A Note on the Number of Non-Cyclic Subgroups of Finite Groups

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Abstract

Let $G$ be a finite group with at most 29 non-cyclic proper subgroups. In this note we show that if $G$ is not isomorphic to the alternating group $A_5$ and the special linear group $\text{SL}_2(5)$ then $G$ must be solvable.

Mathematics Subject Classification: 20D05; 20D10

Keywords: non-cyclic subgroup; non-normal; solvable group

1 Introduction

For the solvability of a finite group with a given number of some particular subgroups, some previous results are mainly focused on the study of the influence of the number of maximal subgroups on the solvability of finite groups, the readers can refer to [2, 5, 7]. As a generalization of those results, the second

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1Z. Wu was supported by The Scientific Research Foundation of Guangxi University Xingjian College of Science and Liberal Arts (Grant No. 2014ZKLX02).

2J. Shi was supported in part by NSFC (11201401, 11361075 and 11561021). Corresponding author
author of this paper and C. Zhang [4] investigated the influence of the number of non-abelian subgroups on the solvability of finite groups. We proved the following result:

**Theorem 1.1** [4, Theorem 3.1] Let $G$ be a group with exactly $\delta(G)$ non-abelian proper subgroups.

1. If $0 \leq \delta(G) \leq 20$, then $G$ is solvable.
2. $G$ is a non-solvable group with $\delta(G) = 21$ if and only if $G \cong A_5$.
3. If $22 \leq \delta(G) \leq 25$, then $G$ is solvable.
4. $G$ is a non-solvable group with $\delta(G) = 26$ if and only if $G \cong \text{SL}_2(5)$.

It is obvious that a non-abelian subgroup must be a non-cyclic subgroup but a non-cyclic subgroup might not be a non-abelian subgroup. Thus as a direct corollary of [4, Theorem 3.1], we can easily get the result:

**Theorem 1.2** Let $G$ be a finite group with exactly $\gamma(G)$ non-cyclic proper subgroups.

1. If $0 \leq \gamma(G) \leq 25$, then $G$ is solvable.
2. $G$ is a non-solvable group with $\gamma(G) = 26$ if and only if $G \cong A_5$ or $\text{SL}_2(5)$.

As a continuation of Theorem 1.2, we in this note have the following result, the proof of which is given in Section 3.

**Theorem 1.3** Let $G$ be a finite group with exactly $\gamma(G)$ non-cyclic proper subgroups. If $27 \leq \gamma(G) \leq 29$, then $G$ is solvable.

As a generalization of [4, Theorem 3.1], the second author of this paper and C. Zhang [6, Theorem 1.2] investigated the influence of the number of non-normal non-abelian proper subgroups on the solvability of finite groups. We proved the result:

**Theorem 1.4** [6, Theorem 1.2] Let $G$ be a finite group having exactly $\tau(G)$ non-normal non-abelian proper subgroups.

1. If $0 \leq \tau(G) \leq 20$ or $22 \leq \tau(G) \leq 25$, then $G$ is solvable.
2. $G$ is a non-solvable group with $\tau(G) = 21$ if and only if $G \cong A_5$.
3. $G$ is a non-solvable group with $\tau(G) = 26$ if and only if $G \cong \text{SL}_2(5)$.

Arguing as in proof of [6, Theorem 1.2], combining Theorems 1.2 and 1.3, we can easily get the following result, here we omit its proof.

**Theorem 1.5** Let $G$ be a finite group having exactly $\eta(G)$ non-normal non-cyclic proper subgroups.

1. If $0 \leq \eta(G) \leq 25$, then $G$ is solvable.
2. $G$ is a non-solvable group with $\eta(G) = 26$ if and only if $G \cong A_5$ or $\text{SL}_2(5)$.
3. If $27 \leq \eta(G) \leq 29$, then $G$ is solvable.
By Theorems 1.2, 1.3 and 1.5, we propose the following question.

**Question 1.6** Let $G_1$ and $G_2$ be two finite groups. If $\gamma(G_1) = \eta(G_2) = n \geq 30$, is it true that $G_1$ and $G_2$ always have the same solvability?

## 2 Preliminaries

We collect some lemmas that are needed in the sequel.

**Lemma 2.1** [1] Let $G$ be a finite group. If every proper subgroup of $G$ is supersolvable, then $G$ is solvable.

**Lemma 2.2** [2] Let $G$ be a finite group. If $G$ has less than $21$ maximal subgroups, then $G$ is solvable.

**Lemma 2.3** [3] Let $G$ be a finite group. If $G$ has an abelian maximal subgroup, then $G$ is solvable.

**Lemma 2.4** [5] Let $G$ be a non-solvable group having exactly $n$ maximal subgroups.

1. If $n = 22$, then $G/\Phi(G) \cong PSL_2(7)$ or $A_5 \times \mathbb{Z}_p$ or $S_5$, where $p$ is a prime, $\Phi(G)$ is the Frattini subgroup of $G$;

2. If $n = 23$, then $G/\Phi(G) \cong PSL_2(7) \times \mathbb{Z}_p$ or $A_5 \times \mathbb{Z}_{pq}$ or $S_5 \times \mathbb{Z}_p$, where $p$ and $q$ are two distinct primes.

