A Note on the Real Division Algebras with Non-trivial Derivations

A. S. Diabang, O. Diankha, M. Ly

Département de Mathématiques et Informatique
Faculté des Sciences et Techniques
Université Cheikh Anta Diop, Dakar, Senegal

A. Rochdi

Département de Mathématiques et Informatique
Faculté des Sciences Ben M’Sik
Université Hassan II, 7955 Casablanca, Morocco

Abstract

We show that every 4-dimensional real division algebra having non-zero derivations is obtained from the real algebra \( \mathbb{C} \) by a special kind of duplication process accompanied by an appropriate isotopy. Also this process produces new examples in dimension 8.

Mathematics Subject Classification: 17A35, 17A36

Keywords: Division algebra, derivation, isotopy, duplication process

1. Introduction

In order to study real division algebras Benkart and Osborn adopted an approach using derivations ([4], [5]). Let \( A \) be such an algebra and let \( \partial \) be an arbitrary derivation of \( A \). For every \( \sigma \) in the field \( \mathbb{C} \) of complex numbers, Benkart and Osborn considered the sub-space

\[
B_\sigma = \{ x \in A : (\partial - \sigma I_A)(\partial - \overline{\sigma} I_A)x = 0 \},
\]
\( \sigma \) being the complex conjugate of \( \sigma \). They showed that

1. \( A \) is the direct sum of those \( B_\sigma \) such that \( \sigma \) is a characteristic root of \( \partial \) [4, p. 1139].
2. \( B_\sigma B_\tau \subseteq B_{\sigma + \tau} + B_{\sigma - \tau} \), so \( B_0 \) is a sub-algebra of \( A \) [4, p. 1139].
3. The roots of \( \partial \) are purely imaginary [4, Lemma 9].
4. For each \( \lambda \in \mathbb{C} \), \( \dim(B_\lambda) \) is even [4, Corollary 10].
5. \( B_0 \neq 0 \) and \( B_0 = \{ x \in A : \partial(x) = 0 \} \) [4, Lemma 11].

Next, in turn of the celebrated \{1, 2, 4, 8\}-Theorem ([6], [8]) they gave all possibilities for the Lie algebra of derivations \( \text{Der}(A) \) [4]:

**Theorem 1.** The following possibilities actually occurs for \( \text{Der}(A) \):

1. \( \dim(A) = 1 \) or \( 2 \) implies \( \text{Der}(A) = 0 \).
2. \( \dim(A) = 4 \), implies \( \text{Der}(A) \) is \( su(2) \) or \( \dim(\text{Der}(A)) \leq 1 \).
3. \( \dim(A) = 8 \), implies \( \text{Der}(A) \) is one of the following Lie algebras:
   - (a) compact \( G_2 \),
   - (b) \( su(3) \),
   - (c) \( su(2) \oplus su(2) \),
   - (d) \( su(2) \oplus N \) where \( N \) is an abelian ideal and \( \dim(N) \leq 1 \),
   - (e) \( N \) where \( N \) is abelian and \( \dim(N) \leq 2 \).

They then determine those real division algebra \( A \), in dimension 4, when \( \text{Der}(A) \) is \( su(2) \), and, in dimension 8, when \( \text{Der}(A) \) is compact \( G_2 \), \( su(3) \) and \( su(2) \oplus su(2) \). However the problem remains still open for the other cases of the Lie algebra of derivations. In particular, the one where \( \dim(\text{Der}(A)) = 1 \) although some particular cases were studied in dimension 4 [4, Theorem 7.7], [2, Corollary 8], [3].

With an appropriate adaptation of the so-called Cayley-Dickson process, we give here a new construction method of real division algebras whose Lie algebra of derivations has dimension \( \geq 1 \). This will allow us to fully determine the algebras in dimension 4 (Corollaries 1, 2) and some new ones in dimension 8 (Corollary 3, Theorem 3, Remark 1).

