A Remark on Certain Filtrations on the Inner Automorphism Groups of Central Division Algebras over Local Number Fields

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Abstract

Let $K$ be a finite extension of the $p$-adic number field $\mathbb{Q}_p$ and $A$ a central division algebra over $K$. We define a filtration $\{G_n\}$ of the inner automorphism group $\text{Inn}(A)$ of $A$ by the action on the valuation ring of $A$ in an analogous way to the ramification groups. We give a relation between $\{G_n\}$ and a filtration $\{\rho(U^n_A)\}$ of $\text{Inn}(A)$ which is induced by the filtration of the unit groups of $A$. We also give a modified filtration which gives the description of $\rho(U^n_A)$ by the action on $A^*$.

Mathematics Subject Classification: 16W60, 16K20, 11S15,

Keywords: division algebra, discrete valuation, inner automorphism, filtration

1 Introduction

Let $\mathbb{Z}$ be the ring of rational integers. Let $p$ be a prime number. Let $K$ be a finite extension of the $p$-adic number field $\mathbb{Q}_p$. Let $v_K$ be the normalized discrete valuation of $K$, $\mathcal{O}_K$ the valuation ring of $K$, $U_K$ the unit group of $K$ and $\kappa_K$ the residue field of $\mathcal{O}_K$. Let $f$ be the residue degree of $K/\mathbb{Q}_p$. 
Let $A$ be a central division algebra of degree $d^2$ over $K$. $A^*$ denotes the multiplicative group of $A$.

Through in this paper we assume $A \neq K$ i.e. $d \geq 2$.

For any $x \in A$, put $v_A(x) = v_K \circ Nrd_{A/K}(x)$ where $Nrd_{A/K}$ is the reduced norm of $A/K$. Then $v_A$ is the normalized discrete valuation of $A$. Let $O_A = \{ x \in A | v_A(x) \geq 0 \}$ be the valuation ring of $A$, $m_A = \{ x \in A | v_A(x) > 0 \}$ the maximal ideal of $A$ and $\kappa_A = O_A/m_A$ the residue field of $O_A$, so that $\kappa_A$ ia a finite field. Let $U_A = \{ x \in A | v_A(x) = 0 \}$ be the unit group of $O_A$ and $U^n_A = \{ x \in A | v_A(x-1) \geq n \}$ the $n$-th unit group of $A$ for each integer $n \geq 1$.

We recall that there is a commutative subfield $L$ of $A$ which is unramified and of degree $d$ over $K$ and $(v_A(A^*) : v_A(L^*)) = (v_A(A^*) : v_A(K^*)) = d$ (cf. [3] Chap. XII, [4]). Here $K^*$ (resp. $L^*$) denotes the multiplicative group of $K$ (resp. $L$). Let $O_L$ be the valuation ring of $L$, $m_L$ the maximal ideal of $O_L$ and $\kappa_L$ the residue field of $O_L$, so that $O_A = O_L + m_A$, $U_A = U_L + m_A$, $\kappa_L = \kappa_A$ and $[\kappa_L : \kappa_K] = d$.

We write $\text{Inn}(A)$ the inner automorphism group of $A$. For each $a \in A^*$ we define the surjective group homomorphism $\rho : A^* \to \text{Inn}(A)$ by $a \mapsto \rho_a$, where $\rho_a(x) = axa^{-1}$ for any $x \in A$. The kernel of $\rho$ is $K^*$. We note that $v_A(\rho_a(x)) = v_A(x)$ for any $x \in A$.

For a subset $S$ of $A^*$, write $\rho(S) = \{ \rho_a | a \in S \}$.

Moreover by [5] Chap. I § 4 Prop. 5 , there are a prime element $\pi_A$ of $A$ and a prime element $\pi_K$ of $K$ such that:
(i) $\pi_A^d = \pi_K$ ,
(ii) $\{ 1, \pi_A, \cdots , \pi_A^{d-1} \}$ is a basis of $A$ as a left $L$-space and generates $O_A$ as a left $O_L$-module,
(iii) the inner automorphism $\rho_{\pi_A} : x \mapsto \pi_A x \pi_A^{-1}$ of $A$ induces on $L$ a generator of the Galois group of $L/K$.
Put $\varphi = \rho_{\pi_A}$. Here we note that $\varphi(\pi_A) = \pi_A$.

