Generalization of Grassmannian and Polylogarithmatic Groups Complex

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Abstract

In this study, the Grassmannian complex and the variant of Catelineau’s complexes are investigated by with the help of new homomorphisms. We discuss weight 5,6, and for the first time, and generalize it for weight “n”. This generalization will definitely enhance the knowledge in the area of polylogarithmic groups and open new gateways for its study. In this new morphism the outcome of our diagram is commutative, while the dimension of points remains constant for increasing weights.

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1 Introduction

Suslin [1] was the first one to define Grassmannian bi-complex of projective configuration of points. Gonchrove defined the geometry between Grassmannian and Bloch-Suslin complexes [2–4] by homomorphisms for weight 2 [5]. In order to connect these two complexes, Gonchrove, made use of the Bloch group $B_2(F)$ [6]. To prove (projected seven terms) the functional equation, the author uses the duality of configuration for $B_3(F)$ and proves that the diagram is bi-complex and commutative for weight 3. The most important part of his work was the generalization of cross ratio. This was the significant work of Gonchrove and is also known as the triple ratio. The analogue of Gonchrove complexes was defined by Cathelineau [7, 8] which was in an additive (both infinitesimal and tangential) setting. This complex are known as Cathelineau complex. The settings of triple ratio in terms of two projected cross ratio was introduced by Siddiqui [9, 10], the author uses new morphisms for this purpose, and also connects Cathelineau infinitesimal complexes and Grassmannian complexes for weight 2 and 3 by his own morphism. Author showed the diagram is commutative and bi-complex.

In the previous paper, Khalid et al. [11] established a new homomorphism for the geometry of the Grassmannian sub complexes and infinitesimal complex for weight 2, 3 and 4, which are analogues of the Gronchove complex. The author also concluded that the associated diagrams for these weights are commutative as well as bi-complex. The key point there was that the dimension of the points remains unchanged and all these results are achieved with less and easy calculation.

In this work we have tried to discover new homomorphism between Grassmannian and variant of Cathelineau complexes for weight 5, 6 and then generalized it for weight $n$. This technique will definitely enhance the knowledge of the researcher of algebraic number theory and they will use these homomorphism to prove the associative diagram is commutative, as we did in this study. The common point in the previous and this study is that with increasing weight, the dimension of points remains unchanged.
2 Preliminary and Background

The authors of this paper already discussed the mathematical background of Grassmannian complex, Polylogarithmic complex and Cathelineau’s complexes with their variant. For detail see [11].

3 Geometry between Grassmannian and Infinitesimal Complexes

3.1 Weight (n = 5)

We define geometry between the sub complex of infinitesimal in weight-5, then we will connect them like given below

\[ G_8(2) \xrightarrow{p} G_7(1) \]
\[ \downarrow d \quad \quad \quad \downarrow d \]
\[ G_7(2) \xrightarrow{p} G_6(1) \]
\[ \downarrow h_5^5 \quad \quad \quad \downarrow h_5^5 \]

\[ \beta_2^D(F) \otimes F^\times \otimes F \otimes B_2(F) \otimes F^\times \xrightarrow{\partial^D} F \otimes F^\times \]

where

\[ h_5^5 : (q_0, ..., q_5) \rightarrow \sum_{i=0}^{5} (-1)^i D \log \Delta(q_i) \otimes \frac{\Delta(q_{i+1})}{\Delta(q_{i+2})} \wedge \frac{\Delta(q_{i+2})}{\Delta(q_{i+3})} \wedge \frac{\Delta(q_{i+3})}{\Delta(q_{i+4})} \wedge \frac{\Delta(q_{i+4})}{\Delta(q_{i+5})} \pmod 6 \]  

(1)

and \( h_5^5 \) as given below.

\[ h_5^5(q_0, ..., q_6) = - \frac{1}{10} \left[ \sum_{i \neq j \neq k}^{6} (-1)^i (r(q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_6) \partial_{i,j,k}^{D} \otimes \prod_{l=0}^{6} \Delta(q_l, q_l) \right] \]

\[ \prod_{j \neq l}^{6} \Delta(q_j, q_l) \wedge \prod_{i \neq l}^{6} \Delta(q_i, q_l) - D \log(\prod_{i \neq l}^{6} \Delta(q_i, q_l)) \otimes \]

\[ [r(q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_6)]_2 \otimes \prod_{j \neq l}^{6} \Delta(q_j, q_l) \wedge \prod_{k \neq i \neq j \neq l}^{6} \Delta(q_k, q_l) \]
\[
+ D \log(\prod_{j \neq l}^{6} \Delta(q_j, q_l)) \otimes [r(q_0, ..., \hat{q_i}, \hat{q_j}, ..., q_6)]_2 \otimes \prod_{k \neq l}^{6} \Delta(q_k, q_l) \wedge \\
\prod_{i \neq l}^{6} \Delta(q_i, q_l) - D \log(\prod_{k \neq l}^{6} \Delta(q_k, q_l)) \otimes [r(q_0, ..., \hat{q_i}, \hat{q_j}, \hat{q_k}, ..., q_6)]_2 \\
\otimes \prod_{i \neq l}^{6} \Delta(q_i, q_l) \wedge \prod_{j \neq l}^{6} \Delta(q_j, q_l)
\]