## 2. Unit-duplication process

All algebras will be of finite-dimension over the field \( \mathbb{R} \) of real numbers. An algebra \( A \) is said to be

1. **flexible** if it satisfies \( (xy)x = x(yx) \) for all \( x, y \) in \( A \),
2. **quadratic** if it contains an unit element \( e \) and for every \( x \) in \( A \) the three elements \( e, x, x^2 \) are linearly dependent,
(3) a division algebra if the linear operators $L_a$, $R_a$ of left and right multiplication by every non-zero element $a$ in $A$ are bijective. In this case the algebra $A_a$ having $A$ as underlying space and product given by:

$$x \odot y = R_a^{-1}(x)L_a^{-1}(y),$$

called Albert-isotope of $A$, is also a division algebra with unit element $e = a^2$ [1].

A linear mapping $\partial : A \rightarrow A$ is said to be a derivation of the algebra $A$ if the equality $\partial(xy) = (\partial x)y + x\partial(y)$ holds for all $x, y$ in $A$. We denote by $\text{Der}(A)$ the well known Lie algebra of derivations of $A$ [10]. Also $\text{Lin}\{x_1, \ldots, x_n\}$ will denote the lineal hull spanned by $x_1, \ldots, x_n \in A$.

**Definition 1. (Unit-duplication process)** Let $B$ be a real algebra having an unit element $e$ and let $\rho, \sigma, \phi, \psi : B \rightarrow B$ be linear mappings such that $\phi(e) = \psi(e) = e$. We define on the space $B \times B$ the product:

$$(x, y) \odot (x', y') = \left(xx' + \rho(\sigma(y')y), y\phi(x') + y'\psi(x)\right). \quad (2.1)$$

The algebra resulting has an unit element $(e, 0)$ and contains $B \times \{0\}$ as sub-algebra. It is said to be obtained from $B$ and $\rho, \sigma, \phi, \psi$ by unit-duplication process and is denoted by $\text{UDP}_B(\rho, \sigma, \phi, \psi)$. This generalizes the classical Cayley-Dickson process as well as the process given in [11, p. 1].

### 3. Four-dimensional real division algebras with non-trivial derivations

We have the following preliminary result taken from [4]:

**Lemma 1.** Let $A$ be a four-dimensional real division algebra admitting a non-zero derivation $\partial$. Then, according to the notations in Section 1, we have:

1. $B_0$ has dimension 2,
2. There exists a non-zero purely imaginary $\sigma = \zeta i$, with $\zeta > 0$, such that $A = B_0 \oplus B_\sigma$,
3. $B_{2\sigma} = \{0\}$ and $B_\sigma^2 = B_0$.

The following additional preliminary result, easy to show by taking into account [4, Theorem 6.3], will help us:

**Lemma 2.** Let $A$ be a four-dimensional real division algebra. Then the following are equivalent:

1. $A$ contains an unit element and $\text{Der}(A) = \text{su}(2)$.
2. $A$ is a quadratic and flexible algebra.
We state now the following result:

**Theorem 2.** Let $A$ be a four-dimensional real division algebra with unit element $e$. If $A$ admits a non-zero derivation $\partial$ then there exists a basis $\{e, u, x_1, x_2\}$ of $A$ for which the multiplication is given by the following table:

<table>
<thead>
<tr>
<th></th>
<th>$e$</th>
<th>$u$</th>
<th>$x_1$</th>
<th>$x_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e$</td>
<td>$e$</td>
<td>$u$</td>
<td>$x_1$</td>
<td>$x_2$</td>
</tr>
<tr>
<td>$u$</td>
<td>$u$</td>
<td>$-e$</td>
<td>$\alpha x_1 + \beta x_2$</td>
<td>$-\beta x_1 + \alpha x_2$</td>
</tr>
<tr>
<td>$x_1$</td>
<td>$x_1$</td>
<td>$\gamma x_1 + \delta x_2$</td>
<td>$\lambda e + \mu u$</td>
<td>$\omega e + \theta u$</td>
</tr>
<tr>
<td>$x_2$</td>
<td>$x_2$</td>
<td>$-\delta x_1 + \gamma x_2$</td>
<td>$-\omega e - \theta u$</td>
<td>$\lambda e + \mu u$</td>
</tr>
</tbody>
</table>

