Finite $p$-Groups with Large Normal Closures of Non-Normal Cyclic Subgroups

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Abstract

In this paper, we completely classify all the finite $p$-groups $G$ such that, for every non-normal cyclic subgroup $H$ of $G$, the quotient group $G/H^G$ is cyclic.

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1 Introduction

All groups considered in this paper are finite.

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It is well-known that the normality of subgroups plays an important role in the research in group theory. Thus it is interesting to investigate the structure of groups by using normalizers or normal closures of non-normal subgroups. For example, some authors investigated \( p \)-groups in which all non-normal cyclic subgroups have index in their normalizers not greater than \( p^m \) (see [7]). Herzog, Longobardi, Maj and Mann investigated groups \( G \) in which \( \langle a, a^g \rangle = \langle a \rangle^G \) for every non-normal cyclic subgroup \( \langle a \rangle \) and every \( g \in G \setminus N_G(\langle a \rangle) \), which are called \( J \)-groups [3]. Then [2] continued to classify all the \( J \)-groups of odd prime power order. It is easy to see that \( |\langle a \rangle^G : \langle a \rangle| \leq p \) for every \( a \in G \) if a \( p \)-group \( G \) is a \( J \)-group. This motivates the investigation of the structure of \( p \)-groups \( G \) in which \( |\langle a \rangle^G : \langle a \rangle| \leq p^m \) for every cyclic subgroup \( \langle a \rangle \) of \( G \), for a fixed \( m \geq 0 \). In fact, the structure of this kind of groups has been investigated in [4] and many interesting results have been given.

As the dual of above investigation, studying the structure of a \( p \)-group \( G \) in which \( |G : \langle a \rangle^G| \leq p^w \) for every non-normal cyclic subgroup \( \langle a \rangle \) of \( G \) is also very interesting. In fact, when \( w = 1 \), Janko give the classification (see [1]). Then [8] investigated these groups and classify all such groups when \( w = 2 \) and \( p > 2 \).

In this paper, we are interested in the structure of a \( p \)-group \( G \) which satisfies the following property:

For every non-normal cyclic subgroup \( H \) of \( G \), \( G/H \) is cyclic.

For convenience, we use (**) to denote this property. And we classify all the finite \( p \)-groups with (**) up to isomorphism.

The terminology and the notation in this paper are standard. Let \( G \) be a finite \( p \)-group. Then \( \Phi(G) \), \( G' \), \( c(G) \) and \( Z(G) \) denote the Frattini subgroup, the commutator subgroup, the nilpotent class and the center of \( G \), respectively. And \( d(G) \) is a positive integer such that \( |G/\Phi(G)| = p^{d(G)} \). \( \Omega(G) = \langle g \in G | g^p = 1 \rangle \). For a subgroup \( H \) of \( G \), the normal closure and the normalizer of \( H \) in \( G \) are denoted by \( H^G \) and \( N_G(H) \). For an element \( a \) in \( G \), \( o(a) \) denotes the order of \( \langle a \rangle \). We use \( C_{p^n} \) to denote a cyclic group of order \( p^n \) and \( Q_8 \) to denote the quaternion group.

## 2 Preliminaries

We gather all the definitions and results used in what follows.

**Definition 2.1.** ([6] Definition 2.3.2) A non-abelian \( p \)-group is called a minimal non-abelian group if its every maximal subgroup is abelian.

**Definition 2.2.** ([6] Definition 2.5.1) Let \( G \) be a group of order \( p^n \) and \( n \geq 3 \). We call \( G \) a \( p \)-group of maximal class if its nilpotent class is \( n - 1 \)

**Definition 2.3.** ([6] Definition 5.1.12) A \( p \)-group is said to be regular if for any \( a, b \in G \), \( (ab)^p = a^pb^pc_3^{\tau_3}...c_m^{\tau_m} \), where \( c_i \in \langle a, b \rangle' \).
Lemma 2.4. ([6] Burnside's basis theorem) Let $G$ be a $p$-group. If $|G/\Phi(G)| = p^{d(G)}$, then every generating set of $G$ with minimum number of generators exactly contains $d(G)$ generators.

Lemma 2.5. ([6] Theorem 5.2.2) Let $G$ be a $p$-group. If $p > 2$ and $G'$ is cyclic, then $G$ is regular.