First we show that \(h_5^0\) and \(h_1^1\) are well defined morphism.

**Lemma 3.1.** \(h_5^0\) is independent of volume form by vectors in \(V_4\).

**Proof:**
\(h_5^0(q_0, ..., q_5)\) can be written as
\[
h_5^0(q_0, ..., q_5) = \sum_{i=0}^{5} (-1)^i \frac{D \Delta(q_i)}{\Delta(q_i)} \otimes \frac{\Delta(q_{i+1})}{\Delta(q_{i+2})} \wedge \frac{\Delta(q_{i+2})}{\Delta(q_{i+3})} \wedge \frac{\Delta(q_{i+3})}{\Delta(q_{i+4})} \wedge \frac{\Delta(q_{i+4})}{\Delta(q_{i+5})}
\]
so if we change volume \(V = \lambda V\) where \(\lambda \in F\), this change will not affect the right side of the equation.

**Lemma 3.2.** \(h_5^0 \circ p\) is independent of length of vectors in \(V_5\).

**Proof:**
Let \(h_5^0 \circ p(q_0, ..., q_6)\) can be written as
\[
h_5^0 \circ p(q_0, ..., q_6) = \sum_{j=0}^{6} (-1)^j \sum_{i \neq j}^{5} (-1)^i \frac{D \Delta(q_j / q_i)}{\Delta(q_j / q_i)} \otimes \frac{\Delta(q_j / q_{i+1})}{\Delta(q_j / q_{i+2})} \wedge \frac{\Delta(q_j / q_{i+2})}{\Delta(q_j / q_{i+3})} \wedge \frac{\Delta(q_j / q_{i+3})}{\Delta(q_j / q_{i+4})} \wedge \frac{\Delta(q_j / q_{i+4})}{\Delta(q_j / q_{i+5})}
\]
so if we change the length of vectors like \((q_0, ..., q_6) = \omega(q_0, ..., q_6)\) where \(\omega \in F\) and use the homomorphism property \(D(\omega q_0) = \omega D(q_0)\) then the difference will be zero. Therefore \(h_5^0\) is independent of length of vectors in \(V_5\).

**Lemma 3.3.** \(h_1^1\) is independent of volume form.

**Proof:**
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\[ h_1^5(q_0, q_1, q_2, q_3, q_4, q_5, q_6) = -\frac{1}{10} \sum_{i\neq j \neq k}^{6} (-1)^i (r(q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_6)_2^D \otimes \]
\[ \prod_{k \neq l}^{6} \Delta(q_k, q_l) \wedge \prod_{j \neq l}^{6} \Delta(q_j, q_l) \wedge \prod_{i \neq l}^{6} \Delta(q_i, q_l) \]
\[ D \log(\prod_{i \neq j \neq k \neq l}^{6} \Delta(q_i, q_l)) \otimes [r(q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_6)]_2^D \]
\[ = \prod_{j \neq l}^{6} \Delta(q_j, q_l) \wedge \prod_{k \neq l}^{6} \Delta(q_k, q_l) + D \log(\prod_{j \neq l}^{6} \Delta(q_j, q_l)) \otimes (4) \]
\[ = [r(q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_6)]_2^D \otimes \prod_{k \neq l}^{6} \Delta(q_k, q_l) \wedge \prod_{i \neq l}^{6} \Delta(q_i, q_l) \]
\[ D \log(\prod_{k \neq l}^{6} \Delta(q_k, q_l)) \otimes [r(q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_6)]_2^D \]
\[ = \prod_{i \neq l}^{6} \Delta(q_i, q_l) \wedge \prod_{j \neq l}^{6} \Delta(q_j, q_l) \]
if we change volume \( V = \gamma V \) where \( \gamma \in \text{field} \), the difference gives us

\[ h_1^5((q_0, q_1, q_2, q_3, q_4, q_5, q_6) = -\frac{1}{10} \sum_{i=0}^{6} (-1)^i [[r(q_0, q_1, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_6)]_2^D \otimes \gamma^6 \]