Table 1

for some scalars $\alpha, \beta, \gamma, \delta, \lambda, \mu, \omega, \theta$. In addition, every algebra $A$ whose multiplication is given by Table 1 admits a non-zero derivation. It is a division algebra if and only if the function

$$
\Phi(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\alpha_1^2 + \alpha_2^2) \left( (\alpha_1 + \alpha \alpha_2)^2 + \beta^2 \alpha_2^2 \right) + Q(\alpha_1, \alpha_2)(\alpha_3^2 + \alpha_4^2) + \delta(\lambda \theta - \omega \mu)(\alpha_3^2 + \alpha_4^2)^2
$$

where

$$
Q(\alpha_1, \alpha_2) = - (\gamma \mu + \delta \theta + \lambda) \alpha_1^2 + \left( \alpha (\gamma \lambda + \delta \omega - \mu) + \beta (\delta \lambda - \gamma \omega + \theta) \right) \alpha_2^2 + \left( - \alpha (\gamma \mu + \delta \theta + \lambda) + \beta (\gamma \theta - \delta \mu + \omega) + \gamma \lambda + \delta \omega - \mu \right) \alpha_1 \alpha_2
$$

vanishes only at $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0, 0, 0, 0)$.

In these conditions $\text{Der}(A) = su(2)$ if and only if $\alpha = \gamma = \mu = \omega = 0$, $\lambda < 0$, $\beta = -\delta \neq 0$ and $\theta = -\beta \lambda \neq 0$.

**Proof.** According to the notations of Lemma 1, we have $A = B_0 \oplus B_i$. Moreover, $B_0$ contains $e$, which is the only non-zero idempotent of algebra $A$ [12, Theorem 1], and is isomorphic to $\mathbb{C}$ ([13, Corollary 1], [9]). So $u^2 = -e$ for some $u$ in $B_0$. On the other hand there exists a basis $\{x_1, x_2\}$ of $B_i$ such that

$$
\partial(x_1) = x_2 \quad \text{and} \quad \partial(x_2) = -x_1.
$$
We obtain a basis $B = \{e, u, x_1, x_2\}$ of $A$. As $u \in \ker(\partial)$, the left and right multiplication operators $L_u$ and $R_u$ commute with $\partial$. So $\text{Im}(\partial)$ is both $L_u$-invariant and $R_u$-invariant. Writing $ux_1 = \alpha x_1 + \beta x_2$ and $x_1u = \gamma x_1 + \delta x_2$ where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, we have

\[
ux_2 = u\partial(x_1) = \partial(ux_1) = \alpha x_2 - \beta x_1, \\
x_2u = \partial(x_1)u = \partial(x_1u) = \gamma x_2 - \delta x_1.
\]

Moreover, $x_1^2 \in B_0^2 = B_0$ and there are some scalars $\lambda, \rho$ such that $x_1^2 = \lambda e + \rho u$. In the other hand:

\[
x_2x_1 + x_1x_2 = \partial(x_1)x_1 + x_1\partial(x_1) = \partial(x_1^2) = 0 \quad \text{and} \\
x_2^2 - x_1^2 = \partial(x_1)x_2 + x_1\partial(x_2) = \partial(x_1x_2) = 0.
\]

We obtain the desired multiplication Table 1.

In order to establish the second assertion, note that the linear mapping $A \rightarrow A$ whose matrix with respect to the basis $\{e, u, x_1\}$ is given by

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}
\]

is a derivation.

