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On Congruences on Ultraproducts of Algebraic Structures[1]

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Abstract

We show that, for any similar algebraic structures A_i, $i \in I$, the ultraproduct of the $\wedge$-semilattices of all congruences of A_i, $i \in I$ is embeddable into the $\wedge$-semilattice of all congruences of the ultraproduct of A_i, $i \in I$. We apply this result for factor algebras and for ultrapowers.

Mathematics Subject Classification: 08A30, 03C20

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1 Introduction

Let I be a nonempty set and $\mathcal{D}$ an ultrafilter over I, that is, $\mathcal{D}$ is a subset of the power set $P(I)$ of I with the following properties:

(1) $I \in \mathcal{D}$ but $\emptyset \notin \mathcal{D}$;

(2) $A \cap B \in \mathcal{D}$ for any $A, B \in \mathcal{D}$;

(3) If $A \in \mathcal{D}$ and $C \subseteq I$ with $A \subseteq C$ then $C \in \mathcal{D}$;

(4) For each $A \subseteq I$, $A \in \mathcal{D}$ or $I \setminus A \in \mathcal{D}$.

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We note that (1) and (2) together imply that, for each $A \subseteq I$, exactly one of the sets A , $I \setminus A$ belongs to $\mathcal{D}$. Moreover, by Corollary 3.13 of Chapter IV of [1], condition (4) can be replaced by the following condition: (4*) For any $A, B \subseteq I$, the assumption $A \cup B \in \mathcal{D}$ implies $A \in \mathcal{D}$ or $B \in \mathcal{D}$.

Let $A_i = (A_i; \Omega)$, $i \in I$ be arbitrary similar algebraic structures and σ_i be a congruence on A_i, $i \in I$. Let $\Pi(A_i|i \in I)$ denote the direct product of A_i. Let $\mathbf{Con}(A_i)$ denote the $\wedge$-semilattice of all congruences of A_i, $i \in I$. Let $\sigma \in \Pi(\mathbf{Con}(A_i)|i \in I)$ be an element for which $\sigma(i) = \sigma_i$, $i \in I$. Let $\Pi(\sigma(i)|i \in I)$ denote the relation on the direct product $\Pi(A_i|i \in I)$ defined by $(a, b) \in \Pi(\sigma(i)|i \in I)$ if and only if $\{i \in I|(a(i), b(i)) \in \sigma(i)\} \in \mathcal{D}$. By the dual of Lemma 3 of §8 of [2], it is easy to see that $\Pi(\sigma(i)|i \in I)$ is a congruence on $\Pi(A_i|i \in I)$. Recall that this congruence is the ultraproduct congruence (denoted by $\mathcal{D}^*$) on $\Pi(A_i|i \in I)$ if $\sigma(i)$ is the identity relation on A_i for all $i \in I$. It is evident that $\mathcal{D}^* \subseteq \Pi(\sigma(i)|i \in I)$ for arbitrary congruences $\sigma(i)$ on A_i, $i \in I$. Then we can consider the congruence $\Pi(\sigma(i)|i \in I)/\mathcal{D}^*$ on the ultraproduct $\Pi_{\mathcal{D}}(A_i|i \in I) = \Pi(A_i|i \in I)/\mathcal{D}^*$ (see Definition 6.13 and Lemma 6.14 of Chapter II of [1]). This congruence will be denoted by $\Pi_{\mathcal{D}}(\sigma(i)|i \in I)$. In this paper we examine congruences $\Pi_{\mathcal{D}}(\sigma(i)|i \in I)$ on the ultraproduct $\Pi_{\mathcal{D}}(A_i|i \in I)$. Let the ultraproduct congruence on $\Pi(\mathbf{Con}(A_i)|i \in I)$ denoted by also $\mathcal{D}^*$. In Section 2, we show that $\Phi : \sigma/\mathcal{D}^* \mapsto \Pi_{\mathcal{D}}(\sigma(i)|i \in I)$ is an embedding of the $\wedge$-semilattice $\Pi_{\mathcal{D}}(\mathbf{Con}(A_i)|i \in I)$ into the $\wedge$-semilattice $\mathbf{Con}(\Pi_{\mathcal{D}}(A_i|i \in I)$, where $\sigma/\mathcal{D}^*$ denotes the ultraproduct congruence class on the direct product $\Pi(\mathbf{Con}(A_i)|i \in I)$ containing $\sigma = (\sigma(i))_{i \in I}$. In Section 3, we prove that the factor algebra $\Pi_{\mathcal{D}}(A_i|i \in I)/\Pi_{\mathcal{D}}(\sigma(i)|i \in I)$ is isomorphic to the ultraproduct $\Pi_{\mathcal{D}}(A_i/\sigma(i)|i \in I)$. In Section 4, we apply our results for the ultrapower $\Pi_{\mathcal{D}}(A|i \in I)$ of an algebraic structure A. Since A can be embedded into its ultrapower $\Pi_{\mathcal{D}}(A|i \in I)$ then, for an arbitrary family $\{\sigma(i)|i \in I\}$ of congruences on A, we can consider the restriction $\Pi_{\mathcal{D}}(\sigma(i)|i \in I)|A$ of the congruence $\Pi_{\mathcal{D}}(\sigma(i)|i \in I)$ on $\Pi_{\mathcal{D}}(A|i \in I)$ to A. We show that if $\{\sigma(i)|i \in I\}$ is an arbitrary family of congruences on A and $\mathcal{D} = \{K_j|j \in J\}$ then $\cup_{j \in J}(\cap_{k \in K_j} \sigma_k)$ is a congruence on A such that $\Pi_{\mathcal{D}}(\sigma(i)|i \in I)|A = \cup_{j \in J}(\cap_{k \in K_j} \sigma_k) = \vee_{j \in J}(\wedge_{k \in K_j} \sigma(k))$.

