Closed Cofinitely Weak Supplemented Modules

Burcu Nişancı Türkmen

Amasya University, Faculty of Art and Science
Ipekköy, Amasya, Turkey

Yıldız Aydın

Ondokuz Mayıs University, Faculty of Art and Science
Kurupelit, Samsun, Turkey

Ali Pancar

Ondokuz Mayıs University, Faculty of Art and Science
Kurupelit, Samsun, Turkey

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Abstract

We say that a module $M$ is a closed cofinitely weak supplemented module or briefly ccws-module if, for every closed cofinite submodule $N$ of $M$, $N$ has a weak supplement in $M$. In this article, the various properties of ccws-modules are given as a generalization of cofinitely weak supplemented modules. In particular, we prove that a left V-ring $R$ is ccws if and only if $R$ is an extending ring. Finally, we show that the notion of cofinitely weak supplemented modules and the notion of ccws-modules are equivalent under some special conditions.

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1 Introduction

Throughout this paper, it is assumed that $R$ is an associative ring with identity and all modules are unital left $R$-modules. Let $R$ be such a ring and let $M$ be an $R$-module. The notation $K \leq M$ ($K < M$) means that $K$ is a (proper) submodule of $M$. A submodule $N$ of $M$ is called cofinite in $M$ if the factor module $M/N$ is finitely generated. A module $M$ is called extending if every submodule is essential in a direct summand of $M$ (see [3]). Here a submodule $K \leq M$ is said to be essential in $M$, denoted as $K \lhd M$, if $K \cap N \neq 0$ for every non-zero submodule $N \leq M$. A closed submodule $N$ of $M$, denoted by $N \leq c M$, is a submodule which has no proper essential extension in $M$. Every direct summand of a module $M$ is a closed submodule of $M$. If $L \leq c N$ and $N \leq c M$, then $L \leq c M$ by [5, Proposition 1.5]. If $N$ is closed and cofinite submodule of $M$, we denote as $N \leq cc M$. As a dual notion of essential submodule, a submodule $S$ of $M$ is called small (in $M$), denoted as $S \ll M$, if $M \neq S + L$ for every proper submodule $L$ of $M$ ([12, 19.1]). The Jacobson radical of $M$ will be denoted by $\text{Rad}(M)$. It is known that $\text{Rad}(M)$ is the sum of all small submodules of $M$ ([12, 21.5]).

A non-zero module $M$ is said to be hollow if every proper submodule of $M$ is small in $M$, and it is said to be local if it is hollow and is finitely generated. A module $M$ is local if and only if it is finitely generated and $\text{Rad}(M)$ is maximal (see [3, 2.12 §2.15]). A ring $R$ is said to be local if $J$ is maximal, where $J$ is the Jacobson radical of $R$.

An $R$-module $M$ is called supplemented if every submodule of $M$ has a supplement in $M$. Here a submodule $K \leq M$ is said to be a supplement of $N$ in $M$ if $K$ is minimal with respect to $N + K = M$, or equivalently, if $N + K = M$ and $N \cap K \ll K$ ([12, page 349]). Every direct summand of a module $M$ is a supplement submodule of $M$, and supplemented modules are a proper generalization of semisimple modules. In addition, every factor module of a supplemented module is again supplemented. As a generalization of supplemented modules, a module $M$ is called weakly supplemented if any submodule $N$ of $M$ has a weak supplement $K$, i.e. there exists a submodule $K$ of $M$ such that $M = N + K$ and $N \cap K \ll M$ as in [7]. Alizade et al. [1] have defined cofinitely supplemented modules as a proper generalization of supplemented modules. They call a module $M$ cofinitely supplemented if every cofinite submodule $N$ of $M$ has a supplement in $M$, and give characterizations of these modules over any rings and commutative domains (see [1]). In particular, it is shown in [1, Theorem 2.8] that a module $M$ is cofinitely supplemented if and only if every maximal submodule of $M$ has a supplement. A module $M$ is called cofinitely weak supplemented if every cofinite submodule has a weak supplement in $M$ ([2]).