use the five term relation it can be reduce to

\[ h_1^5(q_0, ..., q_6) = -\frac{1}{10} (r(q_0, q_4, q_5, q_6)^D - r(q_1, q_4, q_5, q_6)^D + \]
\[ r(q_2, q_4, q_5, q_6)^D - r(q_3, q_4, q_5, q_6)^D + r(q_0, q_3, q_5, q_6)^D) \otimes \gamma^6 \]
\[ = \text{five term relation in } \beta_2^D(F) \text{ and equal to zero. Therefore } h_1^5 \text{ is independent of volume form.} \]
**Lemma 3.4.** \( h_1^5 \) is independent of length of vectors.

**Proof:**

let \((q_0, q_1, q_2, q_3, q_4, q_5, q_6) = (\omega q_0, q_1, q_2, q_3, q_4, q_5, q_6) \), where \( \omega \in F \).

\[
h_1^5 \circ p \left( (q_0, q_1, q_2, q_3, q_4, q_5, q_6) - (\omega q_0, q_1, q_2, q_3, q_4, q_5, q_6) \right) = 0
\]

The difference gives us

\[
h_1^5 \circ p(q_0, q_1, ..., q_6) = -\frac{1}{10} \left( r(q_1, q_2, q_3, q_4)D^2 - r(\gamma q_0, q_2, q_3, q_4)D^2 + r(\gamma q_0, q_1, q_3, q_4)D^2 - r(\gamma q_0, q_1, q_2, q_4)D^2 + r(\gamma q_0, q_2, q_3, q_5)D^2 - r(\gamma q_0, q_1, q_3, q_5)D^2 + r(\gamma q_0, q_2, q_3, q_6)D^2 - r(\gamma q_0, q_1, q_3, q_6)D^2 + r(q_1, q_2, q_3, q_6)D^2 - r(q_0, q_1, q_2, q_6)D^2 + r(q_0, q_1, q_3, q_6)d^2 - r(q_0, q_1, q_2, q_6)d^2 + r(q_0, q_1, q_2, q_3)d \right) \otimes \gamma^6
\]

which form three five term relation in \( \beta^D_2(F) \) and equal to zero. Therefore \( h_1^5 \) is independent of length of vectors.

**Theorem 3.5.** The above diagram \((A)\) is commutative.

**Proof:**

let we have \((q_0, ..., q_6) \in G_7(2)\) now apply morphism \( p \) we get

\[
p(q_0, ..., q_6) = \sum_{j=0}^{6} \left( q_j / q_0, ..., \hat{q}_j, ..., q_6 \right) \tag{7}
\]

now apply \( h_0^5 \)

\[
h_0^5 \circ p(q_0, ..., q_6) = \sum_{j=0}^{6} (-1)^j \sum_{i \neq j}^{5} (-1)^i \frac{D\Delta(q_j/q_i)}{\Delta(q_j/q_i)} \otimes \frac{\Delta(q_j/q_{i+1})}{\Delta(q_j/q_{i+2})} \otimes \frac{\Delta(q_j/q_{i+3})}{\Delta(q_j/q_{i+3})} \otimes \frac{\Delta(q_j/q_{i+4})}{\Delta(q_j/q_{i+4})} \otimes \frac{\Delta(q_j/q_{i+5})}{\Delta(q_j/q_{i+5})} \tag{8}
\]

take again \((q_0, ..., q_6) \in G_7(2)\), we apply morphism \( h_1^5 \) then we get
\[ h_1^5(q_0, \ldots, q_6) = -\frac{1}{10} \left( \sum_{i \neq j \neq k}^6 (-1)^i (r(q_0, \ldots, \hat{q}_i, \hat{q}_j, \hat{q}_k, \ldots, q_6))^D \otimes \prod_{\substack{k \neq l \atop l = 0}}^6 \Delta(q_k, q_l) \right)^\partial \]

\[ \prod_{j \neq l}^6 \Delta(q_j, q_l) \wedge \prod_{i \neq l}^6 \Delta(q_i, q_l) - D \log(\prod_{i \neq l}^6 \Delta(q_i, q_l)) \otimes \]

\[ [r(q_0, \ldots, \hat{q}_i, \hat{q}_j, \hat{q}_k, \ldots, q_6)]^2 \otimes \prod_{j \neq l}^6 \Delta(q_j, q_l) \wedge \prod_{k \neq l}^6 \Delta(q_k, q_l) + \]

\[ D \log(\prod_{j \neq l}^6 \Delta(q_j, q_l)) \otimes [r(q_0, \ldots, \hat{q}_i, \hat{q}_j, \hat{q}_k, \ldots, q_6)]^2 \otimes \]

\[ \prod_{k \neq l}^6 \Delta(q_k, q_l) \wedge \prod_{i \neq j \neq k}^6 \Delta(q_i, q_l) - D \log(\prod_{i \neq l}^6 \Delta(q_i, q_l)) \otimes \]

\[ [r(q_0, \ldots, \hat{q}_i, \hat{q}_j, \hat{q}_k, \ldots, q_6)]^2 \otimes \prod_{j \neq l}^6 \Delta(q_j, q_l) \wedge \prod_{k \neq l}^6 \Delta(q_k, q_l) \]