2 An embedding theorem

Theorem 1 *Let I be a non-empty set and $\mathcal{D}$ an ultrafilter over I. Then, for arbitrary similar algebraic structures A_i, $i \in I$, the mapping*

$$\Phi : \sigma/\mathcal{D}^* \mapsto \Pi_{\mathcal{D}}(\sigma(i)|i \in I)$$

is a $\wedge$-semilattice embedding of $\Pi_{\mathcal{D}}(\mathbf{Con}(A_i)|i \in I)$ into $\mathbf{Con}(\Pi_{\mathcal{D}}(A_i|i \in I))$.

Proof. Let $\alpha = (\alpha(i))_{i\in I}$ and $\beta = (\beta(i))_{i\in I}$ be arbitrary elements of the direct product $\Pi(\mathbf{Con}(A_i)|i \in I)$. First we show that $\alpha/\mathcal{D}^* = \beta/\mathcal{D}^*$ if and only if $\Pi_{\mathcal{D}}(\alpha(i)|i \in I) = \Pi_{\mathcal{D}}(\beta(i)|i \in I)$. By the Correspondence Theorem (Theorem 6.20 of Chapter II of [1]), it is sufficient to show that $\alpha/\mathcal{D}^* = \beta/\mathcal{D}^*$ if and only if $\Pi(\alpha(i)|i \in I) = \Pi(\beta(i)|i \in I)$.

Assume $\alpha/\mathcal{D}^* = \beta/\mathcal{D}^*$. Then $A = \{i \in I : \alpha(i) = \beta(i)\} \in \mathcal{D}$. If $(a, b) \in \Pi(\alpha(i)|i \in I)$ for some elements $a = (a(i))_{i\in I}$ and $b = (b(i))_{i\in I}$ of $\Pi(A_i|i \in I)$ then $B = \{i \in I : (a(i), b(i)) \in \alpha(i)\} \in \mathcal{D}$ and so, for every $i \in A \cap B \in \mathcal{D}$, $(a(i), b(i)) \in \beta(i)$ from which it follows that $\{i \in I : (a(i), b(i)) \in \beta(i)\} \in \mathcal{D}$. Thus $(a, b) \in \Pi(\beta(i)|i \in I)$. Hence $\Pi(\alpha(i)|i \in I) \subseteq \Pi(\beta(i)|i \in I)$. We can prove $\Pi(\beta(i)|i \in I) \subseteq \Pi(\alpha(i)|i \in I)$ in a similar way. Thus $\Pi(\alpha(i)|i \in I) = \Pi(\beta(i)|i \in I)$.

To prove the converse, assume $\Pi(\alpha(i)|i \in I) = \Pi(\beta(i)|i \in I)$. We show that $K = \{i \in I|\alpha(i) = \beta(i)\} \in \mathcal{D}$. Assume, in an indirect way, that $K \notin \mathcal{D}$. Then $I \setminus K \in \mathcal{D}$.