Let $u$ be an element of $U_L$ whose residue class mod $m_L$ generates the residue extension $\kappa_L/\kappa_K$.

$\varphi^n(u) - u \in m_L$ i.e. $\varphi^n(u) - u \in m_A$ if and only if $d$ divides $n$. Moreover, since the $\varphi$ -invariants of $\kappa_L$ are $\kappa_K$, for any $b \in O_A$(resp. $b \in U_A$) we have $b \in O_K + m_A$(resp. $b \in U_K + m_A$) if $v_A(\varphi(b) - b) \geq 1$.

As $\pi_A^d = \pi_K \in K$, $\rho_{\pi_A} = id_A$. Therefore we note $\text{Inn}(A) = \rho(A^*) = \rho(O_A - \{ 0 \})$.

In the following of this paper we use the notations above.
It is well known that every automorphism of $A$ is an inner automorphism of $A$ (the Skolem-Noether theorem). For each integer $n \geq 0$, we define normal subgroups $G_n$’s of $\text{Inn}(A) = \rho(A^*)$ by the action on $O_A$ as the higher ramification groups of the local number fields:

**Definition 1.1** $G_n = \{ \rho_a | a \in O_A - \{0\}, v_A(\rho_a(x) - x) \geq n + 1 \text{ for all } x \in O_A \}$.

We also take $H_n$ as follows to be defined by the action on $A^*$ (for the case of the ramification groups, for example cf. [2] Ch. II §9 Def. 9.3) :

**Definition 1.2** $H_n = \{ \rho_a | a \in O_A - \{0\}, v_A(\rho_a(x)x^{-1} - 1) \geq n \text{ for all } x \in A^* \}$.

$H_n$’s are also normal subgroups of $\text{Inn}(A)$ (cf. [1] §1 the proof of Lemma 1).

We show a periodic relation between $\{G_n\}$ and $\{\rho(U^n_A)\}$ in th. 3.1. In fact, if $d$ divides $n$, then $G_n = \rho(U^n_A)$ and $G_{n+j} = \rho(U^{n+j+1}_A)$ for $j = 1, 2, \cdots, d-1$.

In prop. 3.2 we also remark that $H_n = \rho(U^n_A)$ for $n \geq 1$. This means that $H_n$ gives the description of $\rho(U^n_A)$ by the action on $A^*$. Lastly an example follows.

## 2 Preliminaries

The following criterion holds in the same way as [1] and [3].

**Criterion.** (i) (cf. [3] Chap. IV §1 Lemma 1) Let $a \in O_A - \{0\}$. Then $\rho_a \in G_n$ if and only if $v_A(\rho_a(u) - u) \geq n + 1$ and $v_A(\rho_a(\pi_A) - \pi_A) \geq n + 1$.

(ii) (cf. [1] §1 the proof of Lemma 1) Let $a \in O_A - \{0\}$. Then $\rho_a \in H_n$ if and only if $v_A(\rho_a(u)u^{-1} - 1) = v_A(\rho_a(u) - u) \geq n$ and $v_A(\rho_a(\pi_A)\pi_A^{-1} - 1) \geq n \text{ i.e. } v_A(\rho_a(\pi_A) - \pi_A) \geq n + 1$.

We remark that $G_n \subset H_n \subset G_{n-1} \subset H_{n-1}$ for each integer $n \geq 1$.

In this paper we will use the following elementary lemma.
Lemma 2.1 (i) a) For \( a \in O_A - \{0\} \), we have \( v_A(\rho_a(\pi_A) - \pi_A) = v_A(\varphi(a) - a) + 1 - v_A(a) \geq 1 \).

b) For \( a \in U_A \), we have \( v_A(\rho_a(\pi_A) - \pi_A) \geq 1 \).

Moreover, \( v_A(\rho_a(\pi_A) - \pi_A) \geq 2 \) if and only if \( a \in U_K + m_A \).

c) For \( a = 1 + bn_A(b \in O_A, n \geq 1) \), we have \( v_A(\rho_a(\pi_A) - \pi_A) = v_A(\varphi(b) - b) + n + 1 \geq n + 1 \).

Moreover, \( v_A(\rho_a(\pi_A) - \pi_A) \geq n + 2 \) if and only if \( b \in O_K + m_A \).

(ii) a) For \( a \in O_A - \{0\} \), we have \( v_A(\rho_a(u) - u) = v_A(au - ua) - v_A(a) \geq 0 \).