(9)

Now apply \( \partial^D \)

\[ \partial^D \circ h_1^5(q_0, \ldots, q_6) = -\frac{1}{10} \left( \sum_{i \neq j \neq k}^6 (-1)^i (-D \log(1 - r(q_0, \ldots, \hat{q}_i, \hat{q}_j, \hat{q}_k, \ldots, q_6)) \otimes r(q_0, \ldots, \hat{q}_i, \hat{q}_j, \hat{q}_k, \ldots, q_6) \wedge \prod_{k \neq l}^6 \Delta(q_k, q_l) \right) \wedge \prod_{i \neq l}^6 \Delta(q_i, q_l) \wedge \prod_{j \neq l}^6 \Delta(q_j, q_l) + D \log(\prod_{i \neq l}^6 \Delta(q_i, q_l)) \otimes (1 - r(q_0, \ldots, \hat{q}_i, \hat{q}_j, \hat{q}_k, \ldots, q_6)) \wedge \prod_{k \neq l}^6 \Delta(q_k, q_l) \wedge \prod_{i \neq j \neq k}^6 \Delta(q_i, q_l) - D \log(\prod_{i \neq l}^6 \Delta(q_i, q_l)) \otimes \]
\[
\partial^D h^5 = -\frac{1}{10} \left[ \sum_{\substack{i \neq j \neq k \neq l \neq m \neq n \neq 0 \neq 2 \neq 3 \neq 4 \neq 5 \neq 6}} (-1)^i (-D \log(1 - r(q_6, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_0)) \otimes r(q_0, ...,
\right.
\]
\[
\hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_6) \wedge \prod_{k \neq l}^{6} \Delta(q_k, q_l) \wedge \prod_{j \neq l}^{6} \Delta(q_j, q_l) \wedge \prod_{i \neq l}^{6} \Delta(q_i, q_l)
\]
\[
+ D \log(r(q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_6)) \otimes (1 - r(q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_6)) \wedge
\]
\[
\prod_{k \neq l}^{6} \Delta(q_k, q_l) \wedge \prod_{j \neq l}^{6} \Delta(q_j, q_l) \wedge \prod_{i \neq l}^{6} \Delta(q_i, q_l) - D \log(\prod_{i \neq l}^{6} \Delta(q_i, q_l)) \otimes
\]
\[
(1 - r(q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_6)) \wedge r(q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_6) \wedge \prod_{j \neq l}^{6} \Delta(q_j, q_l) \wedge
\]
\[
\prod_{k \neq l}^{6} \Delta(q_k, q_l) + D \log(\prod_{j \neq l}^{6} \Delta(q_j, q_l)) \otimes (1 - r(q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_6)) \wedge
\]
\[
r(q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_6) \wedge \prod_{k \neq l}^{6} \Delta(q_k, q_l) \wedge \prod_{i \neq l}^{6} \Delta(q_i, q_l) - D \log(\prod_{k \neq l}^{6} \Delta(q_k, q_l)) \otimes (1 - r(q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_6)) \wedge r(q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_6) \wedge
\]
\[
\prod_{i \neq l}^{6} \Delta(q_i, q_l) \wedge \prod_{j \neq l}^{6} \Delta(q_j, q_l)
\]
(10)

after using wedge, tensor and Seigal cross ratio property [12] we get

\[
\partial^D h^5 = \sum_{i=0}^{6} (-1)^i \sum_{i \neq j}^{5} (-1)^i \frac{D \Delta(q_j / q_i)}{\Delta(q_j / q_i)} \otimes \frac{\Delta(q_j / q_{i+1})}{\Delta(q_j / q_{i+2})} \wedge \frac{\Delta(q_j / q_{i+2})}{\Delta(q_j / q_{i+3})} \wedge \frac{\Delta(q_j / q_{i+3})}{\Delta(q_j / q_{i+4})} \wedge \frac{\Delta(q_j / q_{i+4})}{\Delta(q_j / q_{i+5})}
\]
(11)