Let now $x = \alpha_1 e + \alpha_2 u + \alpha_3 x_1 + \alpha_4 x_2$, $y = \beta_1 e + \beta_2 u + \beta_3 x_1 + \beta_4 x_2$ be arbitrary in $A$. A direct calculation gives

\[
xy = \left(\alpha_1 \beta_1 - \alpha_2 \beta_2 + \lambda \alpha_3 \beta_3 + \omega \alpha_3 \beta_4 - \omega \alpha_4 \beta_3 + \lambda \alpha_4 \beta_4\right)e \\
+ \left(\alpha_1 \beta_2 + \alpha_2 \beta_1 + \mu \alpha_3 \beta_3 + \theta \alpha_3 \beta_4 - \theta \alpha_4 \beta_3 + \mu \alpha_4 \beta_4\right)u \\
+ \left(\alpha_1 \beta_3 + \alpha_2 \beta_3 - \beta \alpha_2 \beta_4 + \alpha_3 \beta_1 + \gamma \alpha_3 \beta_2 - \delta \alpha_4 \beta_2\right)x_1 \\
+ \left(\alpha_1 \beta_4 + \beta \alpha_2 \beta_3 - \alpha_2 \beta_4 + \alpha_3 \beta_2 + \gamma \alpha_3 \beta_1 + \gamma \alpha_4 \beta_2\right)x_2.
\]

So

\[
xy = 0 \Leftrightarrow \begin{pmatrix}
\alpha_1 & -\alpha_2 & \lambda \alpha_3 - \omega \alpha_4 & \omega \alpha_3 + \lambda \alpha_4 \\
\alpha_2 & \alpha_1 & \mu \alpha_3 - \theta \alpha_4 & \theta \alpha_3 + \mu \alpha_4 \\
\alpha_3 & \gamma \alpha_3 - \delta \alpha_4 & \alpha_1 + \alpha_2 & -\beta \alpha_2 \\
\alpha_4 & \delta \alpha_3 + \gamma \alpha_4 & \beta \alpha_2 & \alpha_1 + \alpha_2
\end{pmatrix}
\begin{pmatrix}
\beta_1 \\
\beta_2 \\
\beta_3 \\
\beta_4
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}.
\]

A laborious calculation shows that the determinant of the above matrix is
\[(\alpha_1^2 + \alpha_2^2)\left((\alpha_1 + \alpha\alpha_2)^2 + \beta^2 \alpha_2^2\right) + Q(\alpha_1, \alpha_2)(\alpha_3^2 + \alpha_4^2) + \delta(\lambda\theta - \omega\mu)(\alpha_3^2 + \alpha_4^2)^2.\]

This shows the third proposition of the theorem.

Assume now that \(A\) is a division algebra and \(\text{Der}(A) = su(2)\). Then according to Lemma 2 the algebra \(A\) is quadratic and flexible. Now

\[x_1^2 \in \text{Lin}\{e, x_1\}, \quad (u + x_1)^2 \in \text{Lin}\{e, u + x_1\}\]

give

\[\mu = 0 \quad \text{and} \quad \alpha + \gamma = \beta + \delta = 0. \quad (3.2)\]

As \(A\) is a division algebra then clearly \(\delta \neq 0\) and we deduce from \(\mu = 0\) that \(\lambda < 0\) and \(\theta \neq 0\). On the other hand, the equalities

\[(x_1x_2)x_1 = x_1(x_2x_1), \quad (x_1u)x_1 = x_1(ux_1), \quad ((x_1+u)x_2)(x_1+u) = (x_1+u)(x_2(x_1+u))\]

give

\[2\omega + (\alpha + \gamma)\theta = (\beta + \delta)\theta = 0, \quad (3.3)\]

\[(\gamma - \alpha)\lambda - (\beta + \delta)\omega = (\gamma - \alpha)\mu - (\beta + \delta)\theta = 0, \quad (3.4)\]

\[\theta = -\beta\lambda. \quad (3.5)\]

Now, taking into account the equalities (3.2), (3.3), (3.4), we have:

\[\alpha = \gamma = \mu = \omega = 0, \quad \lambda < 0 \quad \text{and} \quad \delta = -\beta, \quad \theta = -\beta\lambda \neq 0.\]

The converse is obvious.

**Corollary 1.** Every four-dimensional real unital division algebra \(A\) having a non-trivial derivation is obtained from the real unital algebra \(\mathbb{C}\) by unit-duplication process.