Moreover, \( v_A(\rho_a(u) - u) \geq 1 \) if and only if \( a \in (O_K - \{0\})U_A \).

b) For \( a \in U_A \), we have \( v_A(\rho_a(u) - u) \geq 1 \).

c) For \( a = 1 + b\pi_A^n(b \in O_A, n \geq 1) \), we have \( v_A(\rho_a(u) - u) = v_A(b\varphi^n(u) - ub) + n \geq n \).

If \( d \) does not divide \( n \) and \( b \in U_A \), then we have \( v_A(\rho_a(u) - u) = n \).

On the other hand, if either \( d \) divides \( n \) or \( b \in m_A \), then we have \( v_A(\rho_a(u) - u) \geq n + 1 \).

Proof. (i) a) \( v_A(\rho_a(\pi_A) - \pi_A) = v_A(\pi_Aa - a\pi_A) - v_A(a) = v_A(\varphi(1 + b\pi_A^n) - (1 + b\pi_A^n)) + v_A(\pi_A) \geq 1 \).

b) For \( a \in U_A \), we have \( v_A(\rho_a(\pi_A) - \pi_A) = v_A(\varphi(a) - a) + 1 \geq 1 \) from a) above. Moreover, \( v_A(\rho_a(\pi_A) - \pi_A) \geq 2 \) if and only if \( v_A(\varphi(a) - a) \geq 1 \) which is equivalent to \( a \in U_K + m_A \).

c) For \( a = 1 + b\pi_A^n(b \in O_A, n \geq 1) \), we have \( v_A(\rho_a(\pi_A) - \pi_A) = v_A((1 + b\pi_A^n)) + v_A(\pi_A) \geq n + 2 \) if and only if \( v_A(\varphi(b) - b) \geq 1 \) i.e. \( b \in O_K + m_A \).

(ii) a) \( v_A(\rho_a(u) - u) = v_A(aua^{-1} - u) = v_A(au - ua) - v_A(a) \geq 0 \).

Let \( a = b\pi_A^n \) with \( b \in U_A, 0 \leq n \in \mathbb{Z} \). Then we have \( v_A(\rho_a(u) - u) = v_A(b\pi_A^n) = v_A(b\pi_A^n u - ub\pi_A^n) \).

Therefore, \( v_A(\rho_a(u) - u) \geq 1 \) if and only if \( v_A(b\varphi^n(u) - ub) \geq 1 \). By reducing mod \( m_A \) and noting that \( \kappa_A = \kappa_L \) is commutative, \( v_A(\rho_a(u) - u) \geq 1 \) if and only if \( d \) divides \( n \) i.e. \( a \in (O_K - \{0\})U_A \).

b) This follows from a) above.

c) For \( a = 1 + b\pi_A^n(b \in O_A, n \geq 1) \), we have \( v_A(\rho_a(u) - u) = v_A((1 + b\pi_A^n)u - u) = v_A(b\pi_A^n u - ub\pi_A^n) \).

If \( d \) does not divide \( n \), then \( v_A(\varphi^n(u) - u) = 0 \). Therefore, for \( b \in U_A \), we have \( v_A(\rho_a(u) - u) = n \).

On the other hand, if \( d \) divides \( n \), then \( v_A(\varphi^n(u) - u) \geq 1 \). Therefore by reducing mod \( m_A \) as above, we have \( v_A(b\varphi^n(u) - ub) \geq 1 \), so that \( v_A(\rho_a(u) - u) \geq n + 1 \). Also if \( b \in m_A \), then \( v_A(\rho_a(u) - u) \geq n + 1 \).

From now on we consider relations of the filtrations \( \{G_n\}, \{H_n\} \) and \( \{\rho(U^n_A)\} \).
of $\text{Inn}(A)$.

**Proposition 2.2** $G_0 = \rho(U_A)$.

Proof. From lemma 2.1, we have $\rho(U_A) \subset G_0$. On the other hand, let $a \in O_A - \{0\}$ such that $\rho_a \in G_0$. Then $v_A(\rho_a(u) - u) \geq 1$. Therefore $a \in (O_K - \{0\})U_A$ by lemma 2.1(ii) and then $\rho_a \in \rho(U_A)$ and $G_0 \subset \rho(U_A)$. Hence $G_0 = \rho(U_A)$.