(8) and (11) are enough for the proof of the theorem.
3.2 Weight \( (n = 6) \)

For this weight we connect the sub complex of infinitesimal in weight-6 and Grassmannian complex.

\[
\begin{array}{ccc}
G_9(2) & \xrightarrow{p} & G_8(1) \\
\downarrow d & & \downarrow d \\
G_8(2) & \xrightarrow{p} & G_7(1)
\end{array}
\]

where

\[
\beta^D_2(F) \otimes \wedge^4 F^\times \oplus F \otimes B_2(F) \otimes \wedge^3 F^\times \xrightarrow{\partial^D} F \otimes \wedge^5 F^\times
\]

and

\[
h_0^6 : (q_0, \ldots, q_6) \to \sum_{i=0}^{6} (-1)^i \text{D log } \Delta(q_i) \otimes \frac{\Delta(q_{i+1})}{\Delta(q_{i+2})} \wedge \frac{\Delta(q_{i+2})}{\Delta(q_{i+3})} \wedge \frac{\Delta(q_{i+3})}{\Delta(q_{i+4})} \wedge \frac{\Delta(q_{i+4})}{\Delta(q_{i+5})} \wedge \frac{\Delta(q_{i+5})}{\Delta(q_{i+6})} \pmod{7}
\]

\[
h_1^6(q_0, \ldots, q_7) = \frac{1}{15} \sum_{i \neq j \neq k \neq l} (-1)^i [r(q_0, \ldots, q_i, \hat{q}_j, \hat{q}_k, \hat{q}_l, \ldots, q_6)]^D_2 \otimes \prod_{i \neq m, m=0}^{7} \Delta(q_i, q_m) \wedge \prod_{k \neq m, m=0}^{7} \Delta(q_k, q_m) \wedge \prod_{j \neq m, m=0}^{7} \Delta(q_j, q_m) \wedge \prod_{l \neq m, m=0}^{7} \Delta(q_l, q_m)
\]

\[- \text{D log} \prod_{i \neq m, m=0}^{7} \Delta(q_i, q_m) \otimes [r(q_0, \ldots, q_i, \hat{q}_j, \hat{q}_k, \hat{q}_l, \ldots, q_6)]^D_2 \otimes \prod_{j \neq m, m=0}^{7} \Delta(q_j, q_m) \wedge \prod_{k \neq m, m=0}^{7} \Delta(q_k, q_m) \wedge \prod_{l \neq m, m=0}^{7} \Delta(q_l, q_m) \wedge \prod_{i \neq m, m=0}^{7} \Delta(q_i, q_m) \]

\[
+ \text{D log} \prod_{j \neq m, m=0}^{7} \Delta(q_j, q_m) \otimes [r(q_0, \ldots, q_i, \hat{q}_j, \hat{q}_k, \hat{q}_l, \ldots, q_6)]^D_2 \otimes \prod_{k \neq m, m=0}^{7} \Delta(q_k, q_m) \wedge \prod_{l \neq m, m=0}^{7} \Delta(q_l, q_m) \wedge \prod_{i \neq m, m=0}^{7} \Delta(q_i, q_m) \]

\[
\prod_{k \neq m, m=0}^{7} \Delta(q_k, q_m) \wedge \prod_{l \neq m, m=0}^{7} \Delta(q_l, q_m) \wedge \prod_{i \neq m, m=0}^{7} \Delta(q_i, q_m) \]

now let us take again (5.5).

Theorem 3.6. The above diagram (B) is commutative.

Proof:
The proof is the same as theorem 5.5.

let us have \((q_0, ..., q_7) \in G_8(2)\) now apply morphism \(p\) we get

\[ p(q_0, ..., q_7) = \sum_{j=0}^{7} (-1)^j (q_j / q_0, ..., \hat{q}_j, ..., q_7) \]  
(14)

now apply \(h_0^6\)

\[ h_0^6 \circ p(q_0, ..., q_6) = \sum_{j=0}^{7} (-1)^j \sum_{i \neq j}^{6} (-1)^i D \log \Delta(q_j / q_i) \otimes \frac{\Delta(q_j / q_{i+1})}{\Delta(q_j / q_{i+2})} \wedge \]  
(15)

now let us take again \((q_0, ..., q_7) \in G_8(2)\) apply \(h_1^6\)

\[ h_1^6(q_0, ..., q_6, q_7) = \frac{1}{15} \left[ \sum_{i \neq j \neq k \neq l}^{7} (-1)^i (r(q_0, ..., q_i, \hat{q}_i, q_j, \hat{q}_j, q_k, \hat{q}_k, \hat{q}_l, ..., q_7)_2 \right] \otimes \prod_{l \neq m}^{7} \Delta(q_l, q_m) \wedge \]