Lemma 2.6. ([6] Corollary 2.2.12) Let $G$ be a 2-group and $N \leq G$. If $N$ is not cyclic and $N \leq \Phi(G)$, then there exists an abelian subgroup $K \leq G$ such that $K \leq N$ and $K \cong C_2 \times C_2$.

Lemma 2.7. ([6] Theorem 2.5.5) A group $G$ is a 2-group of maximal class if and only if $|G : G'| = 4$.

Lemma 2.8. ([5] ) Let $G$ be a minimal non-abelian $p$-group. Then $G$ is isomorphic to one of the following $p$-groups:

1. $Q_8$;
2. $M_p(n,m) = \langle a, b | a^p = b^m = 1, a^b = a^{1+p^{n-1}} \rangle$ with $n \geq 2$; (metacyclic)
3. $M_p(n,m,1) = \langle a, b, c | a^p = b^m = c^p = 1, [a,b] = c, [c,a] = [c,b] = 1 \rangle$ with $n \geq m$, and $n + m \geq 3$ when $p = 2$. (non-metacyclic)

Lemma 2.9. ([6] Theorem 2.3.6) Let $G$ be a finite $p$-group. Then the following propositions are equivalent:

1. $G$ is minimal non-abelian $p$-group;
2. $d(G) = 2$ and $|G'| = p$;
3. $d(G) = 2$ and $Z(G) = \Phi(G)$;

Lemma 2.10. ([6] Proposition 2.1.7 and 2.1.8) Let $G$ be a $p$-group and $G'$ is abelian. For any positive integer $i$ and $j$, we make the following convention:

$$[ia, jb] = [a, b, a, \ldots, a, b, \ldots, b]. \quad a, b \in G$$

Then for any positive integers $m$ and $n$,

$$[a^m, b^n] = \prod_{i=1}^{m} \prod_{j=1}^{n} [ia, jb]^\binom{m}{i} \binom{n}{j},$$

$$(ab^{-1})^m = a^m \prod_{i+j \leq m} [ia, jb]_{i+j} \binom{m}{i+j} b^{-m}, m \geq 2.$$
3 Main Results

In this section we prove our results.

Lemma 3.1. Let $G$ be a $p$-group. If $G$ satisfies (**), then its quotient groups also satisfy (**).

Proof. Pick a normal subgroup $N$ of $G$, we prove that $G/N = \bar{G}$ satisfies (**). For any non-normal cyclic subgroup $\langle a \rangle$ of $\bar{G}$, we see $\langle a \rangle \not\trianglelefteq \bar{G}$. Since $G$ satisfies (**), we see that $G/\langle a \rangle^G$ is cyclic. So $G/\langle a \rangle^G \cong G/(\langle a \rangle^G N) \cong (G/\langle a \rangle^G)/(\langle a \rangle^G N/\langle a \rangle^G)$ is cyclic. Thus $G/N$ satisfies (**). \qed

Lemma 3.2. Let $G$ be a $p$-group. If $G$ satisfies (**), then $G'$ is cyclic.

Proof. Let $G$ be a minimal counterexample. Then $d(G') > 1$ and $G$ is a non-Dedekind group. There exists a non-normal cyclic subgroup $T$ of $G$ such that $G/T^G$ is cyclic and so $d(G) = 2$ by Lemma 2.4. Pick $N \leq Z(G)$ such that $|N| = p$. By Lemma 3.1, we see that $G/N$ satisfies the condition (**). So $G'/N/N$ is cyclic and then $G'$ is abelian and $d(G') = 2$. If $\Phi(G') \neq 1$, then $(G'/\Phi(G'))' = G'/\Phi(G')$ is cyclic. Thus $G'$ is cyclic, a contradiction. So $\Phi(G') = 1$ and therefore $G' \cong C_p \times C_p$. If $G' \not\trianglelefteq Z(G)$, then there exists an element $c \in G'$ such that $\langle c \rangle \not\trianglelefteq G$. By (**), we see that $G/\langle c \rangle^G$ is cyclic. $\langle c \rangle^G \leq G' \leq \Phi(G)$ implies that $G$ is cyclic, a contradiction. If $G' \leq Z(G)$, then $c(G) = 2$ and therefore $G'$ is cyclic by $d(G) = 2$, the final contradiction. \qed

Theorem 3.3. Let $G$ be a non-Dedekind $p$-group. Then $G$ satisfies (**) if and only if all the subgroups of $\Phi(G)$ are normal in $G$ and $d(G) = 2$. Moreover, if $G$ satisfies (**), then $\Phi(G)$ is abelian.

