C_{n+m}$ has a PBW basis, the degree $\ell$ part $(C_{n+m})_\ell$ of $C_{n+m}$ consists of the homogeneous polynomials of degree $\ell$, hence $\dim_k(C_{n+m})_\ell=dim_k(C_n \otimes_q C_m)_\ell$ and $\phi$ is an isomorphism.

We have proved the isomorphism $C_{n+m} \cong C_n \otimes_q C_m$ and $C_n$, $C_m$ can be considered subalgebras of $C_{n+m}$.

In a similar way we define a morphism:

\( \varphi: \mathbb{k} \langle X_1X_2,...,X_n,...,X_{n+m} \rangle \rightarrow C_m \otimes_q C_n \) by $\phi(1)=1 \otimes 1$, for $1 \leq i \leq n$ $\phi(X_i)=1 \otimes X_i$, and for $1 \leq i \leq m$ $\phi(X_{i+n})=X_i \otimes 1$.

We leave to the reader to check that $\varphi$ induces an isomorphism:

\( \overline{\varphi}: C_{n+m}=\mathbb{k} \langle X_1X_2,...,X_n,...,X_{n+m} \rangle / \langle X_jX_i-\hat{c}_{ij}X_iX_j \rangle \rightarrow C_m \otimes_q C_n \).

Since $C_n$ is a subalgebra of $C_{2n}$, there are ring isomorphisms: $(C_n \otimes_q C_n) \otimes_q C_n \cong C_{2n}$ and $C_n \otimes_q (C_n \otimes_q C_n) \cong C_{n} \otimes_q (C_n \otimes_q C_n)$.

We have proved that the twisted tensor product is commutative and associative.

We prove next that $C_n$ has with the twisted tensor product a Hopf structure.

We begin proving that $C_n$ with the twisted tensor product has an algebra structure.

As above $C_{2n}=\mathbb{k} \langle X_1X_2,...,X_n,...,X_{2n} \rangle / \langle X_jX_i-\hat{c}_{ij}X_iX_j \rangle \cong C_n \otimes_q C_n$ is the isomorphism given by $X_i \rightarrow X_i \otimes 1$, $X_{i+n} \rightarrow 1 \otimes X_j$, $1 \leq i \leq n$.

We define a $\mathbb{k}$-algebra map $m: \mathbb{k} \langle X_1X_2,...,X_n,...,X_{2n} \rangle \rightarrow C_n$ by $m(X_i)=X_i$ and $m'(X_{i+n})=X_i$ for $1 \leq i \leq n$.

We check $m'(X_jX_i-\hat{c}_{ij}X_iX_j)=0=0$ and $m'$ induces a map $\overline{m}: \mathbb{k} \langle X_1X_2,...,X_n,...,X_{2n} \rangle \rightarrow C_n$.

Let $m$ be the composition of algebra maps: $C_n \otimes_q C_n \rightarrow \mathbb{k} \langle X_1X_2,...,X_n,...,X_{2n} \rangle / \langle X_jX_i-\hat{c}_{ij}X_iX_j \rangle \cong C_n \otimes_q C_n$.

Let $\psi$ be the map $\psi: \mathbb{k} \langle X_1X_2,...,X_n,...,X_{3n} \rangle \rightarrow C_n \otimes_q C_n$ given for $1 \leq i \leq n$ by $\psi(X_i)=X_i \otimes 1$, $\psi(X_{i+n})=X_i \otimes 1$, we verify $\psi(X_jX_i-\hat{c}_{ij}X_iX_j)=0$, hence $\psi$ induces a map $\mathbb{k} \langle X_1X_2,...,X_n,...,X_{3n} \rangle \rightarrow C_n \otimes_q C_n$. The composition
\[(C_n \otimes_q C_n) \otimes_q C_n \to k \langle X_1 X_2 \ldots X_n \rangle / \langle X_i X_j - c_{ij} X_i X_j \rangle = \psi \Rightarrow C_n \otimes_q C_n\] is an algebra map that will be denoted by \(m \otimes 1\).