**Proposition 2.3** $H_1 = \rho(U^1_A)$ and $H_0 = \rho(A^*)$.

Proof. We have $\rho(U^1_A) \subset H_1$ by lemma 2.1. On the other hand, let $a \in O_A - \{0\}$ such that $\rho_a \in H_1$. By lemma 2.1(ii), $a \in (O_K - \{0\})U_A$, so that we can write $a = b\pi_K^n$ with $b \in U_A$, $0 \leq n \in \mathbb{Z}$. As $\rho_a = \rho_b\rho_{\pi_K^n} = \rho_b$, we may assume $a = b \in U_A$. As $\rho_a \in H_1$, $v_A(\rho_a(\pi_A) - \pi_A) = v_A(\rho_a(\pi_A)\pi_A^{-1} - 1) + v_A(\pi_A) \geq 2$. By lemma 2.1(i) $a \in U_K + m_A$, so that we can write $a = w + g\pi_A$ with $w \in U_K$ and $g \in O_A$. Put $a' = 1 + w^{-1}g\pi_A \in U^1_A$. As $a = wa'$, we have $\rho_a = \rho_{wa'} = \rho_w\rho_a' = \rho_a'$. Thus $H_1 \subset \rho(U^1_A)$. Hence $H_1 = \rho(U^1_A)$.

Finally, lemma 2.1 shows $\rho(A^*) = \rho(O_A - \{0\}) = H_0$.

**Proposition 2.4** For each integer $n \geq 1$, $\rho(U^n_A)$ is a subgroup of $G_{n-1}$ and $H_n$.

Moreover $\rho(U^n_A)$ is a subgroup of $G_n$ if and only if $d$ divides $n$.

Particularly $G_n \subseteq G_{n-1}$ if $d$ does not divide $n$.

Proof. Let $n \geq 1$ be an integer and $a = 1 + b\pi_A^n, (b \in O_A)$.
We have $v_A(\rho_a(\pi_A) - \pi_A) = v_A(\rho_a(\pi_A)\pi_A^{-1} - 1) + 1 \geq n + 1$ and $v_A(\rho_a(u) - u) = v_A(\rho_a(u)u^{-1} - 1) \geq n$ by lemma 2.1. Hence $\rho_a \in G_{n-1}$.

Suppose that $\rho_a \in G_n$, so that $v_A(\rho_a(u) - u) \geq n + 1$.
By lemma 2.1(ii), this means that either $d$ divides $n$ or $a \in U_A^{n+1}$. Therefore $\rho(U^n_A) \not\subset G_n$ if $d$ does not divide $n$.

Now on the other hand we assume that $d$ divides $n$. From lemma 2.1(ii) we have $v_A(\rho_a(u) - u) \geq n + 1$.
We also have $v_A(\rho_a(\pi_A) - \pi_A) \geq n + 1$ by lemma 2.1(i).
Therefore we get $\rho_a \in G_n$ and $\rho(U^n_A) \subset G_n$.
Hence $\rho(U^n_A) \subset G_n$ if and only if $d$ divides $n$. 

Proposition 2.5 Suppose $d$ does not divide $n$, then $\rho(U_A^n) \cap G_n = \rho(U_A^{n+1})$.

Proof. Let $a = 1 + b \pi_A^n (b \in O_A)$. Suppose $\rho_a \in G_n$. Then by lemma 2.1(ii),
$v_A((b \varphi^n(u) - ub)\pi_A^n) \geq n + 1$. As $v_A(\varphi^n(u) - u) = 0$, $v_A(b) \geq 1$ and $a \in U_A^{n+1}$.
Thus we have $\rho(U_A^n) \cap G_n \subset \rho(U_A^{n+1})$.
On the other hand we have $\rho(U_A^n) \cap G_n \supset \rho(U_A^{n+1})$ by prop. 2.4.
Hence we get $\rho(U_A^n) \cap G_n = \rho(U_A^{n+1})$.

Proposition 2.6 Suppose $d$ divides $n$, then $\rho(U_A) \cap G_j = \rho(U_A^{j+1})$ for $j = 1, 2, \cdots, d - 1$, therefore $G_j = \rho(U_A^{j+1})$ for $j = 1, 2, \cdots, d - 1$.

