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On the Automorphisms of Absolute-Valued Algebras

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Abstract

We characterize the non-triviality of the group of automorphisms for all absolute-valued algebras of finite dimension ≥ 4.

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1. Introduction

Since the discovery of quaternions $\mathbb{H}$ by Hamilton [11, Chap. 7] and octonions $\mathbb{O}$ by Graves and Cayley ([11], Chap. 9], [3]) the absolute-valued algebras have continued to fascinate mathematicians and physicists by their beauty and diversity. The works of illustrious pioneering ([14], [1], [22], [21], [20], [13]) have helped lay the foundations of the theory. There is a compilation of the work around the theory prior to the year 2014 in ([19], [4]).

Albert [1, Theorem 2] proved that every finite-dimensional absolute-valued algebra A is isotopic to either $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$. So A has dimension 1, 2, 4, or 8, and the absolute value of A comes from an inner product. It follows easily that $\mathbb{R}$, $\mathbb{C}$, ${}^*\mathbb{C}$, $\mathbb{C}^*$, $\overset{*}{\mathbb{C}}$ are the only absolute-valued algebras of dimension ≤ 2 [19, p. 107]. Other works has allowed a classification in dimension 4 ([15], [6]). In addition, [15, Proposition 3.2] contains a precision of those algebras which contain two-dimensional sub-algebras. Attempts in dimension 8 resulted in a classification for algebras having a one-sided unit [17] or a non-zero central idempotent [18]. The existence of sub-algebras of dimension 4, in both works, is characterized by the non-triviality of the group of automorphisms ([17, Theorem 6.3], [18, Theorem 4.8])

Recent work has resulted in a classification of all finite-dimensional absolute-valued algebras [5]. These algebras are precisely of the form $\mathbb{A}_{f,g}$, where $\mathbb{A}$ stands for either $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$, $f, g : \mathbb{A} \to \mathbb{A}$ being linear isometries fixing 1, and $\mathbb{A}_{f,g}$ denotes the absolute-valued algebra obtained by endowing the normed space of $\mathbb{A}$ with the product $x \odot y := f(x)g(y)$. However, the non-triviality of the group of automorphisms, in dimension 8, was partially characterized [5, Corollary 6.5]:

For arbitrary linear isometries f, g of the Euclidean space $\mathbb{O}$ fixing 1, *the following are equivalent:*

(1) $\mathbb{O}_{f,g}$ *contains a* 4*-dimensional sub-algebra containing* 1.

(2) $\mathbb{O}$ *contains a* $\{f, g\}$*-invariant* 4*-dimensional sub-algebra.*

(3) $Aut(\mathbb{O}_{f,g})$ *contains a reflection fixing* 1.

In the present work we show that the spectrum of an automorphism of every four-dimensional absolute-valued algebra is not empty (Corollary 1). This has allowed us to give a characterization of the non-triviality of the group of automorphisms through the existence of 2-dimensional sub-algebras (Theorem 1). Next, we specify those four-dimensional sub-algebras of the pseudo-octonion algebra $\mathbb{P}$ [13] (Corollary 3). This provides us with examples of four-dimensional sub-algebras of $\mathbb{P}$ which does not contain a non-zero idempotent arbitrary selected (Corollary 5). This completes the Corollary 6.5 in [5]. Finally, we show that every absolute-valued algebra of dimension $n \geq 4$ with a non-zero flexible idempotent contains always 2-dimensional sub-algebras (Proposition 4) unlike, in dimension 8, for which there exists absolute-valued algebras even with a one-sided unit ([17, Remark 5.3], [19, p. 108]) or a non-zero central idempotent [18, Theorem 4.8], which contain no 4-dimensional sub-algebras.

2. **Notations and preliminary results**

Let A be an arbitrary non-associative algebra over a field $\mathbb{K}$ contained in $\mathbb{C}$ and let f be an automorphism of A. For every $\lambda \in \mathbb{K}$ we denote by $E_\lambda(f)$ the kernel of $f - \lambda I_A$, I_A being the identity operator of A. The automorphism f is said to be a reflection of A if $f \neq I_A$ and $f^2 = I_A$. The algebra A is said to be flexible if it satisfies to $(xy)x = x(yx)$ for all x, y in A. It is well known that every flexible algebra is power-commutative [16], that is, the algebra $\mathbb{K}[x]$ generated by every x in A is commutative.