First assume $R = \{i \in I : \alpha(i) \subset \beta(i)\} \in \mathcal{D}$. Then $(I \setminus R) \notin \mathcal{D}$. For $i \in R$, let (a_i, b_i) be an element of $A_i \times A_i$ such that $(a_i, b_i) \in \beta(i)$ and $(a_i, b_i) \notin \alpha(i)$. For the indexes $i \in I \setminus R$, let $(a_i, b_i) \in A_i \times A_i$ be arbitrary. Let a and b be the elements of $\Pi(A_i|i \in I)$ for which $a(i) = a_i$ and $b(i) = b_i$. As $(a(i), b(i)) \in \beta(i)$ for every $i \in R \in \mathcal{D}$, we have $\{i \in I : (a(i), b(i)) \in \beta(i)\} \in \mathcal{D}$ and so $(a, b) \in \Pi(\beta(i)|i \in I)$. We show that $(a, b) \notin \Pi(\alpha(i)|i \in I)$. Assume, in an indirect way, that $(a, b) \in \Pi(\alpha(i)|i \in I)$. Then $V = \{i \in I : (a(i), b(i)) \in \alpha(i)\} \in \mathcal{D}$. As $V \subseteq I \setminus R$, we have $I \setminus R \in \mathcal{D}$ and so $R \notin \mathcal{D}$ which is a contradiction. Hence $(a, b) \notin \Pi(\alpha(i)|i \in I)$. This implies $\Pi(\alpha(i)|i \in I) \neq \Pi\beta(i)|i \in I)$ which is a contradiction.

Next consider the case when $R = \{i \in I : \alpha(i) \subset \beta(i)\} \notin \mathcal{D}$. Then $\{i \in I : \alpha(i) \subseteq \beta(i)\} = K \cup R \notin \mathcal{D}$. For every $i \in I \setminus (K \cup R) \in \mathcal{D}$, there are elements a_i, b_i of A_i such that $(a_i, b_i) \in \alpha(i), \quad (a_i, b_i) \notin \beta(i)$. For an index $i \in K \cup R$, let the elements $a_i, b_i \in A_i$ be arbitrary. Let a and b be the elements of $\Pi(A_i|i \in I)$ for which $a(i) = a_i$ and $b(i) = b_i$. It is clear that $(a, b) \in \Pi(\alpha(i)|i \in I)$. We show that $(a, b) \notin \Pi_{\mathcal{D}}(\beta(i)|i \in I)$. Assume, in an indirect way, that $(a, b) \in \Pi(\beta(i)|i \in I)$. Then $\{i \in I : (a(i), b(i)) \in \beta(i)\} \in \mathcal{D}$. As $\{i \in I : (a(i), b(i)) \in \beta(i)\} \subseteq K \cup R$, we have $K \cup R \in \mathcal{D}$ which is a contradiction. Thus $(a, b) \notin \Pi(\beta(i)|i \in I)$. Hence $\Pi(\alpha(i)|i \in I) \neq \Pi(\beta(i)|i \in I)$ which is a contradiction. We got a contradiction in both cases. Thus $K = \{i \in I : \alpha(i) = \beta(i)\} \in \mathcal{D}$. Consequently $\alpha/\mathcal{D}^* = \beta/\mathcal{D}^*$.

Thus

$$\Phi : \sigma/\mathcal{D}^* \mapsto \Pi_{\mathcal{D}}(\sigma(i)|i \in I)$$

is a well defined injective mapping. We show that Φ is a $\wedge$-homomorphism. For arbitrary $a, b \in \Pi(A_i|i \in I)$, let $I_{\alpha\wedge\beta} = \{i \in I : (a(i), b(i)) \in (\alpha\wedge\beta)(i) = \alpha(i)\wedge \beta(i)\}$, $I_\alpha = \{i \in I : (a(i), b(i)) \in \alpha(i)\}$ and $I_\beta = \{i \in I : (a(i), b(i)) \in \beta(i)\}$. It is easy to see that $I_{\alpha\wedge\beta} = I_\alpha \cap I_\beta$. As $\mathcal{D}$ is an ultrafilter, $I_\alpha \cap I_\beta \in \mathcal{D}$

if and only if $I_\alpha \in \mathcal{D}$ and $I_\beta \in \mathcal{D}$. From the above it follows that $(a,b) \in \Pi((\alpha \wedge \beta)(i)|i \in I)$ if and only if $(a,b) \in \Pi(\alpha(i)|i \in I) \wedge \Pi(\beta(i)|i \in I)$. Hence $\Pi((\alpha \wedge \beta)(i)|i \in I) = \Pi(\alpha(i)|i \in I) \wedge \Pi(\beta(i)|i \in I)$. By the Correspondence Theorem (Theorem 6.20 of Chapter II of [1]), we have $\Pi_{\mathcal{D}}((\alpha \wedge \beta)(i)|i \in I) = \Pi_{\mathcal{D}}(\alpha(i)|i \in I) \wedge \Pi_{\mathcal{D}}(\beta(i)|i \in I)$. Then