A module $M$ is called lifting (or $D_1$-module) if, for every submodule $N$ of
$M$, there exists a direct summand $K$ of $M$ such that $K \leq N$ and $\frac{N}{K} \ll \frac{M}{K}$ [3, 22.2]. Mohamed and Miiller has generalized the concept of lifting modules to $\oplus$-supplemented modules. $M$ is called $\oplus$-supplemented if every submodule $N$ of $M$ has a supplement that is a direct summand of $M$ ([8]). Clearly every $\oplus$-supplemented module is supplemented, but a supplemented module need not be $\oplus$-supplemented in general (see [8, Lemma A.4 (2)]). It is shown in [8, Proposition A.7 and Proposition A.8] that if $R$ is a dedekind domain, every supplemented $R$-module is $\oplus$-supplemented. Hollow modules are $\oplus$-supplemented.

In [4], Çalışıcı and Pancar call a module $M$ $\oplus$-cofinitely supplemented if every cofinite submodule of $M$ has a supplement that is a direct summand of $M$. They gave in the same paper some properties of these modules. In addition, it is proven in [4, Theorem 2.9] that a ring $R$ is semiperfect (that is, $_RR$ is supplemented) if and only if every free left $R$-module is $\oplus$-cofinitely supplemented.

In this paper, we introduce the notion of closed cofinitely weak supplemented modules, or briefly ccws-modules. We provide some properties of these modules. An example is given to separate ccws-modules and (cofinitely) weak supplemented modules. In general, a finite direct sum of ccws-modules need not be a ccws-module (see Example 2.13). We study on the finite sums of ccws-modules. We prove that every cofinite direct summand of a ccws-module is a ccws-module. We obtain a new characterization of left V-rings by using ccws-modules. We prove that a cofinitely refinable module $M$ is $\oplus$-cofinitely supplemented if and only if $M$ is a ccws-module under some special conditions.

## 2 Ccws-Modules

Recall from [9] that a module $M$ is called closed weak supplemented if every closed submodule of $M$ has a weak supplement in $M$. In this section, we define the concept of ccws-modules as a generalization of cofinitely weak supplemented modules, and give various properties of them.

**Definition 2.1.** Let $M$ be a module. Then $M$ is called a closed cofinitely weak supplemented module or briefly a ccws-module if, for every $N \leq_{cc} M$, there exists a submodule $K$ of $M$ such that $M = K + N$ and $K \cap N \ll M$.

Under given definitions, we clearly have the following implications on modules:
It follows from [3, 1.16] that a module $M$ is extending if and only if every closed submodule is a direct summand of $M$. Applying this fact, we obtain that every extending module is a ccws-module.

**Example 2.2.** Let $\mathbb{Z}$ be the ring of all integers. Since $\mathbb{Z}$ is extending, it is a ccws-module. But $\mathbb{Z}$ is not (cofinitely) weak supplemented since, for any $n \geq 2$, $n\mathbb{Z}$ has no weak supplement in $\mathbb{Z}$.

**Proposition 2.3.** Let $M$ be a ccws-module. Then, every cofinite direct summand of $M$ is a ccws-module.

**Proof.** Let $N$ be any cofinite direct summand of $M$ and $K \leq_{cc} N$. Since $N \leq_{c} M$, we obtain that $K \leq_{c} M$. As $\frac{M}{K} \cong \frac{N}{N}$, we obtain that $K \leq_{cc} M$. By the hypothesis, there exists a submodule $L$ of $M$ such that $M = K + L$ and $K \cap L \ll M$. Applying the modular law, $N = N \cap (K + L) = K + (N \cap L)$.

Therefore $K \cap (N \cap L) = K \cap L \ll M$. Since $N$ is a direct summand of $M$ and $K \cap L \subseteq N$, we have $K \cap (N \cap L) \ll N$ by [12, 19. (3.5)]. Thus $N$ is a ccws-module.