\[ \prod_{k \neq m}^{7} \Delta(q_k, q_m) \wedge \prod_{j \neq m}^{7} \Delta(q_j, q_m) \wedge \prod_{i \neq m}^{7} \Delta(q_i, q_m) \]

\[ - D \log \left( \prod_{i \neq m}^{7} \Delta(q_i, q_m) \right) \otimes [r(q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, ..., q_6)]_2 \otimes \]
\[
\cdots \prod_{j \neq m} \triangle(q_j, q_m) \wedge \prod_{k \neq m} \triangle(q_k, q_m) \wedge \prod_{l \neq m} \triangle(q_l, q_m)
\]

\[+ D\log(\prod_{j \neq m} \triangle(q_j, q_m)) \otimes \left[ r(q_0, \ldots, \hat{q}_i, \hat{q}_j, \hat{q}_k, \hat{q}_l, \ldots, q_7) \right]_2 \otimes \]

\[\prod_{k \neq m} \triangle(q_k, q_m) \wedge \prod_{l \neq m} \triangle(q_l, q_m) \wedge \prod_{i \neq m} \triangle(q_i, q_m)\]

\[- D\log(\prod_{k \neq m} \triangle(q_k, q_m)) \otimes \left[ r(q_0, \ldots, \hat{q}_i, \hat{q}_j, \hat{q}_k, \hat{q}_l, \ldots, q_7) \right]_2 \otimes \]

\[\prod_{i \neq m} \triangle(q_i, q_m) \wedge \prod_{j \neq m} \triangle(q_j, q_m) \wedge \prod_{k \neq m} \triangle(q_k, q_m)\]

\[
\text{now apply } \partial^D.
\]

\[
\partial^D \circ h^6(q_0, \ldots, q_7) = -\frac{1}{15} \left[ \sum_{i \neq j \neq k \neq l}^{7} (-1)^i (1 - r(q_0, \ldots, \hat{q}_i, \hat{q}_j, \hat{q}_k, \hat{q}_l, \ldots, q_7)) \otimes r(q_0, \ldots, \hat{q}_i, \hat{q}_j, \hat{q}_k, \hat{q}_l, \ldots, q_7) \right.
\]

\[
\hat{q}_i, \hat{q}_j, \hat{q}_k, \hat{q}_l, \ldots, q_7) \wedge \prod_{l \neq m} \left( q_l, q_m \right) \prod_{k \neq m} \left( q_k, q_m \right) \prod_{j \neq m} \left( q_j, q_m \right) \prod_{i \neq m} \left( q_i, q_m \right)
\]

\[+ D\log(r(q_0, \ldots, \hat{q}_i, \hat{q}_j, \hat{q}_k, \hat{q}_l, \ldots, q_7) \otimes (1 - r(q_0, \ldots, \hat{q}_i, \hat{q}_j, \hat{q}_k, \hat{q}_l, \ldots, q_7)) \wedge \]

\[\prod_{l \neq m} \left( q_l, q_m \right) \wedge \prod_{k \neq m} \left( q_k, q_m \right) \wedge \prod_{j \neq m} \left( q_j, q_m \right) \wedge \prod_{i \neq m} \left( q_i, q_m \right)\]

\[- D\log(\prod_{i \neq m} \left( q_i, q_m \right)) \otimes (1 - r(q_0, \ldots, \hat{q}_i, \hat{q}_j, \hat{q}_k, \hat{q}_l, \ldots, q_7) \otimes r(q_0, \ldots, \hat{q}_i, \hat{q}_j, \hat{q}_k, \hat{q}_l, \ldots, q_7) \ldots \]

...
For this weight we connect $G^3$. Weight $(n = N)$, after simplifications we get

\[ \hat{q}_k, \hat{q}_l, ... , q_7 \wedge r(q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, \hat{q}_l, ..., q_7) \wedge r(q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, \hat{q}_l, ..., q_7) \wedge \prod_{i \neq m} (q_i, q_m) - D \log(\prod_{k \neq m} (q_k, q_m)) \otimes (1 - r(q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, \hat{q}_l, ..., q_7) \otimes r(q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, \hat{q}_l, ..., q_7) \wedge \prod_{i \neq m} (q_i, q_m) \wedge \prod_{k \neq m} (q_k, q_m) \wedge \prod_{l \neq m} (q_l, q_m) \] 

\[ \prod_{i \neq m} (q_i, q_m) - D \log(\prod_{k \neq m} (q_k, q_m)) \otimes (1 - r(q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, \hat{q}_l, ..., q_7) \otimes r(q_0, ..., \hat{q}_i, \hat{q}_j, \hat{q}_k, \hat{q}_l, ..., q_7) \wedge \prod_{i \neq m} (q_i, q_m) \wedge \prod_{k \neq m} (q_k, q_m) \wedge \prod_{l \neq m} (q_l, q_m) ] \]