Proof. Since $G$ is a non-Dedekind $p$-group, there exists a non-normal cyclic subgroup $T$ of $G$. If $G$ satisfies (**), then $G/T^G$ is cyclic and so $d(G) = 2$ by Lemma 2.4. For any cyclic subgroup $H \leq \Phi(G)$, if $H \not\trianglelefteq G$, then $G/H^G$ is cyclic by the property (**). $H^G \leq HG' \leq \Phi(G)$ implies that $G$ is cyclic, a contradiction. So all the subgroups of $\Phi(G)$ are normal in $G$. Conversely, For any cyclic subgroup $K \not\trianglelefteq G$, we have $K \not\trianglelefteq \Phi(G)$. Then $G/K^G$ is cyclic by $d(G) = 2$. So $G$ satisfies (**).

By the above, $\Phi(G)$ is Dedekind $p$-group. If $\Phi(G)$ is not abelian, then $Q_8 \not\leq \Phi(G)$ and $Q_8 \not\leq G$. Since $Q_8$ has no subgroup of type $C_p \times C_p$, we see $Q_8$ is cyclic by Lemma 2.6, a contradiction. Therefore $\Phi(G)$ is abelian. \qed

Theorem 3.4. Let $G$ be a $p$-group with $p > 2$. Then $G$ satisfies (**) if and only if $G$ is an abelian $p$-group or a minimal non-abelian $p$-group.

Proof. If $G$ is abelian, then all the subgroups are normal in $G$ and $G$ satisfies (**). When $G$ is a minimal non-abelian $p$-group, we see that $\Phi(G) = Z(G)$
by Lemma 2.9 and then $G$ satisfies (***) by Theorem 3.3. Now we prove the necessity part. If all cyclic subgroups are normal in $G$, then $G$ is a Dedekind $p$-group. Since $p > 2$, we see $G$ is abelian. When there exists a cyclic subgroup which is not normal in $G$, it is easy to see that $d(G) = 2$. Then we only need to prove that $|G'| = p$ by Lemma 2.9. Since $G'$ is cyclic and $p > 2$, $G$ is regular by Lemma 2.5 and so there exist $a$ and $b$ such that $[a, b] = c$, $\langle a \rangle \cap \langle c \rangle = 1$ and $G' = \langle c \rangle$ (By Theorem 5.5.5 in [6]). If $o(a) = p$, then it follows from $1 = [a^p, b] = [a, b]^p[b, a]^p(a^{(\frac{1}{2})}[a, b, a]^p(\frac{1}{2})... = c^{p(1+kp)}$ that $c^p = 1$. If $o(a) \geq p^2$ and $o(c) > p$, then $\langle a^p \rangle$ is not normal in $G$, which contradicts to Theorem 3.3. So $o(c) = p$. Therefore $|G'| = p$ and then $G$ is a minimal non-abelian $p$-group.

Now we investigate 2-groups $G$ with (**). Since all the groups of order $2^3$ satisfy (**) by using theorem 3.3, we only consider the 2 groups of order $\geq 2^4$ and give the following theorem:

**Theorem 3.5.** Let $G$ be a 2-group and $|G| \geq 2^4$. Then $G$ satisfies (**) if and only if $G$ is one of the following 2-groups:

(I) $\langle a, b, c | a^4 = 1, b^{2^n} = 1, c^2 = b^4, [b, a] = c^{-1}, [c, a] = c^{-2}, [c, b] = 1 \rangle$, where $n \geq 3$;

(II) $\langle a, b, c | a^4 = 1, b^2 = 1, c^{2^n} = 1, [b, a] = c, [c, a] = [c, b] = c^{-2} \rangle$, where $n \geq 2$;

(III) $\langle a, b, c | a^4 = 1, b^2 = c^{2^n-1}, c^{2^n} = 1, [b, a] = c, [c, a] = [c, b] = c^{-2} \rangle$, where $n \geq 2$;

(IV) $\langle a, b | a^4 = 1, b^{2^n} = 1, [b, a] = b^{-2} \rangle$, where $n \geq 3$;

(V) $\langle a, b | a^4 = 1, b^{2^n} = 1, [b, a] = b^{2^{n-1} - 2} \rangle$, where $n \geq 3$;

(VI) 2-groups of maximal class of order $\geq 2^4$;

(VII) minimal non-abelian 2-groups of order $\geq 2^4$;

(VIII) Dedekind 2-groups of order $\geq 2^4$.