In a similar way we define the algebra map \(1 \otimes m\).

It is easy to see that the following diagram
\[
\begin{array}{ccc}
C_n \otimes_q C_n \otimes_q C_n & \overset{m \otimes 1}{\rightarrow} & C_n \otimes_q C_n \\
\downarrow 1 \otimes m & & \downarrow m \\
C_n \otimes_q C_n & \rightarrow & C_n
\end{array}
\]
commutes.

We have a map of algebras \(\mu : k \to C_n\) given by \(\mu(1) = 1\) and a map
\(\varphi : k \langle X_1 X_2 \ldots X_n \rangle \to C_n \otimes_q C_n\) given by \(\varphi(X_i) = 1 \otimes X_i\), it is clear \(\varphi(X_i X_j - c_{ij} X_i X_j) = 0\) and we have a well defined homomorphism \(k \otimes C_n \cong k \langle X_1 X_2 \ldots X_n \rangle / \langle X_i X_j - c_{ij} X_i X_j \rangle \Rightarrow C_n \otimes_q C_n\) which we denote by \(\mu \otimes 1\), similarly we have a map \(1 \otimes \mu : C_n \otimes k \to C_n \otimes_q C_n\).

We check that the diagrams:
\[
\begin{array}{ccc}
k \otimes C_n & \overset{\mu \otimes 1}{\rightarrow} & C_n \otimes_q C_n \\
\downarrow \cong & & \downarrow \cong \\
C_n & \overset{1 \otimes \mu}{\rightarrow} & C_n \otimes_q C_n
\end{array}
\]
commute.

We have proved that \(C_n\) is an algebra with the twisted tensor product.

Let us check first that \(\Delta : C_n \to C_n \otimes_q C_n\) given by \(\Delta(X_j) = X_j \otimes 1 + 1 \otimes X_j\) and \(\Delta(1) = 1 \otimes 1\) induces a co algebra structure.

We define a map \(\psi : k \langle X_1 X_2 \ldots X_n \rangle \to C_n \otimes_q C_n \otimes_q C_n\) by \(\psi(1) = 1 \otimes 1 \otimes 1\), for \(1 \leq i \leq n\) \(\psi(X_i) = X_i \otimes 1 \otimes 1 + 1 \otimes X_i \otimes 1\) and \(\psi(X_{i+n}) = 1 \otimes 1 \otimes X_i\) we leave to the reader to check that for \(1 \leq i,j \leq 2n\) \(\psi(X_j X_i - \hat{c}_{ij} X_i X_j) = 0\). Hence there is an induced ring homomorphism:
\[\overline{\psi} : k \langle X_1 X_2 \ldots X_n \rangle / \langle X_j X_i - \hat{c}_{ij} X_i X_j \rangle \to C_n \otimes_q C_n \otimes_q C_n,\] composing \(\overline{\psi}\) with the isomorphism \(C_n \otimes_q C_n \rightarrow k \langle X_1 X_2 \ldots X_n \rangle / \langle X_j X_i - \hat{c}_{ij} X_i X_j \rangle\) given for \(1 \leq i \leq n\) by: \(X_i \rightarrow X_i \otimes 1\) and \(X_{i+n} \rightarrow 1 \otimes X_i\) we obtain a ring homomorphism:
\[C_n \otimes_q C_n \rightarrow C_n \otimes_q C_n \otimes_q C_n\] given in the generators by: \(1 \otimes 1 \otimes 1, X_i \otimes 1 \otimes 1 + 1 \otimes X_i \otimes 1\) and \(1 \otimes X_i \rightarrow 1 \otimes 1 \otimes X_i\). We denote this map by \(\Delta \otimes 1\).