Proof. We first prove the assertion for the case of $j = 1$.
We take $a \in U_A$. Suppose $\rho_a \in G_1$. Then $v_A(\rho_a(\pi_A) - \pi_A) \geq 2$.
By lemma 2.1(i) we have $a \in U_K + m_A$. In the same way as in the proof of prop.
2.3 we can show that there are some $w \in U_K$ and $a' \in U_A^j$ such that $a = wa'$.
Therefore $\rho_a = \rho_a'$. Hence by prop. 2.5, we have $\rho_a \in \rho(U_A) \cap G_1 = \rho(U_A^2)$ and
$\rho(U_A) \cap G_1 \subset \rho(U_A^2)$. On the other hand, $\rho(U_A) \cap G_1 \supset \rho(U_A^2)$ by prop. 2.4. Thus we get $\rho(U_A) \cap G_1 = \rho(U_A^2)$.
Suppose $d \geq 3$, then $\rho(U_A) \cap G_2 = \rho(U_A) \cap (G_1 \cap G_2) = (\rho(U_A) \cap G_1) \cap G_2$
$= \rho(U_A^2) \cap G_2$. As $d$ does not divide 2, by prop. 2.5, $\rho(U_A^2) \cap G_2 = \rho(U_A^3)$. Thus if $d \geq 3$, $\rho(U_A) \cap G_2 = \rho(U_A^3)$
By repeating the same argument, we get $\rho(U_A) \cap G_j = \rho(U_A^{j+1})$ for $j = 1, 2, \cdots, d - 1$. As $\rho(U_A) = G_0 \supset G_j$ by prop. 2.2, the last statement follows from above.

Proposition 2.7 Let $n$ be an integer with $n \geq 1$. Suppose $d$ divides $n$, then
$\rho(U_A^n) \cap G_{n+j} = \rho(U_A^{n+j+1})$ for $j = 1, 2, \cdots, d - 1$.

Proof. We first prove the assertion for the case $j = 1$.
Let $a = 1 + b \pi_A^n (b \in O_A)$. Suppose that $\rho_a \in G_{n+1}$. Then we have
$v_A(\rho_a(\pi_A) - \pi_A) \geq n + 2$. By lemma 2.1(i), this implies that $b \in O_K + m_A$.
Therefore we can write $b = c + g \pi_A$ with $c \in O_K$ and $g \in O_A$, so that
$a = 1 + c \pi_A^n + g \pi_A^{n+1}$. Write $n = dr$ with $1 \leq r \in \mathbb{Z}$. $1 + c \pi_A^n = 1 + c \pi_A$ has
the inverse in $U_K^n$. Calling $w = 1 + c \pi_A^n \in U_K$ and $a' = 1 + w^{-1}g \pi_A^{n+1} \in U_A^{n+1}$,
we have $a = wa'$. Consequently $\rho_a = \rho_w \rho_a' = \rho_a' \in \rho(U_A^{n+1})$.
By assumption, we have $\rho_a \in G_{n+1} \cap \rho(U_A^{n+1})$. Since $d$ does not divide $n + 1$,
$\rho_a \in \rho(U_A^{n+2}) = \rho(U_A^{n+1}) \cap G_{n+1}$ by prop. 2.5.
Hence \( \rho(U_A^n) \cap G_{n+1} \subset \rho(U_A^{n+2}) \).

On the other hand we have \( \rho(U_A^n) \cap G_{n+1} \supset \rho(U_A^{n+2}) \) by prop. 2.4.

Therefore we have shown \( \rho(U_A^n) \cap G_{n+1} = \rho(U_A^{n+2}) \).

Now suppose \( d \geq 3 \) and we will show the assertion for the case \( j = 2 \) similarly as the proof of prop. 2.6. In fact, by assumption, we have \( \rho(U_{A}^{n}) \cap G_{n+2} = \rho(U_{A}^{n}) \cap (G_{n+1} \cap G_{n+2}) = (\rho(U_{A}^{n}) \cap G_{n+1}) \cap G_{n+2} = \rho(U_{A}^{n+2}) \cap G_{n+2} \). As \( d \) does not divide \( n+2 \), \( \rho(U_{A}^{n+2}) \cap G_{n+2} = \rho(U_{A}^{n+3}) \) by prop. 2.5. This implies \( \rho(U_A^n) \cap G_{n+2} = \rho(U_A^{n+3}) \) if \( d \geq 3 \).