Assume now that $\mathbb{K} = \mathbb{R}$ then A is said to be

(1) A *real division algebra* if the operators

$$L_x : A \to A \quad y \mapsto xy, \qquad R_x : A \to A \quad y \mapsto yx$$

are bijective, for all $x \in A$, $x \neq 0$.

(2) An *absolute-valued algebra* if it satisfies $||xy|| = ||x||\,||y||$ for a given norm $||.||$ on A, and all $x, y \in A$.

An element a in an arbitrary algebra A is said to be central if $[a, A] = 0$ where $[., .]$ means the commutator.

Let's start with the following preliminary results:

Lemma 1. *Let A be an algebra over $\mathbb{K}$ and let f be an automorphism of A. Then $E_1(f)$, $E_1(f) + E_{-1}(f)$ are sub-algebras of A and $E_1(f) + E_{-1}(f)$ is a direct sum. If, in addition, $f \neq I_A$ then the following are equivalent:*

(1) *f est une reflection of A,*

(2) *$A = E_1(f) \oplus E_{-1}(f)$.*

Proof. Clearly $E_1(f)$ is a sub-algebra of A and we have the inclusions

$$(2.1)\ E_1(f)E_{-1}(f),\ E_{-1}(f)E_1(f) \subseteq E_{-1}(f),\ E_{-1}(f)E_{-1}(f) \subseteq E_1(f).$$

So $E_1(f) + E_{-1}(f)$ is a sub-algebra of A containing $E_1(f)$. Let us show the equivalence of two statements:

(1) $\Rightarrow$ (2) ? We have

$$\begin{aligned} A &= \ker(f^2 - I_A) \\ &= \ker\Big((f - I_A) \circ (f + I_A)\Big) \\ &= \ker(f - I_A) \oplus \ker(f + I_A) \\ &= E_1(f) \oplus E_{-1}(f). \end{aligned}$$

(2) $\Rightarrow$ **(1) ?** Every x in A is written $x_1 + x_{-1}$ with $x_1 \in E_1(f)$, $x_1 \in E_1(f)$ and we have:

$$\begin{aligned} f^2(x) &= f\big(f(x_1) + f(x_{-1})\big) \\ &= f(x_1 - x_{-1}) \\ &= x_1 + x_{-1} \\ &= x. \end{aligned}$$

So $f^2 = I_A$. ▪

Lemma 2. *Let A be a finite-dimensional algebra over $\mathbb{K}$ and let f be an automorphism of A. Then the spectrum $sp(f)$ of f is a subset of all complex numbers of modulus one. In particular, $sp(f) \subseteq \{1, -1\}$ if $\mathbb{K} \subseteq \mathbb{R}$.*

Proof. Obviously $sp(f)$ contains no zero element and we can assume it not empty. Moreover, $sp(f)$ is, clearly, a multiplicative set. Now, for every λ in $sp(f)$, we have $\{\lambda^k : k \geq 1\} \subset sp(f) \subset \mathbb{K} \subset \mathbb{C}$. Moreover $sp(f)$ is finite, because A has finite dimension, so $\{\lambda^k : k \geq 1\}$ is finite and then the modulus $|\lambda|$ of λ is one. ▪

Lemma 3. *Let A be a finite-dimensional real division algebra and let f be an automorphism of A. Then the following propositions are equivalent:*

(1) $sp(f) = \{1, -1\}$.
(2) $-1 \in sp(f)$.
(3) $\dim E_1(f) = \dim E_{-1}(f) \geq 1$.
(4) $sp(f) \neq \emptyset$ *and the following equalities hold:*

$$E_1(f)E_{-1}(f) = E_{-1}(f)E_1(f) = E_{-1}(f), \quad E_{-1}(f)E_{-1}(f) = E_1(f). \tag{2.2}$$

Proof. The implication **(3)** $\Rightarrow$ **(1)** follows by taking into account the Lemma **2** and the implication **(1)** $\Rightarrow$ **(2)** is clear.