It can be proved in a similar way that there is a ring homomorphism:
\[1 \otimes \Delta : C_n \otimes_q C_n \rightarrow C_n \otimes_q C_n \otimes_q C_n\] given in the generators by: \(1 \otimes 1 \otimes 1, 1 \otimes \Delta(X_i \otimes 1) = X_i \otimes 1 \otimes 1 + 1 \otimes 1 \otimes X_i\) and \(1 \otimes \Delta(1 \otimes X_i) = 1 \otimes X_i \otimes 1 + 1 \otimes 1 \otimes X_i\).

To prove that the square
\[
\begin{array}{ccc}
C_n & \overset{\Delta}{\rightarrow} & C_n \otimes_q C_n \\
\downarrow \Delta & & \downarrow 1 \otimes \Delta \\
C_n \otimes_q C_n & \overset{\Delta \otimes 1}{\rightarrow} & C_n \otimes_q C_n \otimes_q C_n
\end{array}
\]
commutes, it is enough to check it in the generators, but it is clear \((\Delta \otimes 1)\Delta(X_i) = X_i \otimes 1 \otimes 1 + 1 \otimes X_i \otimes 1 + 1 \otimes 1 \otimes X_i = (1 \otimes \Delta)(\Delta(X_i))\). We define the counity \(\varepsilon: C_n \to k\) by \(\varepsilon(1) = 1\) and \(\varepsilon(X_i) = 0\).

There are ring isomorphisms \(C_n \otimes_q k \cong C_n\) and \(k \otimes_q C_n = k \otimes_k C_n \cong C_n\). Using the isomorphism \(k < X_1 X_2 \ldots X_n \ldots, X_2 n > \cong C_n \otimes X_1\), it is clear that there are homomorphisms \(\varepsilon \otimes 1: C_n \otimes_q C_n \to k \otimes k C_n\) and \(1 \otimes \varepsilon: C_n \otimes q C_n \to C_n \otimes k\) given by \(\varepsilon \otimes 1(1 \otimes 1) = 1 \otimes 1\), \(\varepsilon \otimes 1(X_i \otimes 1) = 0\) and \(\varepsilon \otimes 1(1 \otimes X_i) = 1 \otimes X_i\), \(1 \otimes \varepsilon((1 \otimes 1) = 1 \otimes 1\), \(1 \otimes \varepsilon(X_i \otimes 1) = X_i \otimes 1\), \(1 \otimes \varepsilon(1 \otimes X_i) = 0\).

We check in the generators that the diagram

\[
\begin{array}{ccc}
C_n \otimes_q C_n & \xrightarrow{\varepsilon \otimes 1} & 1 \otimes \varepsilon \\
\| & \Delta \downarrow & \| \\
k \otimes_k C_n & \approx & C_n \otimes_k k
\end{array}
\]

commutes.

Hence \((C_n, \Delta, \varepsilon)\) is a coalgebra.

To define the antipode we proceed as above, starting with a ring map \(S: k < X_1 X_2 \ldots X_n \ldots, X_2 n > \to C_n\), given by \(S(1) = 1\), \(S(X_i) = -X_i\), we check \(S(X_j X_i - c_{ij} X_i X_j) = 0\). Hence; \(S\) induces a ring isomorphism \(S: C_n \to C_n\), given by \(S(1) = 1\) and \(S(X_i) = -X_i\).

It is clear \(S^2 = 1\).

Using as above the isomorphism \(k < X_1 X_2 \ldots X_n \ldots, X_2 n > \cong C_n \otimes_q C_n\), it is clear that there is a ring homomorphism \(C_n \otimes_q C_n \to C_n \otimes_q C_n\) given in generators by \(1 \to 1\), \(X_i \otimes 1 \to X_i \otimes 1\) and \(1 \otimes X_i \to 1 \otimes X_i\). We denote this map by \(S \otimes 1\) and define similarly \(1 \otimes S\) by \(1 \to 1\), \(X_i \otimes 1 \to X_i \otimes 1\) and \(1 \otimes X_i \to -1 \otimes X_i\).