(17)

After simplifications we get

\[ h_0^6 \circ p = \partial^D \circ h_1^6 \] 

(18)

### 3.3 Weight $(n = N)$

For this weight we connect $G_{N+3}(2), G_{N+2}(2), G_{N+2}(1)$ and $G_{N+1}(1)$ and the sub complex of infinitesimal in weight-$N$, therefore we will connect them like given below

\[ G_{N+3}(2) \xrightarrow{p} G_{N+2}(1) \quad (C) \]

\[ G_{N+2}(2) \xrightarrow{p} G_{N+1}(1) \]

\[ \beta_2^D(F) \otimes \wedge^{N-2} F^\times \oplus F \otimes B_2(F) \otimes \wedge^{N-3} F^\times \partial^D \rightarrow F \otimes \wedge^{N-1} F^\times \]
where

\[ h_0^N : (q_0, \ldots, q_N) \rightarrow \sum_{i=0}^{N} (-1)^i \text{D} \log(q_i) \otimes \frac{(q_{i+1})}{(q_{i+2})} \wedge \frac{(q_{i+3})}{(q_{i+4})} \wedge \ldots \wedge \frac{(q_{i+N-1})}{(q_{i+N})} \pmod{(N+1)} \] (19)

and

\[ h_1^N(q_0, \ldots, q_{N+1}) = (-1)^N \frac{1}{N!} \sum_{\substack{i \neq i_1 \neq \ldots \neq i_{N+1} \neq t \neq 0 \neq \ldots \neq 0 \neq \ldots \neq \ldots \neq 0}}^{N+1} (-1)^i (+q_0, \ldots, \hat{q}_i, \hat{q}_{i+1}, \ldots, \hat{q}_{i+N-2}, \ldots q_{N+1}) \otimes \wedge \left( \prod_{t=0}^{N+1} (q_{N+1}, q_t) \wedge \ldots \wedge \prod_{t=0}^{N+1} (q_i, q_t) \right) - \text{D} \log(\prod_{t=0}^{N+1} (q_i, q_t)) \otimes [q_0, \ldots, \hat{q}_i, \hat{q}_{i+1}, \ldots, \hat{q}_{i+N-2}, \ldots q_{N+1}]_2 \otimes \prod_{t=0}^{N+1} (q_i, q_t) \wedge \ldots \wedge \prod_{t=0}^{N+1} (q_{N+1}, q_t) \left( -1 \right)^{N+1} \text{D} \log(\prod_{t=0}^{N+1} (q_{N+1}, q_t)) \otimes [q_0, \ldots, \hat{q}_i, \hat{q}_{i+1}, \ldots, \hat{q}_{i+N-2}, \ldots q_{N+1}]_2 \otimes \prod_{t=0}^{N+1} (q_i, q_t) \right] \] (20)

**Theorem 3.7.** \( h_0^N \circ p = \partial^D \circ h_1^N \)

**Proof:**

let us have \((q_0, \ldots, q_{N+1}) \in G_{N+2}(2)\) and by applying morphism \(p\) we get

\[ p(q_0, \ldots, q_{N+1}) = (q_0/q_1, \ldots, q_{N+1}) - (q_1/q_0, \ldots, q_{N+1}) + \ldots + (-1)^{N+1}(q_{N+1}/q_0, \ldots, q_N) \] (21)