In general, a finite direct sum of ccws-modules need not be a ccws-module. Later, we shall give an example of such a module.

**Lemma 2.4.** (See [2, Lemma 2.11]) Let $N$ and $L$ be cofinite submodules of $M$ such that $N + L$ has a weak supplement $H$ in $M$ and $N \cap (H + L)$ has a weak supplement $G$ in $N$. Then $H + G$ is a weak supplement of $L$ in $M$.

**Proposition 2.5.** For ccws-modules $M_1$ and $M_2$, let $M = M_1 \oplus M_2$. Suppose that $M_i \cap (M_j + L) \leq_{cc} M_i$ and $M_j \cap (L + K) \leq_{cc} M_j$, where $K$ is a weak supplement of $M_i \cap (M_j + L)$ in $M_i$, $i \neq j$, for any $L \leq_{c} M$. Then $M$ is a ccws-module.

**Proof.** Let $L \leq_{cc} M$, then $M = M_1 + M_2 + L$ has a weak supplement 0 in $M$.

Since $M_i \cap (M_j + L) \leq_{cc} M_i$ and $M_i$ is a ccws-module, then there exists a submodule $K$ of $M_j$ such that $M_1 = M_1 \cap (M_j + L) + K$ and $M_1 \cap (M_2 + L) \cap K = K \cap (M_2 + L) \ll M_1$. By Lemma 2.4, $K$ is a weak supplement of $M_2 + L$ in $M$. It follows from the assumption that $M_2 \cap (K + L)$ has a weak supplement of $L$ in $M$. Hence $M$ is a ccws-module.
Proposition 2.6. Let $M = M_1 + M_2$, where $M_1$ is a ccws-module. Suppose that for any $N \leq_{cc} M$, $N \cap M_1 \leq_{cc} M_1$. Then $M$ is a ccws-module if and only if every $N \leq_{cc} M$ with $M_2$ not contained in $N$ has a weak supplement.

Proof. ($\implies$) It is clear.

($\impliedby$) Let $N \leq_{cc} M$ with $M_2 \leq N$. Then $M = M_1 + M_2 = M_1 + N$ and $M_1 + N$ has a weak supplement 0 in $M$. Since $N \cap M_1 \leq_{cc} M_1$ and $M_1$ is a ccws-module, $N \cap M_1$ has a weak supplement $H$ in $M_1$. By Lemma 2.4, $H$ is a weak supplement of $N$ in $M$. By the hypothesis, $M$ is a ccws-module.

Recall from [6, page 185] that a left $R$-module $M$ is said to be singular (respectively, non-singular) if $Z(M) = M$ (respectively, $Z(M) = 0$), where $Z(M) = \{ m \in M \mid \text{Ann}(m) \trianglerighteq R \}$.

Corollary 2.7. Let $M = M_1 + M_2$ be a non-singular module with $M_1$ ccws and $M_2$ any $R$-module. Then $M$ is a ccws-module if and only if every $N \leq_{cc} M$ with $M_2$ not contained in $N$ has a weak supplement.

Proof. As a non-singular module $M$, $N \cap L \leq_{cc} L$ for any submodule $L$ of $M$ and $N \leq_{cc} M$. Thus by Proposition 2.6 the proof is clear.

Corollary 2.8. Let $M$ be a non-singular ccws-module. Then $M$ is a cofinitely weak supplemented module.

Proposition 2.9. Let $M$ be a module and $X$ be a submodule of $M$ with $M/X$ is non-singular. Then $M/X$ is a ccws-module.

Proof. Let $N \leq_{cc} M$. Consider the natural homomorphism $\pi : M \longrightarrow M/X$. Since $M/X$ is non-singular, $N \leq_{cc} M$. As $M/X \cong M/N$, $N \leq_{cc} M$. By the hypothesis, there exists a submodule $K$ of $M$ such that $M = N + K$, $N \cap K \ll M$. It follows that $M/X = N/X + K/X$ and $N/X \cap K/X \ll M/X$. So $M/X$ is a ccws-module.