Moreover, the groups of different types, or of the same type but with different values of parameters, are not isomorphic.

In order to prove the Theorem 3.5, we need the following lemmas.

**Lemma 3.6.** The groups listed in Theorem 3.5 satisfy (**) and are not isomorphic to each other.

**Proof.** Assume that $G = \langle a, b \rangle$ is in type (I) groups. It is easy to see that $a^2 \in Z(G)$ from $[b, a^n] = [b, a^2][b, a, a] = c^{-2}c^2 = 1$. Then $\Phi(G) = \langle a^2, b^2, c \rangle$ is abelian. For any cyclic subgroup $\langle x \rangle$ of $\Phi(G)$, we may assume that $x = c^ia^{2j}b^{2k}$, where $i, j, k$ are integers. Since $[x, a] = [c^ib^{2k}, a] = [c^i, a]b^{2ki}b^{2k}a = c^{-2i}c^{-2k} = x^{-2}$ and $[x, b] = 1$, we see that $\langle x \rangle \leq G$. So $G$ satisfies (**) by Theorem 3.3. Thus type (I) groups satisfy (**).
Let $G = \langle a, b \rangle$ be in type (II) groups. We see that $a^2 \in Z(G)$ from $[b, a^2] = [b, a]^2[b, a, a] = c^2c^{-2} = 1$. Then $\Phi(G) = \langle a^2, c \rangle$ is abelian. For any cyclic subgroup $\langle x \rangle$ of $\Phi(G)$, we may assume that $x = c^i a^j$, where $i, j$ are integers. Since $[x, a] = [c^i, a] = [c^i, a] = c^{-2i} = x^{-2}$ and $[x, b] = c^{-2i} = x^{-2}$, we see that $\langle x \rangle \leq G$. So type (II) groups satisfy (**) unless $\langle x \rangle = \langle y \rangle$. Similarly, type (III) groups also satisfy (**) unless $\langle x \rangle = \langle y \rangle$.

Assume that $G = \langle a, b \rangle$ is in type (IV) groups. It is easy to see that $a^2 \in Z(G)$ from $[b, a^2] = [b, a]^2[b, a, a] = b^{-1}b^4 = 1$. Then $\Phi(G) = \langle a^2, b^2 \rangle$ is abelian. For any cyclic subgroup $\langle x \rangle$ of $\Phi(G)$, we may assume that $x = a^i b^j$, where $i, j$ are integers. Since $[x, a] = [b^j, a] = b^{-4j} = x^{-2}$ and $[x, b] = 1$, we see that $\langle x \rangle \leq G$. So type (IV) groups satisfy (**) unless $\langle x \rangle = \langle y \rangle$. Similarly, type (V) groups also satisfy (**) unless $\langle x \rangle = \langle y \rangle$.

If $G$ is in the type (VI) groups, then $\Phi(G) = G'$ is cyclic. So $G$ satisfy (**) unless $\langle x \rangle = \langle y \rangle$. When $G$ is in the type (VII) group, then $\Phi(G) = Z(G)$ and therefore $G$ satisfy (**) unless $\langle x \rangle = \langle y \rangle$.

For type (VIII) groups, the conclusion is trivial.

Now, we prove that the groups listed in theorem 3.5 are not isomorphic to each other. By using the order of groups, we see the groups of the same type but with different values of parameters are not isomorphic. By considering the nilpotent class and maximal subgroup, we only need to prove that groups in type (II), (III), (IV) or (V) with the same order are not isomorphic to each other. If $G$ is a type (II) group, then $\Omega(G) = \langle a^2, b, c \rangle$ which is of order $2^{n+2}$. However, $\Omega(G) = \langle a^2, c^{2n+1} \rangle$ which is of order 4 when $G$ is a type (III) group.

So groups in type (II) are not isomorphic to groups in type (III). Since type (IV), (V) groups are metacyclic and type (II), (III) groups are not metacyclic, we only need to prove that groups in type (IV) are not isomorphic to groups in type (V).