Finally, by repeating the same argument, we get our conclusion that \( \rho(U_A^n) \cap G_{n+j} = \rho(U_A^{n+j+1}) \) for \( j = 1, 2, \cdots, d-1 \).

As the kernel of \( \rho \) is \( K^* \), we have the isomorphisms

\[
\rho(U_A) \simeq U_A / U_A \cap K^*, \rho(U_A^n) \simeq U_A^n / U_A^n \cap K^* etc.
\]

We also have the isomorphism \( A^* / U_A \simeq v_A(A^*) = Z \) induced by the valuation \( v_A \). Accordingly lemma 2.8 below follows applying the group isomorphism theorem.

**Lemma 2.8** (i) \( \rho(A^*) / \rho(U_A) \simeq Z / dZ \).

(ii) \( \rho(U_A) / \rho(U_A^n) \simeq \kappa_A^* / \kappa_K^* \).

(iii) For each integer \( n \geq 1 \), we have

\[
\rho(U_A^n) / \rho(U_A^{n+1}) \simeq \begin{cases} 
\kappa_{A+}, \text{ (} d \text{ does not divide } n \text{)} \\
\kappa_{A+} / \kappa_{K+}, \text{ (} d \text{ divides } n \text{)} 
\end{cases}
\]

Here we mean by \( \kappa_{A+} \) (resp. \( \kappa_{K+} \)) the additive groups \( \kappa_A \) (resp. \( \kappa_K \)).

3 Main Results

**Theorem 3.1** The \( G_n \)'s form a descending sequence \( \{ G_n \} \) of normal subgroups of \( Int(A) \) such that:

\[
G_0(= \rho(U_A)) \supseteq G_1(= \rho(U_A^2)) \supseteq G_2(= \rho(U_A^3)) \supseteq \cdots \supseteq G_{d-1} = G_d(= \rho(U_A^d)) \supseteq G_{d+1}(= \rho(U_A^{d+2})) \supseteq G_{d+2}(= \rho(U_A^{d+3})) \supseteq \cdots \supseteq G_{2d-1} = G_{2d}(= \rho(U_A^{2d})) \supseteq G_{rd+1}(= \rho(U_A^{rd+2})) \supseteq \cdots \supseteq G_{rd+1}(= \rho(U_A^{rd+j+1})) \supseteq \cdots \supseteq G_{rd+(d-1)} = G_{(r+1)d}(= \rho(U_A^{(r+1)d+1})) \supseteq G_{(r+1)d+1}(= \rho(U_A^{(r+1)d+2})) \supseteq \cdots .
\]

Moreover, for each integer \( r \geq 0 \), we have :

\[
\left\{ 
\begin{array}{ll}
(G_0 : G_1) &= p^{d_{rl}} p^{a-1} \\
(G_{rd+j} : G_{rd+j+1}) &= p^{d_{rl}} \quad (j = 1, 2, \cdots, d-2), \\
(G_{rd+(d-1)} : G_{(r+1)d}) &= p^{d_{rl}}, \\
(G_{(r+1)d} : G_{(r+1)d+1}) &= p^{d_{rl}}(2d-1),
\end{array}
\right.
\]
Proof. By props. 2.2, 2.4 and 2.6, we have $G_0 = \rho(U_A) \supsetneq G_1 = \rho(U_A^2) \supsetneq G_2 = \rho(U_A^3) \supsetneq \cdots \supsetneq G_j = \rho(U_A^{j+1}) \supsetneq \cdots \supsetneq G_{d-1} = \rho(U_A^d)$.

By props. 2.4 and 2.6, we have $G_{d-1} = \rho(U_A^d) \subsetneq G_d \subsetneq G_{d-1}$, which implies $G_{d-1} = G_d = \rho(U_A^d)$.

Substituting $d$ for $n$ in prop. 2.7, we have $\rho(U_{A}^{d+j+1}) = \rho(U_{A}^{d}) \cap G_{d+j} = G_d \cap G_{d+j} = G_{d+j}$ for $j = 1, 2, \cdots, d - 1$.

Similarly as above we have $G_{d+1} = \rho(U_{A}^{d+2}) \supsetneq G_{d+2} = \rho(U_{A}^{d+3}) \supsetneq \cdots \supsetneq G_{2d-1} = \rho(U_{A}^{2d})$. Also we have $G_{2d-1} = \rho(U_{A}^{2d}) \subsetneq G_{2d} \subsetneq G_{2d-1}$, so that $G_{2d-1} = G_{2d} = \rho(U_{A}^{2d})$.