$$\Phi(\alpha/\mathcal{D}^* \wedge \beta/\mathcal{D}^*) = \Phi((\alpha \wedge \beta)/\mathcal{D}^*) = \Pi_{\mathcal{D}}((\alpha \wedge \beta)(i)|i \in I) =$$

$$= \Pi_{\mathcal{D}}(\alpha(i)|i \in I) \wedge \Pi_{\mathcal{D}}(\beta(i)|i \in I) = \Phi(\alpha/\mathcal{D}^*) \wedge \Phi(\beta/\mathcal{D}^*),$$

and so Φ is a $\wedge$-homomorphism. Thus Φ is an embedding of the $\wedge$-semilattice $\Pi_{\mathcal{D}}(\mathbf{Con}(A_i)|i \in I)$ into the $\wedge$-semilattice $\mathbf{Con}(\Pi_{\mathcal{D}}(A_i|i \in I))$. □

3 An isomorphism theorem

Theorem 2 *Let I be a non-empty set and $\mathcal{D}$ an ultrafilter over I. Then, for arbitrary similar algebraic structures A_i, $i \in I$ and an arbitrary congruence $\sigma(i)$ on A_i $(i \in I)$, $\Pi_{\mathcal{D}}(A_i|i \in I)/\Pi_{\mathcal{D}}(\sigma(i)|i \in I) \cong \Pi_{\mathcal{D}}(A_i/\sigma(i)|i \in I)$.*

Proof. As $\Pi_{\mathcal{D}}(A_i|i \in I)/\Pi_{\mathcal{D}}(\sigma(i)|i \in I) \cong \Pi(A_i|i \in I)/\Pi(\sigma(i)|i \in I)$ by the Second Isomorphic Theorem (Theorem 6.15 of Chapter II of [1]), it is sufficient to show that $\Pi(A_i|i \in I)/\Pi(\sigma(i)|i \in I) \cong \Pi_{\mathcal{D}}(A_i/\sigma(i)|i \in I)$. For an element $a = (a(i))_{i\in I}$ of the direct product $\Pi(A_i|i \in I)$, let $\Delta(a) = ([a(i)]_{\sigma(i)})_{i\in I}/\mathcal{D}^*$, where $[a(i)]_{\sigma(i)}$ denotes the $\sigma(i)$-class of A_i containing $a(i)$. It is obvious that $\Delta : \Pi(A_i|i \in I) \mapsto \Pi_{\mathcal{D}}(A_i/\sigma(i)|i \in I)$ is a well defined surjective mapping. Let $a_j = (a_j(i))_{i\in I}$, $(j = 1, \ldots, n)$ be arbitrary elements of $\Pi(A_i|i \in I)$. If $\omega_n \in \Omega$ is an n-ary operation then

$$\omega_n(\Delta(a_1), \ldots, \Delta(a_n)) = \omega_n(([a_1(i)]_{\sigma(i)})_{i\in I}/\mathcal{D}^*, \ldots, ([a_n(i)]_{\sigma(i)})_{i\in I}/\mathcal{D}^*) =$$

$$= \omega_n(([a_1(i)]_{\sigma(i)})_{i\in I}, \ldots, ([a_n(i)]_{\sigma(i)})_{i\in I})/\mathcal{D}^* =$$

$$= (\omega_n([a_1(i)]_{\sigma(i)}, \ldots, [a_n(i)]_{\sigma(i)}))_{i\in I}/\mathcal{D}^* = ([\omega_n(a_1(i), \ldots, a_n(i))]_{\sigma(i)})_{i\in I}/\mathcal{D}^* =$$

$$= (([\omega_n(a_1, \ldots, a_n)(i)]_{\sigma(i)})_{i\in I})/\mathcal{D}^* = \Delta(\omega_n(a_1, \ldots, a_n)).$$

Thus Δ is a homomorphism. For elements $a = (a(i))_{i\in I}$ and $b = (b(i))_{i\in I}$ of $\Pi(A_i|i \in I)$, $\Delta(a) = \Delta(b)$ if and only if $([a(i)]_{\sigma(i)})_{i\in I}/\mathcal{D}^* = ([b(i)]_{\sigma(i)})_{i\in I}/\mathcal{D}^*$, that is, $\{i \in I | [a(i)]_{\sigma(i)} = [b(i)]_{\sigma(i)}\} \in \mathcal{D}$. This last condition is equivalent to the condition that $\{i \in I | (a(i), b(i)) \in \sigma(i)\} \in \mathcal{D}$, that is, $(a,b) \in \Pi(\sigma(i)|i \in I)$. Thus the kernel of Δ is $\Pi(\sigma(i)|i \in I)$. Our assertion follows from the Homomorphism Theorem (Theorem 6.12 of Chapter II of [1]). □