If $G$ is in type (IV) or (V) groups of order $2^{n+2}$, then $Z(G) = \langle a^2 \rangle \times \langle b^{2n-1} \rangle$. We consider the factor group of order $2^{n+1}$.

For the type (IV) group:

$G/\langle a^2 \rangle \cong \langle b \bar{a}^2 = 1, \bar{b}^{2n} = 1, [\bar{b}, \bar{a}] = \bar{b}^{-2} \rangle$, which is dihedral group.

$G/\langle b^{2n-1} \rangle \cong \langle \bar{a}, \bar{b} \bar{a}^4 = 1, \bar{b}^{2n-1} = 1, [\bar{b}, \bar{a}] = \bar{b}^{-2} \rangle$, which is type (IV) group.

$G/\langle a^2 b^{2n-1} \rangle \cong \langle \bar{a}, \bar{b} \bar{b}^{2n} = 1, \bar{a}^2 = \bar{b}^{2n-1}, [\bar{b}, \bar{a}] = \bar{b}^{-2} \rangle$, which is generalized quaternion group.

For the type (V) group:

$G/\langle a^2 \rangle \cong \langle \bar{a}, \bar{b} \bar{a}^2 = 1, \bar{b}^{2n} = 1, [\bar{b}, \bar{a}] = \bar{b}^{2n-1-2} \rangle$, which is semidihedral group.

$G/\langle b^{2n-1} \rangle \cong \langle \bar{a}, \bar{b} \bar{a}^4 = 1, \bar{b}^{2n-1} = 1, [\bar{b}, \bar{a}] = \bar{b}^{-2} \rangle$, which is type (IV) group.

$G/\langle a^2 b^{2n-1} \rangle \cong \langle \bar{a}, \bar{b} \bar{b}^{2n} = 1, \bar{a}^2 = \bar{b}^{2n-1}, [\bar{b}, \bar{a}] = \bar{b}^{2n-1-2} \rangle$, which is semidihedral group.

So groups in type (IV) are not isomorphic to groups in type (V). The proof is complete.
Lemma 3.7. Let $G$ be a 2-group with (**). If there exists an element $x \in Z(G) \cap G'$ such that $o(x) = 2$ and $\bar{G} = G/\langle x \rangle$ is of maximal class, Then $G$ is also a 2-group of maximal class.

Proof. $\bar{G}$ is a 2-group of maximal class. By Lemma 2.7, we see $|\bar{G} : G'| = 4$. Then $|G : G'| = 4$ and therefore $G$ is also a 2-group of maximal class. $\square$

Lemma 3.8. Let $G$ be a 2-group with (**). If there exists an element $x \in Z(G) \cap G'$ such that $o(x) = 2$ and $\bar{G} = G/\langle x \rangle$ is a Dedekind group, Then $G$ is a minimal non-abelian 2-group.

Proof. By the hypotheses, It is easy to see that $d(G) = 2$. If $\bar{G}$ is not abelian, then $G \cong \langle a, b | a^2 = 1, b^2 = a^2, [\bar{a}, \bar{b}] = \bar{a}^{-2} \rangle \times \bar{A}$, where $\bar{A}$ is elementary abelian 2-group. And $G = \langle a, b, \bar{A} \rangle$. Since $G'$ is cyclic, we see $a^4 = x \in Z(G)$. Then $1 = [b^2, b] = [a^2, b] = [a, b]^2[a, b, a] = a^{-4} = x$, a contradiction. So $\bar{G}$ is abelian and then $|G'| = 2$. Thus $G$ is minimal non-abelian 2-group by Lemma 2.9. $\square$

Lemma 3.9. Let $G$ be a 2-group with (**). If there exists an element $x \in Z(G) \cap G'$ such that $o(x) = 2$ and $\bar{G} = G/\langle x \rangle$ is a minimal non-abelian 2-group, Then $|G| \leq 2^6$ and $G$ is one of type (I–VI) groups listed in theorem 3.5.

Proof. We proceed in the two cases by Lemma 2.8.