**Proof.** Let \(\sigma : x \mapsto \overline{x}\) be the standard involution of the real algebra \(\mathbb{C}\) and let \(\alpha, \beta, \gamma, \delta, \lambda, \mu, \omega, \theta\) be real numbers. We consider the \(\mathbb{R}\)-linear mappings \(\rho, \phi, \psi : \mathbb{C} \rightarrow \mathbb{C}\) whose matrices with respect to the canonical basis \(\{1, i\}\) are given, respectively, by:

\[
\begin{pmatrix}
\lambda & -\omega \\
\mu & -\theta
\end{pmatrix}, \quad
\begin{pmatrix}
1 & \gamma \\
0 & \delta
\end{pmatrix}, \quad
\begin{pmatrix}
1 & \alpha \\
0 & \beta
\end{pmatrix}.
\]
Now, the multiplication table of algebra $UDP_C(\rho, \sigma, \phi, \psi)$ with respect to the basis $\{(1, 0), (i, 0), (0, 1), (0, i)\}$ is identical to Table 1.

The study of non-unital case can be reduced to the study of unital one by using Albert-isotopy:

**Lemma 3.** Let $A$ be a division algebra of dimension $\geq 4$ having a non-trivial derivation $\partial$ and let $a$ be non-zero in the kernel of $\partial$. Then $\partial$ is a derivation of the unital division algebra $A_a$.

**Proof.** We have $[\partial, L_x] = L_{\partial(x)}$ for all $x \in A$. In the other hand, the operators of left and right multiplication by any $x$ in algebra $A_a$ are given, respectively, by $L^\circ_x = L_{R_a^{-1}(x)} \circ L^{-1}_a$, $R^\circ_x = R_{L_a^{-1}(x)} \circ R^{-1}_a$. As $\partial(a) = 0$, $\partial$ commutes with $L_a$ and we have:

$$
[\partial, L^\circ_x] = \partial \circ L_{R_a^{-1}(x)} \circ L^{-1}_a - L_{R_a^{-1}(x)} \circ L^{-1}_a \circ \partial
$$

$$
= \partial \circ L_{R_a^{-1}(x)} \circ L^{-1}_a - L_{R_a^{-1}(x)} \circ \partial \circ L^{-1}_a
$$

$$
= [\partial, L_{R_a^{-1}(x)}] \circ L^{-1}_a
$$

$$
= L_{\partial(R_a^{-1}(x))} \circ L^{-1}_a
$$

$$
= L_{R_a^{-1}(\partial x)} \circ L^{-1}_a
$$

$$
= L^\circ_{\partial x}.
$$

This shows that $\partial$ is a derivation of the algebra $A_a$. $
$

**Corollary 2.** Every four-dimensional real division algebra whose Lie algebra of derivations has dimension one is obtained from the real algebra $\mathbb{C}$ by unit-duplication process accompanied by an Albert-isotopy.

4. New examples in dimension 8

We start with the following useful definition:

**Definition 2.** Let $A$ be an algebra.

(1) A linear mapping $f : A \rightarrow A$ is said to be an involution if $f^2 = I_A$ (the identity operator of $A$) and $f(xy) = f(y)f(x)$ for all $x, y$ in $A$.

(2) If $A$ contains an unit element $e$ and is provided with an involution $\sigma_A : A \rightarrow A$ $x \mapsto \overline{x}$, it is called a Cayley algebra if $x + \overline{x} := T(x)$, $x\overline{x} := N(x) \in \mathbb{R}e$ for all $x, y$ in $A$. Clearly $\sigma_A(e) = e$. 

It is well known that every Cayley algebra $A$ is a quadratic algebra and has a decomposition $A = \mathbb{R}e \oplus \text{Im}(A)$ into a sum of sub-space of scalars $\mathbb{R}e$ and sub-space of imaginary elements $\text{Im}(A) = \{ x \in A : x^2 \in \mathbb{R}e, x \notin \mathbb{R} - \{0\} \}$ [7]. For every $x$ in $A$ we will denote by $s(x), \text{im}(x)$, respectively, the scalar and imaginary parts of $x$.