(2) $\Rightarrow$ **(4) ?** $E_{-1} \neq \{0\}$ and for every non-zero u in $E_{-1}(f)$ we have:

$$\begin{aligned} R_u\big(E_1(f)\big) &= E_1(f)u \subseteq E_1(f)E_{-1}(f) \overset{(2.1)}{\subseteq} E_{-1}(f) \text{ and} \\ R_u\big(E_{-1}(f)\big) &= E_{-1}(f)u \subseteq E_{-1}(f)E_{-1}(f) \overset{(2.1)}{\subseteq} E_1(f). \end{aligned}$$

Now

$$\begin{aligned} \dim E_1(f) &= R_u\big(E_1(f)\big) \text{ because } R_u \text{ is injective} \\ &\leq \dim E_{-1}(f). \end{aligned}$$

Likewise $\dim E_{-1}(f) \leq \dim E_1(f)$ and then $\dim E_{-1}(f) = \dim E_1(f)$. Consequently, the inclusions in **(2.1)** are equalities.

(4) $\Rightarrow$ (3) ? The equalities **(2.2)** show that E_1, E_{-1} are non-trivial sub-spaces. As A is a division algebra, a similar reasoning as above shows that these two sub-spaces have the same dimension. ∎

Proposition 1. *Let A be a real division algebra of dimension $n \geq 4$ and let f be a non-identity automorphism of A having two free eigenvectors u, v. Then f leaves invariant a proper sub-algebra of dimension ≥ 2.*

Proof. The sub-algebra $E_1(f^2) := B$ is invariant under f and contains both vectors u, v. So we can assume that B has dimension ≥ 4 and we distinguish the following two cases:

(1) If $B = A$ then f becames a reflection and then $E_1(f)$ is an f-invariant proper sub-algebra of A.

(2) If $B \neq A$ then B answers the question. ∎

Remark 1. Let A a real division algebra of dimension $n \geq 4$ with an automorphism f such that $sp(f) = \{1, -1\}$. Then the inclusions between the following sub-algebras of A are strict: $\{0\} \subset E_1(f) \subset E_1(f) + E_{-1}(f)$. In this case

$$\dim E_1(f) = n \geq 1, \quad \text{and} \quad \dim\big(E_1(f) + E_{-1}(f)\big) = 2n \text{ ∎}$$

Remark 2. The spectrum of an automorphism may be empty even for absolute-valued algebras of dimension 2. Indeed, the well known McClay algebra $\overset{*}{\mathbb{C}}$ ([2], [19, p. 107]) has three non-zero idempotents, namely 1, $\frac{-1+i\sqrt{3}}{2} := j$, $\frac{-1-i\sqrt{3}}{2}$, $\{1, i\}$ being the canonical basis of real space $\mathbb{C}$. Now, the mapping $\mathbb{C} \to \mathbb{C}$ $z \mapsto zj$ is an automorphism of the real algebra $\overset{*}{\mathbb{C}}$ with empty spectrum. ∎

3. Study of the non-triviality of $Aut(A)$ in dimension 4

For arbitrary norm-one $a, b \in \mathbb{H}$, let $\mathbb{H}(a,b)$, ${}^*\mathbb{H}(a,b)$, $\mathbb{H}^*(a,b)$ and $\overset{*}{\mathbb{H}}(a,b)$ be the *principal isotopes* of $\mathbb{H}$ [15], that is, the algebras having $\mathbb{H}$ as underlying space and products $x \odot y$ given respectively by $axyb$, $\overline{x}ayb$, $axb\overline{y}$, $a\overline{x}\,\overline{y}b$ where $x \mapsto \overline{x}$ stands for the standard involution of algebra $\mathbb{H}$.