If not, $n + m \geq 5$. If $n = 2$, then $m \geq 3$ and $[a, b^2] = 1$. There is $\langle a^2b^{2m-2} \rangle \leq \Phi(G)$ which is not normal in $G$ when $b^{2m} = 1$, which contradicts to Theorem 3.3. For $b^{2m} = x$, there exists $\langle a^2b^{2m-1} \rangle \not\unlhd G$, a contradiction. So $n \geq 3, [a, b^2] = x$ and then $m \geq 2$. If $b^{2m} = 1$, then $\langle b^2 \rangle \not\unlhd G$, a contradiction. If $b^{2m} = x$, then there exists $\langle a^2b^{2m-n+1} \rangle \not\unlhd G$ when $m > n, \langle a^2b^2 \rangle \not\unlhd G$ when $m = n$ and $\langle b^2a^{-2m-1} \rangle \not\unlhd G$ when $m < n$, which are contradictions.

If $n = 1$ and $n = 2$, then $G$ is of maximal class and $|G| = 2^4$ by Lemma 3.7.

If $m = 1$ and $n = 3$, then $1 = [a, b^2] = [a, b]^2[a^4, b] = [a, b]^2 = x$, a contradiction.

The remaining case is $n = m = 2$. If $b^4 = x$, then $a^2b^2 \in \Phi(G)$ is of order 2. But $[a^2b^2, b] = [a^2, b] = x$, which contradicts to Theorem 3.3. So $b^4 = 1$. Therefore $G$ is type (V) group when $[a, b] = a^2$ and $G$ is type (IV) group when $[a, b] = a^{-2}$.

Case(2) $G \cong \langle a, b, \bar{c} \bar{a}^{2n} = 1, \bar{b}^{2n} = 1, \bar{c}^2 = 1, [\bar{b}, \bar{a}] = \bar{c}, [\bar{c}, \bar{a}] = [\bar{c}, \bar{b}] = \bar{c}^{-2} \rangle$, where $n \geq m$. And $G = \langle a, b, c, x \rangle$. Since $G'$ is cyclic, we may assume that $c^{2n} = x$. We claim that $n + m \leq 4$.

If not, $n + m \geq 5$. Then $n \geq 3$. If $a^{2n} = 1$, then $[c, a] = x$ by $[a^2, b] = x[c, a] \in \langle a^2 \rangle$. However we see $[c, a] = 1$ from $[a^{2n-2}, c, a] = [c, a] \in \langle a^{2n-2} \rangle$, a
contradiction. Thus $a^{2^n} = x$. It follows that $[c, a] = [c, b] = 1$ from $(a^{2^n-1}c) \leq G$. If $m = 1$, then $1 = [a, b^2] = [a, b]^2 = x$, a contradiction. So $m \geq 2$. If $b^{2^m} = 1$, then there exists $\langle b^2 \rangle \not\leq G$ since $[a, b^2] = [a, b]^2 = x$, a contradiction. If $b^{2^m} = x$, then $\langle b^2 a^{2^m-1} \rangle \not\leq G$, a contradiction.

If $n = 1$, then $m = 1$. we see that $G$ is of maximal class and $|G| = 2^4$ by Lemma 3.7.

If $n = 2$ and $m = 1$, then $[c, b] = [c, a] = x$ by $1 = [a, b^2] = c^2[c, a]$. If $a^4 = 1$, then $[c, a] = x$ by $[a^2, b] = [a, b]^2[c, a] \in \langle a^2 \rangle$. Thus $G$ is type (II) group when $b^2 = 1$ and $G$ is type (III) group when $b^2 = x$. If $a^4 = x$, then $[c, a] = b$ by $[a^2c, a] = [c, a] \in \langle a^2c \rangle$. Setting $a_1 = ab, b_1 = b$, we see that $G = \langle a_1, b_1 \rangle$ is type (II) group when $b^2 = 1$ and $G$ is type (III) group when $b^2 = x$.

When $n = 2$ and $m = 2$, if $a^4 = b^4 = x$, then $a^2b^2, a^2c \in Z(G)$. And therefore $[a^2b^2, a] = 1 = [a^2b^2, b]$. Thus we see that $[c, b] = [c, a] = x$. Then $1 = [a^2c, b] = [a^2, b][c, b] = [a, b]^2 = x$, a contradiction. So we may assume that $a^4 = 1$. Since $a^2 \in Z(G)$, we see $[c, a] = x$ from $1 = [a^2, b] = x[c, a]$. If $b^4 = 1$, then $[c, b] = x$ by $b^2 \in Z(G)$. Setting $b_1 = ab, a_1 = a$, we see that $G = \langle a_1, b_1 \rangle$ is type (I) group. If $b^4 = x$, then $[c, b] = 1$ by $b^2c \in Z(G)$. Thus $G$ is also type (I) group.