By repeating the same argument, we get the required properties of the descending sequence $\{G_n\}$.

Finally, the assertion for group indexes follow from lemma 2.8.

Here we note $(G_0 : G_1) = (\rho(U_A) : \rho(U_A^1) ) \cdot (\rho(U_A^1) : \rho(U_A^2) )$ and $(G_{(r+1)d} : G_{(r+1)d+1}) = (\rho(U_A^{(r+1)d}) : \rho(U_A^{(r+1)d+1}) \cdot (\rho(U_A^{(r+1)d+1}) : \rho(U_A^{(r+1)d+2}))$.

\textbf{Proposition 3.2} \textit{Let $n$ be an integer with $n \geq 1$. Then $H_n = \rho(U_{A}^n)$. Moreover, for each integer $n \geq 1$, we have:}

$(H_0 : H_1) = d\frac{d^d - 1}{d - 1}$,

$(H_n : H_{n+1}) = p^{f_d}$ if $d$ does not divide $n$,

$(H_n : H_{n+1}) = p^{f(d-1)}$ if $d$ divides $n$.

Proof. We first prove the assertion that, for each integer $n \geq 1$, $\rho(U_{A}^n) \cap H_{n+1} = \rho(U_{A}^{n+1})$.

If $d$ does not divide $n$, we have $\rho(U_{A}^n) \cap H_{n+1} \subset \rho(U_{A}^{n}) \cap G_n = \rho(U_{A}^{n+1})$ by prop. 2.5. Now we assume that $d$ divides $n$. Let $a = 1 + b\pi_{n}(b \in O_{A})$. Suppose that $\rho_{a} \in H_{n+1}$. Then $v_{A}(\rho_{a}(\pi_{A}) - \pi_{A}) \geq n + 2$. Therefore $b \in O_{A} + m_{A}$ by lemma 2.1(i). As in the proof of prop. 2.7, we have $\rho_{a} \in \rho(U_{A}^{n+1})$, so that $\rho(U_{A}^n) \cap H_{n+1} \subset \rho(U_{A}^{n+1})$.

On the other hand we have, for each integer $n \geq 1$, $\rho(U_{A}^n) \cap H_{n+1} \supset \rho(U_{A}^{n+1})$ by prop. 2.4. Thus we have $\rho(U_{A}^n) \cap H_{n+1} = \rho(U_{A}^{n+1})$.

From now on we prove the proposition.

From prop. 2.3, we have $\rho(U_{A}^1) = H_1$ i.e. the assertion for $n = 1$.

Suppose that $H_n = \rho(U_{A}^n)$. By the assertion above we have $\rho(U_{A}^{n+1}) = \rho(U_{A}^{n}) \cap H_{n+1} = H_n \cap H_{n+1} = H_{n+1}$, and therefore we have $H_{n+1} = \rho(U_{A}^{n+1})$.

By induction we get our conclusion.

Finally, the assertion for group indexes follows from lemma 2.8.
Example 3.3 For $K = \mathbb{Q}_p$ and $d = 2$, we have:

for each integer $n \geq 1$, we have $(G_0 : G_1) = p^2(p+1), (G_{2n-1} : G_{2n}) = 1, (G_{2n} : G_{2n+1}) = p^3, (H_0 : H_1) = 2(p+1), (H_{2n-1} : H_{2n}) = p^2$ and $(H_{2n} : H_{2n+1}) = p$.

\[
\begin{array}{cccc}
G_0 & \supsetneq & G_1 & = \quad G_2 & \supsetneq & G_3 & = \quad G_4 & \supsetneq & G_5 \\
\rho(A^*) & \supsetneq & \rho(U_{A}) & \supsetneq & \rho(U_{A}^1) & \supsetneq & \rho(U_{A}^2) & \supsetneq & \rho(U_{A}^3) & \supsetneq & \rho(U_{A}^4) & \supsetneq & \rho(U_{A}^5) \\
H_0 & \supsetneq & H_1 & \supsetneq & H_2 & \supsetneq & H_3 & \supsetneq & H_4 & \supsetneq & H_5
\end{array}
\]

References


Received: January 28, 2016; Published: March 18, 2016