Let now A be an arbitrary algebra, we denote by A^{op} the opposite algebra of A that is the algebra obtained by endowing the space A with the product $x \odot y = yx$. If, moreover, A is an absolute-valued algebra with absolute-value $|.|$, we consider the sub-spaces

$$\begin{aligned} Z(A) &= \{u \in A : ux = xu \text{ for all } x \in A\} \text{ (the center of } A) \\ \mathcal{D}_r(A) &= \{u \in A : (ux)x = |x|^2 u \text{ for all } x \in A\} \\ \mathcal{D}_l(A) &= \{u \in A : x(xu) = |x|^2 u \text{ for all } x \in A\} \end{aligned}$$

We start with the following preliminary key result:

Lemma 4. *Let f be an automorphism of any one of the isotopes $\mathbb{H}(a,b)$, ${}^*\mathbb{H}(a,b)$, $\mathbb{H}^*(a,b)$, $\overset{*}{\mathbb{H}}(a,b)$ of algebra $\mathbb{H}$. Then $f(1) \in \{1,-1\}$.*

Proof. Let A be any one of above algebras. We denote by $|.|$, $S_{\mathbb{H}}$ the absolute-value and unit-sphere of $\mathbb{H}$.

It is well known that the center of the isotopes $\mathbb{H}(a,b)$, $\overset{*}{\mathbb{H}}(a,b)$ is $\mathbb{R}$ [5, p. 202]. So, for A equal to either $\mathbb{H}(a,b)$ or $\overset{*}{\mathbb{H}}(a,b)$ we have $f(1) \in Z(A) \cap S_{\mathbb{H}} = \{1,-1\}$.

Let now $\odot_l$ be the product in ${}^*\mathbb{H}(a,b)$. We have

$$\begin{aligned} u \in \mathcal{D}_r\Big({}^*\mathbb{H}(a,b)\Big) &\Leftrightarrow (u \odot_l x) \odot_l x = |x|^2 u \text{ for all } x \in \mathbb{H} \\ &\Leftrightarrow \overline{\overline{u}\, axb}\; axb = |x|^2 u \text{ for all } x \in \mathbb{H} \\ &\Leftrightarrow \overline{b}\,\overline{x}\,\overline{a}u\; axb = |x|^2 u \text{ for all } x \in \mathbb{H} \\ &\Leftrightarrow \overline{x}\,\overline{a}u\; ax = |x|^2 bu\overline{b} \text{ for all } x \in \mathbb{H} \\ &\Leftrightarrow \overline{a}u\; ax = xbu\overline{b} \text{ for all } x \in \mathbb{H} \\ &\Leftrightarrow \overline{a}ua = bu\overline{b} \in \mathbb{R} \\ &\Leftrightarrow u \in \mathbb{R}. \end{aligned}$$

Moreover, the equality $(1 \odot_l x) \odot_l x = |x|^2$ hold for all x in $\mathbb{H}$. So, for $A = {}^*\mathbb{H}(a,b)$, we have $f(1) \in \mathcal{D}_r(A) \cap S_{\mathbb{H}} = \{1,-1\}$. On the other hand,

$\mathcal{D}_l(A) = \mathcal{D}_r(A^{op})$. So the result is concluded by taking into account that the mapping $\left({}^{*}\mathbb{H}(a,b)\right)^{op} \to \mathbb{H}^{*}(\bar{b},\bar{a}) \quad x \mapsto \overline{x}$ is an isomorphism of algebras. ▪

Corollary 1. *Let A be a 4-dimensional absolute-valued algebra. Then the spectrum of every automorphism of algebra A is not empty.* ▪

We will say that an algebra is ***ugly*** if it has a trivial automorphism group, and ***beautiful*** if not. We can now state the following:

Theorem 1. *Let A be a four-dimensional absolute-valued algebra. Then the following assertions are equivalent:*

(1) *A is beautiful.*
(2) *-1 belongs to the spectrum of some automorphism of algebra A.*
(3) *$Aut(A)$ contains a reflection.*
(4) *A contains a two-dimensional sub-algebra.*
(5) *A is obtained by duplication.*

Proof. The implications **3)** $\Rightarrow$ **2)** $\Rightarrow$ **1)** are clear.