The last case is $n = 3$ and $m = 1$. If $a^8 = 1$, then $[c, a] = x$ by $[a^2, b] \in \langle a^2 \rangle$. Then we see $\langle a^2 \rangle \not\leq G$ from $[a^2, a] = x$, a contradiction. If $a^8 = x$, then $[c, b] = 1$ by $\langle a^4 \rangle \leq G$. Thus $1 = [a, b^2] = [a, b]^2 = x$, a contradiction. The proof is complete. \hfill \Box

Lemma 3.10. Let $G$ be a 2-group and $G$ satisfies (**). If there exists an element $x \in Z(G) \cap G'$ such that $o(x) = 2$ and $G = G/\langle x \rangle$ is one of type (I–V) groups listed in theorem 3.5, Then $G$ is one of type (I–V) groups listed in theorem 3.5.

Proof. We split into five cases by $G$.

Case(1) $G \cong \langle a, b, c | a^4 = 1, b^{2^n} = 1, c^2 = b^4, [b, a] = c^{-1}, [c, a] = c^{-2}, [c, b] = 1 \rangle$, where $n \geq 3$. And $G = \langle a, b, c, x \rangle$. $G'$ is cyclic by Lemma 3.2. Then we may assume that $c^{2n-1} = x$. If $a^4 = x$, then the element $a^2c^{2n-2} \in \Phi(G)$ is of order 2 since $[c, a]^2 = [c, a]^2[a, c, a] = c^{-4}c^4 = 1$ and $a^4 = c^{2n-1} = x$. Noting that $[c^2, a^2, a] = [c^{2n-2}, a] = c^{2n-1} = x$, we see that $\langle a^2c^{2n-2} \rangle \not\leq G$, which contradicts to Theorem 3.3. So $a^4 = 1$. If $[c, a] = c^{2n-2} = 2$, then $[b, a]^2 = [b, a]^2[b, a, a] = c^{-2}[c, a] = x$. Thus $\langle a^2 \rangle \not\leq G$, a contradiction. Therefore $[c, a] = c^{2n-1-2}x = c^{-2}$. When $[c, b] = x$, by $[c, b^2] = [c, b]^2[c, b, b] = 1$, we see that the element $cb^{-2}$ or $cb^{2n-1-2} \in \Phi(G)$ is of order 2. And $[cb^l, b] = [c, b] = x$, where $l$ is an integer. So there exists $\langle cb^{-2} \rangle$ or $\langle cb^{2n-1-2} \rangle$ which is in $\Phi(G)$ and is not normal in $G$, which contradicts to Theorem 3.3. So $[c, b] = 1$. If $c^2 = b^4$, then it is easy to see that $G$ is type (I) group of order $2^n + 4$. If $c^2 = b^4x = b^{2n+4}$, then, setting $b_1 = b^{2n-2+1}$, we have $G = \langle a, b_1 \rangle$ is also type (I) group of order $2^n + 4$. 


Case (2) $\bar{G} \cong \langle \bar{a}, \bar{b}, \bar{c}|\bar{a}^4 = 1, \bar{c}^{2n} = 1, \bar{b}^2 = 1, [\bar{b}, \bar{a}] = \bar{c}, [\bar{c}, \bar{a}] = [\bar{c}, \bar{b}] = \bar{c}^{-2}\rangle$, where $n \geq 2$. And $G = \langle a, b, c, x \rangle$. Since $G'$ is cyclic, we may assume that $c^{2n} = x$. If $[c, b] = c^{2n-2}$, then $1 = [b^2, a] = [b, a]^2[b, a, a] = c^2[c, b] = c^{2n} = x$, a contradiction. Thus $[c, b] = c^{-2}$. If $a^4 = x$, then using $[c^{2n-1}, a] = [c, a]^{2n-1} = c^{-2n} = x$, we see $[c^{2n-1}, a^2] = 1$ and therefore $a^2c^{2n-1} \in \Phi(G)$ is of order 2. But $[a^2c^{2n-1}, a] = [c^{2n-1}, a] = x$ implies that $\langle a^2c^{2n-1} \rangle \not\subseteq G$, which contradicts to Theorem 3.3. So $a^4 = 1$. If $[c, a] = c^{2n-2}$, then $[b, a^2] = [b, a]^2[b, a, a] = x$. So $\langle a^2 \rangle \not\subseteq G$, a contradiction. Thus $[c, a] = c^{-2}$. If $b^2 = 1$, then $G = \langle a, b \rangle$ is type (II) group of order $2^{n+4}$. When $b^2 = x = c^{2n}$, it is easy to see that $G$ is type (III) group of order $2^{n+4}$.