The map just defined is an antipode, this means the following diagram commutes:

\[
\begin{array}{ccc}
C_n \otimes_q C_n & \xrightarrow{m} & C_n \\
\| & \mu \varepsilon \downarrow & \| \\
1 \otimes S & \xrightarrow{\mu S} & S \otimes 1 \\
\| & \Delta \downarrow & \| \\
C_n \otimes_q C_n & \xrightarrow{\Delta} & C_n \otimes_q C_n
\end{array}
\]

Since \(\mu \varepsilon(1) = 1\), \(\mu \varepsilon(X_i) = 0\), \(m \otimes S \Delta(1) = m S \otimes 1 \Delta(1) = \mu \varepsilon(1) = 1\) and \(m \otimes S \Delta(X_i) = m S \otimes 1 \Delta(X_i) = \mu \varepsilon(X_i) = 0\).

Since all the maps are algebra homomorphisms this means that the diagram commutes.

We have proved:
Theorem 3.11. Let \( C_n \) be a quantized polynomial algebra with twisted tensor product \( C_n \otimes_q C_n \) and \( \Delta: C_n \rightarrow C_n \otimes_q C_n \), \( \mu: C_n \otimes_q C_n \rightarrow C_n \), \( \varepsilon: C_n \rightarrow \mathbb{k} \), \( \varepsilon, \mu \) a unit, \( \Delta \) comultiplication, \( \varepsilon, \mu \) counit and \( S \) the antipode.

In a similar way we consider the quantized exterior algebra \( C_n^t = \mathbb{k} \langle X_1, X_2, \ldots, X_n, \rangle / \langle X_1^2, X_1 X_j + c_{ij} X_j X_i, c_{ij} \in \mathbb{k} \{0\} \rangle \).

Given \( n \leq m \) we define the twisted tensor product \( C_n \otimes_q C_m^t \) if \( v_1, v_2 \) are homogeneous elements of \( C_n \) and \( w_1, w_2 \) are elements of \( C_m \). \( \ell(v_i), \ell(w_i) \) is the length then \( (v_1 \otimes w_1)(v_2 \otimes w_2) = (-1)^{\ell(v_2)\ell(w_1)} c_{\rho_i\rho}(v_1 v_2 \otimes w_1 w_2) \).

We check as above, for \( v = X_i X_j \ldots X_k \), \( \rho(v) = i \).

As before we obtain a ring isomorphism \( C_n \otimes_q (C_n \otimes_q C_n^t) \cong (C_n \otimes_q C_m^t) \otimes C_n^t \).

Proceeding as above we check that there is an homomorphism of \( \mathbb{k} \)-algebras \( m: C_n \otimes_q C_m^t \rightarrow C_m^t \) given by \( m(X_i \otimes 1) = m(1 \otimes X_i) = X_i \), \( m(1 \otimes 1) = 1 \) a unity \( \mu: \mathbb{k} \rightarrow C_n^t \) given by \( \mu(1) = 1 \) a co multiplication \( \Delta: C_n^t \rightarrow C_n \otimes_q C_m^t \) given by \( \Delta(X_i) = X_i \) a co unity \( \varepsilon: C_n^t \rightarrow \mathbb{k} \), given by \( \varepsilon(1) = 1 \), \( \varepsilon(X_i) = 0 \) and the antipode \( S: C_n^t \rightarrow C_n^t \) given in an homogeneous element \( v \) by \( S(v) = (-1)^{\ell(v)} v \).

We leave to the reader to prove the following:

Theorem 3.12. Let \( C_n^t \) be a quantized exterior algebra with twisted tensor product \( C_n^t \otimes_q C_n^t \) and \( \Delta: C_n^t \rightarrow C_n^t \otimes_q C_n^t \), \( \mu: C_n^t \otimes_q C_n^t \rightarrow C_n^t \), \( \varepsilon: C_n^t \rightarrow \mathbb{k} \), \( \varepsilon, \mu \) a unit, \( \Delta \) comultiplication, \( \varepsilon, \mu \) counit and \( S \) the antipode.

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