Let now $A$ be a Cayley algebra with involution $\sigma_A : x \mapsto \overline{x}$ and let $\kappa$ be in $A$. We denote by $E_{\kappa}(A)$ the algebra having underlying space $A \times A$ and product

$$(x, y)(x', y') = \left( xx' + \kappa \overline{y} y, y\overline{x}' + y' x \right).$$

This is the algebra $UDP_A(L_\kappa, \sigma_A, \sigma_A, I_A)$.

We have the following preliminary result:

**Lemma 4.** Let $A$ be an associative Cayley algebra, of unit $e$ and involution $x \mapsto \overline{x}$, and let $\kappa$ be in $A - \mathbb{R}e$. Then $E_{\kappa}(A)$ is a division algebra if and only if $A$ is a division algebra. In this case, the equation $(x, y)^2 = -(e, 0)$ in $E_{\kappa}(A)$ is equivalent to $x^2 = -e$ and $y = 0$.

**Proof.** It remains only to show the "if" part. Let $x, y, x', y'$ be arbitrary elements of $A$ with $(x, y) \neq (0, 0)$ such that $(x, y)(x', y') = (0, 0)$. We can assume, without lost of generality, that $y \neq 0$. We have

$$(x, y)(x', y') = \left( xx' + \kappa \overline{y} y, y\overline{x}' + y' x \right).$$

So

$$(x, y)(x', y') = (0, 0) \iff \begin{cases} xx' + \kappa \overline{y} y = 0 \\ y\overline{x}' + y' x = 0 \end{cases}$$

This gives

$$0 = (y\overline{x}' + y' x)x' - y'(xx' + \kappa \overline{y} y) = (N(x') - y'\kappa \overline{y})y$$

$$= \left(N(x') - s(\kappa)N(y') - y' \text{im}(\kappa)\overline{y}\right)y$$

and then

$$\left(N(x') - s(\kappa)N(y')\right) - y' \text{im}(\kappa)\overline{y} = 0. \quad (4.6)$$

Multiplying equality $(4.6)$ on the left by $\overline{y}$ and right by $y'$ we get:
Real division algebras with non-trivial derivations

0 = \overline{y'} \left((N(x') - s(\kappa)N(y')) - y' \text{im}(\kappa)\overline{y'}\right) y' \\
= N(y') \left((N(x') - s(\kappa)N(y')) - N(y')\text{im}(\kappa)\right).

If \(N(y') \neq 0\) then

\((N(x') - s(\kappa)N(y')) - N(y')\text{im}(\kappa) = 0.\)

As \(N(x'), s(\kappa)N(y')\) are scalars, \(\text{im}(\kappa)\) must vanish, that is, \(\kappa \in \mathbb{R}e\), which is absurd. Therefore \(y' = 0 = x'\).

Let now \(x, y\) be in \(A\), we have

\((x, y)^2 = -(e, 0) \iff \begin{cases} x^2 + N(y)\kappa = -e \\ T(x)y = 0 \end{cases} \)

because \(\eta \neq 0\). If \(y \neq 0\) then \(T(x) = 0\), that is \(x \in \text{Im}(A)\) and \(\kappa = -N(y)^{-1}(e + x^2)\) must be a scalar, a contradiction. This shows the second proposition.

**Corollary 3.** Let \(\mathbb{H}\) be the quaternion algebra and let \(\kappa\) be in \(\mathbb{H} - \mathbb{R}\). Then

1. \(E_\kappa(\mathbb{H})\) is a division algebra.
2. Every two-dimensional subalgebra of \(E_\kappa(\mathbb{H})\) is contained in \(\mathbb{H} \times \{0\}\).

**Proof.** The first proposition is given in Lemma 4. The second one is consequence of Lemma 4 by taking into account that every 2-dimensional subalgebra of \(E_\kappa(\mathbb{H})\) contains the unit element \((1, 0)\) [12, Theorem 1] and is isomorphic to \(\mathbb{C}\) ([13, Corollary 1], [9]).

**Theorem 3.** Let \(\mathbb{H}\) be the quaternion algebra and let \(\kappa\) be in \(\mathbb{H} - \mathbb{R}\). Then \(\text{Der}(E_\kappa(\mathbb{H})) = su(2) \oplus N\) where \(N\) is 1-dimensional Lie algebra.