Case (3) $\bar{G} \cong \langle \bar{a}, \bar{b}, \bar{c}|\bar{a}^4 = 1, \bar{c}^{2n} = 1, \bar{b}^2 = \bar{c}^{2n-1}, [\bar{b}, \bar{a}] = \bar{c}, [\bar{c}, \bar{a}] = [\bar{c}, \bar{b}] = \bar{c}^{-2}\rangle$, where $n \geq 2$. And $G = \langle a, b, c, x \rangle$. Since $G'$ is cyclic, we may assume that $c^{2n} = x$. By $x \in Z(G)$, we see $1 = [b^2, b] = [c^{2n-1}, b] = [c, b]^{2n-1} = c^n = x$, which is a contradiction.

Case (4) $\bar{G} \cong \langle \bar{a}, \bar{b}, \bar{c}|\bar{a}^4 = 1, \bar{b}^{2n} = 1, [\bar{b}, \bar{a}] = \bar{b}^{-2}\rangle$, where $n \geq 3$. And $G = \langle a, b, x \rangle$. Since $G'$ is cyclic, we may assume that $b^{2n} = x$. If $a^4 = x$, then $a^2b^{2n-1} \in \Phi(G)$ is of order 2 since $[b, a^2] = [b, a]^2[b, a, a] = b^{-4}b^4 = 1$. And $[a^2b^{2n-1}, a] = [b^{2n-1}, a] = [b, a]^{2n-1} = b^{2n} = x$, which implies that $\langle a^2b^{2n-1} \rangle \not\subseteq G$, a contradiction. So $a^4 = 1$. $G$ is type (V) group when $[b, a] = b^{2n-2}$ and $G$ is type (IV) group when $[b, a] = b^{-2}$.

Case (5) $\bar{G} \cong \langle \bar{a}, \bar{b}, \bar{c}|\bar{a}^4 = 1, \bar{b}^{2n} = 1, [\bar{b}, \bar{a}] = \bar{b}^{2n-2}\rangle$, where $n \geq 3$. And $G = \langle a, b, x \rangle$. Since $G'$ is cyclic, we may assume that $b^{2n} = x$. By calculation, we see that $[b, a^2] = [b, a]^2[b, a, a] = b^{2n-4}b^4 = b^{2n} = x \quad (*)$

If $a^4 = 1$, then $\langle a^2 \rangle \not\subseteq G$ by $(*)$, a contradiction. If $a^4 = x$, then the element $a^2b^{2n-1}$ is of order 2 by $(*)$. And $[b, a^2b^{2n-1}] = [b, a^2] = x$ implies that $\langle a^2b^{2n-1} \rangle \not\subseteq G$, a contradiction. The proof is complete. \[ \]

Now we give the proof of Theorem 3.5.

**Proof of Theorem 3.5**: By the Lemma 3.6, we only need to prove the necessity part.

Firstly, we prove that the theorem is true when $|G| = 2^4$. If $|G| = 2^4$ and $G$ is not Dedekind group, then we pick $x \in Z(G) \cap G'$ such that $o(x) = 2$ and consider $\bar{G} = G/\langle x \rangle$. Since $|\bar{G}| = 2^5$, $\bar{G}$ is abelian or minimal non-abelian group. By the Lemma 3.8 and 3.9, $G$ is of maximal class or $G$ is a minimal non-abelian group.

Let $G$ be a minimal counterexample. Then $|G| \geq 2^5$ and $G$ is a non-Dedekind 2-group. Pick $x \in Z(G) \cap G'$ such that $o(x) = 2$ and consider $\bar{G} = G/\langle x \rangle$. Since $G$ satisfies the condition (**), $\bar{G}$ is one of the groups listed in the theorem. By Lemma 3.7, 3.8, 3.9 and 3.10, it is easy to see that $G$ is also one of the groups listed in the theorem, a contradiction. The proof is complete.
References


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