1) $\Rightarrow$ **4) ?** Let f be in $Aut(A)$. By Corollary **1** $sp(f) \neq \emptyset$ and f has an eigenvector u. As f is a linear isometry, it leaves invariant the orthogonal tri-dimensional space $u^{\perp} := H$ and its restriction $f_{//H} : H \to H$ admit an eigenvector linearly independent to u. If, moreover, $f \neq I_A$ the Proposition 1 shows that A contains a two-dimensional sub-algebra.

The implication **4)** $\Rightarrow$ **5)** is shown as in [5, Theorem 6.4].

5) $\Rightarrow$ **3) ?** If A is obtained by duplication: $A = (\mathbb{C} \times \mathbb{C}, \odot)$ then the mapping $(x,y) \mapsto (x,-y)$ is a reflection of A. ▪

Remark 3. Among the isotopes $\mathbb{H}(a,b)$, ${}^{*}\mathbb{H}(a,b)$, $\mathbb{H}^{*}(a,b)$, $\overset{*}{\mathbb{H}}(a,b)$ of algebra $\mathbb{H}$ the ugly ones are specified via a geometrical interpretation in [10, p. 14]. A norm-one element in $\mathbb{H}$ is considered as a rotation in the euclidian space $\mathbb{H}$. It is called proper if it is not the identity [10, p. 12].

Let A be one of the above algebras, then A is ugly if and only if both a, b are proper and one of the two following conditions holds for a, b :

(1) their axes are orthogonal and have angles which are not π.
(2) their axes are neither parallel nor orthogonal, and they have angles which are not both π.

4. **Study in dimension** 8

It was shown in [5, Theorem 6.4] the following result:

Theorem 2. *Let A be an eight-dimensional absolute-valued algebra. Then the following assertions are equivalent:*

(1) *$Aut(A)$ contains a reflection.*
(2) *A contains a four-dimensional sub-algebra.*
(3) *A is obtained by duplication.*

As consequence [5, Corollary 6.5]:

Corollary 2. *For arbitrary linear isometries f, g of the Euclidean space $\mathbb{O}$ fixing* 1, *the following are equivalent:*

(1) *$\mathbb{O}_{f,g}$ contains a* 4*-dimensional sub-algebra containing* 1.
(2) *$\mathbb{O}$ contains a $\{f, g\}$-invariant* 4*-dimensional sub-algebra.*
(3) *$Aut(\mathbb{O}_{f,g})$ contains a reflection fixing* 1.

Let now $\mathbb{P}$ be the well known eight-dimensional flexible algebra of pseudo-octonions [13]. We denote by $I(\mathbb{P})$ its set of all non-zero idempotents.

To provide additional clarification on [5, Corollary 6.5], we need the following:

Proposition 2. *$\mathbb{P}$ coincides with the sub-space spanned by $I(\mathbb{P})$.*

Proof. Let $x \in \mathbb{P}$ be not linearly dependent to an idempotent. Then the sub-algebra $\mathbb{R}[x]$ is commutative [16] of dimension ≤ 2 [12]. So $\mathbb{R}[x]$ has dimension 2 and is isomorphic to $\overset{*}{\mathbb{C}}$ [8, Lemme 2.4, Théorème 2.7]. We deduce that x is a linear combination of two (non-zero) idempotents. ∎

Corollary 3. *For any sub-space E of $\mathbb{P}$, of dimension ≤ 7 there exists a non-zero idempotent which is not contained in E.* ∎

We have the following key result:

Proposition 3. *For any non-zero idempotent e in $\mathbb{P}$ the set $\mathcal{C}(e)$, of all elements commuting with e, is a four-dimensional sub-algebra of $\mathbb{P}$.*