**Proof.** Corollary 3 shows that the equation \((x, y)^2 = -(1, 0)\) in algebra \(E_\kappa(\mathbb{H})\) cannot have solutions outside a sub-space of dimension 3 unlike the case of algebras whose multiplications are given by tables \((2.1), (4.2), (5.2)\) in [5]. Moreover, an algebra whose multiplication is given by \((3.1)\) in [4] cannot have an unit-element. So \(\text{Der}(E_\kappa(\mathbb{H}))\) cannot be equal to \(G_2\) compact, \(su(3)\) or \(su(2) \oplus su(2)\).

On the other hand, a simple calculation shows that for every non-zero \(d\) in \(\text{Im}(\mathbb{H})\) the mapping

\[ \Delta_d : E_\kappa(\mathbb{H}) \to E_\kappa(\mathbb{H}) \quad (x, y) \mapsto (0, dy) \]
is a derivation with kernel $\mathbb{H} \times \{0\}$. So $\text{Der}(E_\kappa(\mathbb{H}))$ has dimension $\geq 3$ and cannot be abelian. Moreover, for every non-zero derivation $\partial$ of $\mathbb{H}$ that vanish in $\kappa$ the mapping

$$D_\partial : E_\kappa(\mathbb{H}) \to E_\kappa(\mathbb{H}) \quad (x,y) \mapsto (\partial x, \partial y)$$

is a derivation with kernel

$$\ker(D_\partial) = \text{Lin}\{(1,0), (\kappa,0), (0,1), (0,\kappa)\}.$$

In particular, $D_\partial$ cannot belong to $\{\Delta_d : d \in \text{Im}(\mathbb{H})\}$.

Thus $\text{Der}(E_\kappa(\mathbb{H}))$ has dimension 4 and coincides with $su(2) \oplus N$ where $N$ is abelian of dimension 1.

**Remark 1.** Let $\mathcal{C}$ be the class of eight-dimensional division algebras $A$ with $\text{Der}(A) = su(2) \oplus N$ where $N$ is a 1-dimensional Lie algebra. For every $\theta$ in $]0,\pi[$ the algebra $E_\kappa(\mathbb{H})$, where $\kappa = \cos \theta + i \sin \theta$, belongs to $\mathcal{C}$. The multiplication table of algebra $E_\kappa(\mathbb{H})$ relative to the canonical table

$$\{(1,0), (i,0), (j,0), (k,0), (0,1), (0,i), (0,j), (0,k)\},$$

which we will denote $\{1, i, j, k, f, if, jf, kf\}$, is given by:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>i</th>
<th>j</th>
<th>k</th>
<th>f</th>
<th>if</th>
<th>jf</th>
<th>kf</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>i</td>
<td>j</td>
<td>k</td>
<td>f</td>
<td>-f</td>
<td>-if</td>
<td>-kj</td>
</tr>
<tr>
<td>i</td>
<td>i</td>
<td>-1</td>
<td>k</td>
<td>-j</td>
<td>i</td>
<td>-f</td>
<td>if</td>
<td>-kf</td>
</tr>
<tr>
<td>j</td>
<td>j</td>
<td>-k</td>
<td>-1</td>
<td>i</td>
<td>if</td>
<td>kf</td>
<td>-j</td>
<td>if</td>
</tr>
<tr>
<td>k</td>
<td>k</td>
<td>j</td>
<td>-i</td>
<td>-1</td>
<td>kf</td>
<td>-if</td>
<td>if</td>
<td>i</td>
</tr>
</tbody>
</table>

This example using the unit-duplication process is different from the one in [4, Theorem 6.9]. It illustrates the immensity of the class $\mathcal{C}$.

**Acknowledgements.** This paper has benefited from several remarks and suggestions by Professor Antonio Jesús Calderón. The authors are very grateful to him.
Real division algebras with non-trivial derivations

REFERENCES


Received: November 21, 2015; Published: January 7, 2016