Proposition 2.10. Let $M$ be a module with $\text{Rad}(M) = 0$. Then, $M$ is a ccws-module if and only if every cofinite closed submodule is a direct summand of $M$.

Proof. ($\implies$) Let $N \leq_{cc} M$. By the hypothesis, there exists a submodule $K$ of $M$ such that $M = N + K$ and $N \cap K \ll M$. So $N \cap K \subseteq \text{Rad}(M)$. Since $\text{Rad}(M) = 0$, $M = N \oplus K$. Therefore $N$ is a direct summand of $M$.

($\impliedby$) The converse is clear.

Using Proposition 2.9 and [3, 1.16], we obtain the following fact.

Corollary 2.11. Let $M$ be a finitely generated $R$-module with $\text{Rad}(M) = 0$. Then the following are equivalent.
1. $M$ is a ccws-module.

2. $M$ is extending.

Recall [12, page 192] that a ring $R$ is called a left $V$-ring if every simple left $R$-module is injective.

**Corollary 2.12.** Let $R$ be a left $V$-ring. Then, $R^2$ is a ccws-module if and only if $R$ is an extending ring.

Any finite of ccws-modules need not to be a ccws-module. The following Example shows this.

**Example 2.13.** Let $R = \mathbb{Z}[x]$, where $\mathbb{Z}$ is the ring of all integers. It can be seen that the left $R$-module $R$ is a ccws-module and $M = R \oplus R$ is an extending $R$-module. As $\text{Rad}(M) = 0$, by Corollary 2.11, $M$ is not a ccws-module.

Recall from [11] that a module $M$ is called cofinitely refinable if, for any cofinite submodule $U$ of $M$ and any submodule $V$ of $M$ with $M = U + V$, there exists a direct summand $U'$ of $M$ with $U' \leq U$ and $M = U' + V$.

**Theorem 2.14.** Let $M$ be a cofinitely refinable module. Suppose that for any cofinite submodule $N$ of $M$, there exists a submodule $L$ of $M$ such that $L \leq_{c} M$ and $N = L + T$ or $L = N + T$ for some $T \ll M$. Then the following statements are equivalent.

1. $M$ is a $\oplus$-cofinitely supplemented module;
2. $M$ is a cofinitely supplemented module;
3. $M$ is a cofinitely weak supplemented module;
4. $M$ is a ccws-module.

**Proof.** $(1) \Rightarrow (2)$, $(2) \Rightarrow (3)$ and $(3) \Rightarrow (4)$ is clear.

$(4) \Rightarrow (1)$ Let $N$ be a cofinite submodule of $M$. Suppose that there exists a submodule $L$ of $M$ such that $L \leq_{c} M$ and $N = L + T$ or $L = N + T$ for some $T \ll M$. Since $M$ is a ccws-module, then there exists a submodule $K$ of $M$ such that $M = N + K$ and $N \cap K \ll M$. Since $M$ is cofinitely refinable, there exists a direct summand $U$ of $M$ such that $U \leq N$ and $M = U + K$. So $U \cap K \leq N \cap K \ll M$. As $U$ is a direct summand of $M$, we obtain that $U \cap K \ll U$, which shows that $M$ is $\oplus$-cofinitely supplemented.

Suppose that there exists a closed submodule $L$ of $M$ such that $L = N + T$ for some $T \ll M$. Since $M$ is a ccws-module, there exists a submodule $K$ of $M$ such that $M = L + K$ and $L \cap K \ll M$. As $M$ is cofinitely refinable, then there is a direct summand $U$ of $M$ such that $U \leq N$ and $M = U + K$. Thus $U \cap K \leq N \cap K \ll U$, since $U$ is a direct summand of $M$. Therefore $M$ is $\oplus$-cofinitely supplemented.
References


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