now apply \(h_0^N\)
\[ h_N^0 \circ p(q_0, q_1, \ldots, q_{N+1}) = D \log(q_0, q_1) \otimes \frac{q_0, q_2}{q_0, q_3} \wedge \ldots \wedge \frac{q_0, q_{N-1}}{q_0, q_N} \wedge \frac{q_0, q_N}{q_0, q_{N+1}} - \]
\[ D \log(q_0, q_2) \otimes \frac{q_0, q_1}{q_0, q_3} \wedge \ldots \wedge \frac{q_0, q_{N-1}}{q_0, q_N} \wedge \frac{q_0, q_N}{q_0, q_{N+1}} + \]
\[ \ldots \]
\[ (-1)^{N+1} D \log(q_0, q_{N+1}) \otimes \frac{q_1, q_{N+1}}{q_2, q_{N+1}} \wedge \frac{q_{N+2}, q_{N-2}}{q_{N+1}, q_{N-1}} \wedge \frac{q_{N+1}, q_{N+2}}{q_{N+1}, q_N} \]
\[ \text{now let us consider again } (q_0, \ldots, q_{N+1}) \in G_{N+2}(2) \text{ apply } h_1^N. \]
\[ h_1^N(q_0, \ldots, q_{N+1}) = (-1)^N \frac{1}{NC_2} \left[ \sum_{i \neq i+1 \neq \ldots \neq N+1}^{N+1} (-1)^i (q_0, \ldots, \hat{q}_i, \hat{q}_{i+1}, \ldots, \hat{q}_{i+N-2}, \ldots q_{N+1}) \otimes \right. \]
\[ \prod_{t=0}^{N+1} (q_{N+1}, q_t) \wedge \ldots \wedge \prod_{t=0}^{N+1} (q_i, q_t) \]
\[ - D \log(\prod_{i \neq t}^{N+1} (q_i, q_t)) \otimes [q_0, \ldots, \hat{q}_i, \hat{q}_{i+1}, \ldots, \hat{q}_{i+N-2}, \ldots q_{N+1}] \]
\[ \prod_{t=0}^{N+1} (q_{N+1}, q_t) \wedge \ldots \prod_{t=0}^{N+1} (q_N, q_t) \]
\[ \ldots \]
\[ \left. (-1)^{N+1} D \log(\prod_{i \neq t}^{N+1} (q_{N+1}, q_t)) \otimes [q_0, \hat{q}_i, \hat{q}_j, \hat{q}_k, \hat{q}_l, \ldots, q_{N+1}] \right. \]
\[ \prod_{t=0}^{N+1} (q_i, q_t) \ldots \prod_{t=0}^{N+1} (q_N, q_t) \right] \]
\[ (22) \]
now apply $\partial^D$

$$h_1^N(q_0, \ldots, q_{N+1}) = (-1)^N \cdot \frac{1}{N!} \sum_{i \neq i+1 \neq \ldots \neq N+1} \left[ (-1)^i \left(- D \log(1 - q_0, \ldots, \hat{q}_i, \hat{q}_{i+1}, \ldots, q_{N+1}) \right) \right]$$

$$\cdots$$

$$\hat{q}_{i+N-2}, \ldots q_{N+1}) \otimes (q_0, \ldots, \hat{q}_i, \hat{q}_{i+1}, \ldots, \hat{q}_{i+N-2}, \ldots, q_{N+1}) \wedge \prod_{t=0}^{N+1} (q_N+1, q_t)$$

$$\wedge \ldots \wedge \prod_{t=0}^{N+1} (q_i, q_t) + D \log(q_0, \ldots, \hat{q}_i, \hat{q}_{i+1}, \ldots, \hat{q}_{i+N-2}, \ldots, q_{N+1}) \otimes$$

$$\prod_{t=0}^{N+1} (q_i, q_t) - D \log(\prod_{t=0}^{N+1} (q_i, q_t)) \otimes (1 - (q_0, \ldots, \hat{q}_i, \hat{q}_{i+1}, \ldots, \hat{q}_{i+N-2}, \ldots, q_{N+1}))$$

$$\wedge (q_0, \ldots, \hat{q}_i, \hat{q}_{i+1}, \ldots, \hat{q}_{i+N-2}, \ldots, q_{N+1}) \wedge \prod_{t=0}^{N+1} (q_{N+1}, q_t) \wedge \ldots \wedge \prod_{t=0}^{N+1} (q_{N+1}, q_t)$$

$$\ldots$$

$$(-1)^{N+1} \cdot D \log(\prod_{t=0}^{N+1} \prod_{t=0}^{N+1} (q_{N+1}, q_t)) \otimes (1 - (q_0, \ldots, \hat{q}_i, \hat{q}_{i+1}, \ldots, \hat{q}_{i+N-2}, \ldots, q_{N+1})) \wedge$$

$$\prod_{t=0}^{N+1} (q_i, q_t), \ldots, \prod_{t=0}^{N+1} (q_N, q_t) \right]$$

(24) after using cross ratio, Seigal cross ratio [12], wedge and tensor properties, we observe that $h_1^N \circ p = \partial^D \circ h_1^N$

4 Conclusion

Here we have defined morphisms between Grassmanian and variant of Cathe-lineau complex for weight 5 and 6 and proved that the associated diagrams are commutative. Then we generalized these morphism as $h_0^n$ and $h_1^n$, while also
proving that the generalized diagram is commutative i.e \( h_0^N \circ p = \partial D \circ h_1^N \)

References


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