4 Ultrapowers

Let A be an algebraic structure. If $\mathcal{D}$ is an ultrafilter over a non-empty set I, we can consider the ultrapower of A modulo $\mathcal{D}$ as the ultraproduct $\Pi_{\mathcal{D}}(A_i|i \in I)$, where $A_i = A$ for all $i \in I$. For an arbitrary element $a \in A$, let $\xi(a)$ denote the $\mathcal{D}^*$-class of the direct product $\Pi(A|i \in I)$ which contains the constant function with value a. It is known (Lemma 2.10 of Chapter V of [1]) that $\xi : a \mapsto \xi(a)$ is an embedding (the natural embedding) of A into the ultrapower $\Pi_{\mathcal{D}}(A|i \in I)$. Identify A and $\xi(A)$. Let $\sigma(i)$, $i \in I$ be arbitrary congruences on A. Let $\Pi_{\mathcal{D}}(\sigma(i)|i \in I)|A$ denote the restriction of the congruence $\Pi_{\mathcal{D}}(\sigma(i)|i \in I)$ to A. It is clear that, for some $a, b \in A$, $(a, b) \in \Pi_{\mathcal{D}}(\sigma(i)|i \in I)|A$ if and only if $\{i \in I|(a, b) \in \sigma(i)\} \in \mathcal{D}$.

Theorem 3 *Let I be a non-empty set and $\mathcal{D} = \{K_j|j \in J\}$ an ultrafilter over I. Then, for an arbitrary family $\{\sigma(i)|i \in I\}$ of congruences $\sigma(i)$ on an algebraic structure A, $\cup_{j\in J}(\cap_{k\in K_j}\sigma(k))$ is a congruence on A such that $\Pi_{\mathcal{D}}(\sigma(i)|i \in I)|_A = \cup_{j\in J}(\cap_{k\in K_j}\sigma(k)) = \vee_{j\in J}(\wedge_{k\in K_j}\sigma(k))$.*

Proof. For a couple (a, b) of elements a and b of $A \subseteq \Pi_{\mathcal{D}}(A|i \in I)$, let $K_{a,b} = \{i \in I|(a, b) \in \sigma(i)\}$. Assume $(a, b) \in \Pi_{\mathcal{D}}(\sigma(i)|i \in I)$ for some $a, b \in A$. Then $K_{a,b} \in \mathcal{D}$ and so $(a, b) \in \cap_{k\in K_{a,b}}\sigma(k)$ from which it follows that $(a, b) \in \cup_{j\in J}(\cap_{k\in K_j}\sigma(k))$. Thus $\Pi_{\mathcal{D}}(\sigma(i)|i \in I)|_A \subseteq \cup_{j\in J}(\cap_{k\in K_j}\sigma(k))$.

Conversely, assume $(a, b) \in \cup_{j\in J}(\cap_{k\in K_j}\sigma(k))$ for some $a, b \in A$. Then there is an index $j \in J$ such that $(a, b) \in (\cap_{k\in K_j}\sigma(k))$. Thus $K_j \subseteq K_{a,b}$ and so $K_{a,b} \in \mathcal{D}$. Hence $(a, b) \in \Pi_{\mathcal{D}}(\sigma(i)|i \in I)|_A$. Thus $\cup_{j\in J}(\cap_{k\in K_j}\sigma(k) \subseteq \Pi_{\mathcal{D}}(\sigma(i)|i \in I)|_A$. Consequently $\Pi_{\mathcal{D}}(\sigma(i)|i \in I)|_A = \cup_{j\in J}(\cap_{k\in K_j}\sigma(k))$. Thus $\cup_{j\in J}(\cap_{k\in K_j}\sigma(k))$ is a congruence on A. It is obvious that $\cup_{j\in J}(\cap_{k\in K_j}\sigma(k))$ is a common upper bound of the congruences $\sigma_j = \cap_{k\in K_j}\sigma(k)$, $j \in J$. Moreover, for every common upper bound β of σ_j, $j \in J$, we have $\cup_{j\in J}(\cap_{k\in K_j}\sigma(k)) \subseteq \beta$ and so $\cup_{j\in J}(\cap_{k\in K_j}\sigma(k)) = \vee_{j\in J}(\wedge_{k\in K_j}\sigma(k))$. Hence $\Pi_{\mathcal{D}}(\sigma(i)|i \in I)|_A = \cup_{j\in J}(\cap_{k\in K_j}\sigma(k)) = \vee_{j\in J}(\wedge_{k\in K_j}\sigma(k))$. □

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