Proof. $\mathcal{C}(e)$ is a sub-algebra of A [8, Lemme 2.11]. Moreover, e is a central idempotent for algebra $\mathcal{C}(e)$, so $\dim \mathcal{C}(e) \leq 4$ [4]. Assume now that $\dim \mathcal{C}(e) < 4$ and let $f \in I(\mathbb{P}) - \mathcal{C}(e)$. There exists $g \in I(\mathbb{P})$ such that $[g, e] = [g, f] = 0$ [8, Lemme 2.3]. So $\mathcal{C}(e)$ has dimension 2 [1], it coincides with $Lin\{e, g\}$ and is contained strictly in $\mathcal{C}(g)$. Above sub-algebra has dimension 4. Note that $\mathcal{C}(e)$ is isomorphic to $\overset{*}{\mathbb{C}}$ and then contains a non-zero idempotent h other than e and g. So $\mathcal{C}(e) \subseteq \mathcal{C}(h)$. Thus $\dim\big(\mathcal{C}(h)+\mathcal{C}(g)\big) \leq 6$ and there exists $e_0 \in I(\mathbb{P}) - \big(\mathcal{C}(h) \cup \mathcal{C}(g)\big)$. The idempotent e_0 does not commute with any of the three idempotents e, g, h. There is finally $f_0 \in I(\mathbb{P})$ such that $[f_0, e] = [f_0, e_0] = 0$. Such idempotent f_0 belongs to $\mathcal{C}(e)$, however

$$\begin{aligned}
f_0 &\neq e \text{ because } [e_0, e] \neq 0 \text{ and } [e_0, f_0] = 0\\
f_0 &\neq g \text{ because } [e_0, g] \neq 0 \text{ and } [e_0, f_0] = 0\\
f_0 &\neq h \text{ because } [e_0, h] \neq 0 \text{ and } [e_0, f_0] = 0.
\end{aligned}$$

This is absurd. ▪

Corollary 4. *Sub-algebras of $\mathbb{P}$ of dimension 4 are exactly the ones of the form $\mathcal{C}(e)$ with $e \in I(\mathbb{P})$.*

Proof. Note that every 4-dimensional sub-algebra of $\mathbb{P}$ is isomorphic to $\overset{*}{\mathbb{H}}$ and, consequently, contains a non-zero central idempotent. The result is then obtained via Proposition **3**. ▪

Corollary 5. *For any non-zero idempotent e in $\mathbb{P}$ there is at least one 4-dimensional sub-algebra of $\mathbb{P}$ which does not contain e.* ▪

Corollary 6. *Let f, g be two linear isometries of the euclidian space $\mathbb{O}$ fixing 1 such that $\mathbb{P} = \mathbb{O}_{f,g}$. Then there exists a four-dimensional sub-algebra of $\mathbb{O}_{f,g}$ which does not contain the idempotent 1.* ▪

There are eight-dimensional absolute-valued algebras even with left-unit ([17, Remark 5.3], [19, p. 108]), or a non-zero central idempotent [18, Theorem 4.8], which contain no 4-dimensional sub-algebras. This is not the case, in dimension 4, where the existence of 2-dimensional sub-algebras is ensured only in the case, more general, of existence of a non-zero flexible idempotent.

Proposition 4. *Let A be an absolute-valued algebra of dimension $n \geq 4$ with a non-zero flexible idempotent e. Then A contains a 2-dimensional sub-algebra invariant under both L_e and R_e.*

Proof. The mappings L_e, $R_e : A \to A$ are linear isometries fixing e and induce linear isometries on the sub-space $Lin\{e\}^{\perp} := E$ orthogonal to e. As E has an odd dimension and L_e commute with R_e, there exists norm-one eigenvector $u \in E$ common for both L_e and R_e :

$$L_e(u) = \alpha u, \quad R_e(u) = \beta u \text{ with } \alpha, \beta \in \{1, -1\}.$$

Now, the algebra $(A, \odot)$ obtained by providing the normed space A with the product

$$x \odot y = L_e(x)R_e(y)$$

is an absolute-valued algebra with central idempotent e. So $u \odot u = -e$ [9, Lemma 3.3] that is $u^2 = -\alpha\beta e$. Finally, the sub-space $Lin\{e, u\}$, of algebra A, answers the question. ∎

It is appropriate to raise the following problem:

Problem 1. *Let A be an eight-dimensional beautiful absolute-valued algebra. Does A contain a four-dimensional sub-algebra ?*

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