Arguing as in [5], we can also easily obtain the following result, here we omit its proof.

**Lemma 2.5** Let $G$ be a non-solvable group having exactly $24$ maximal subgroups, then $G/\Phi(G) \cong PSL_2(7) \times \mathbb{Z}_{pq}$ or $A_5 \times \mathbb{Z}_{pq}$ or $A_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $S_5 \times \mathbb{Z}_{pq}$, where $p$, $q$ and $r$ are three distinct primes.

**Lemma 2.6** [7] Let $G$ be a non-solvable group having exactly $21$ maximal subgroups, then $G/\Phi(G) \cong A_5$.

## 3 Proof of Theorem 1.3

**Proof.** (1) Suppose $\gamma(G) = 27$. Assume that $G$ is non-solvable. By Lemma 2.3, one has that all maximal subgroups of $G$ are non-cyclic. Then $G$ might have exactly $21$ or $22$ or $23$ or $24$ or $25$ or $26$ or $27$ maximal subgroups by Lemma 2.2.

We divide it into four cases:
(i) If $G$ has exactly 27 maximal subgroups. It follows that every maximal subgroup of $G$ is a minimal non-cyclic group and then every maximal subgroup of $G$ is supersolvable. By Lemma 2.1, one has that $G$ is solvable, a contradiction.

(ii) If $G$ has exactly 23 or 24 or 25 or 26 maximal subgroups. It implies that $G$ has at least one non-cyclic proper subgroup that is not a maximal subgroup of $G$. Let $H$ be a non-cyclic proper subgroup of $G$ that is not a maximal subgroup of $G$. Then $|G : N_G(H)| = m \leq 4$, that is, the number of conjugates of $H$ is less than 5. Let $N_G(H) = K$.

If $2 \leq m \leq 4$. One has that $G/K_G$ is isomorphic to a subgroup of the symmetric group $S_m$, where $K_G$ is the largest normal subgroup of $G$ contained in $K$. It is easy to see that $\gamma(K_G) < 26$ and then $K_G$ is solvable by Theorem 1.2 (1). It follows that $G$ is solvable since $S_m$ is solvable for $2 \leq m \leq 4$, a contradiction.

If $m = 1$. It implies that $H \trianglelefteq G$. It is obvious that $H$ is solvable since $\gamma(H) < 26$. Next we claim that $G/H$ is solvable. Otherwise, assume that $G/H$ is non-solvable. By Theorem 1.2 (2), we have $G/H \cong A_5$ or $SL_2(5)$. Since $H$ is non-cyclic, one has that every proper subgroup $L$ of $G$ satisfying $H < L$ is non-cyclic. It follows that $\gamma(G) > 27$, a contradiction. Therefore $G/H$ is solvable, and so $G$ is solvable, also a contradiction.

(iii) If $G$ has exactly 22 maximal subgroups. By Lemma 2.4 (1), it is easy to show that $\gamma(G/\Phi(G)) > 27$, which implies that $\gamma(G) > 27$, a contradiction.

(iv) If $G$ has exactly 21 maximal subgroups. By Lemma 2.6, one has $G/\Phi(G) \cong A_5$. Note that $\gamma(A_5) = 26$. It follows that $G$ has exactly 26 non-cyclic proper subgroups that contain $\Phi(G)$ and $G$ has exactly one non-cyclic proper subgroup $M$ of $G$ that is not a conjugate of any other 26 non-cyclic proper subgroups of $G$. One has $M \trianglelefteq G$. It is easy to see that $\Phi(G)$ is cyclic. Then $M \ntrianglelefteq \Phi(G)$. It follows that $M\Phi(G)/\Phi(G)$ is a non-trivial normal subgroup of $G/\Phi(G)$, this contradicts that $G/\Phi(G)$ is a non-abelian simple group.

All above arguments imply that our assumption is not true and so $G$ is solvable for $\gamma(G) = 27$.

(2) Arguing as in (1), we can also get that $G$ is solvable if $\gamma(G) = 28$ or 29. Here we omit it. Note that Lemma 2.5 is needed in the proof of the case $\gamma(G) = 29$. □

References

http://dx.doi.org/10.1007/bf01187387


Received: February 1, 2016; Published: March 24, 2016