New Homomorphism between Grassmannian and Infinitesimal Complexes

M. Khalid¹, Javed Khan

Department of Mathematical Sciences
Federal Urdu University of Arts, Sciences & Technology
Karachi-75300, Pakistan

Azhar Iqbal

Department of Basic Sciences
Dawood University of Engineering & Technology
Karachi-74800, Pakistan

Copyright © 2016 M. Khalid, Javed Khan and Azhar Iqbal. This article is distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this paper, new homomorphisms are defined between the Grassmannian complex of projective configuration and a variant of Cathlineau’s (infinitesimal) polylogarithmatic complex. Then we investigate these homomorphisms for Weight 2,3 and 4. Weight 4 is of greater significance because until now, researchers have investigated different homomorphism between Grassmannian and infinitesimal polylogarithmatic complex for weights 2 and 3 only. In this new homomorphism, the dimensions of the points remain unchanged while weight increase. The associative diagrams have been also proven to be commutative and bi-complex.

Mathematics Subject Classification: 19L20, 22E10, 11G55

Keywords: Homomorphism, Grassmannian, Variant of Cathlineau, Polylogarithm

¹Corresponding author
1 Introduction

The Grassmannian complex of projective configurations was first defined by Suslin [1] and morphisms between Grassmannian and Bloch-Suslin complex [2] for Di-logarithm weight $n = 2$ by Goncharov [3, 4, 5]. For this geometry, Goncharov’s uses Bloch group $B_2(F)$. Goncharov’s also uses the duality of configurations in order to prove (projected seven-term) functional equation for the trilogarithmic group $B_3(F)$ and verifies that a Complex forms among Grassmannian and Goncharov’s Complexes in weight 3 is commutative and Bi complex (see [3]). Goncharov introduces a triple-ratio (call it generalized cross-ratio) which is one of the the most important ingredient of his work. Cathelineau [2, 6] observed an alternative of Goncharov’s complexes in the additive (both infinitesimal and tangential) setting called Cathelineau’s complexes. Siddiqui [7] found that triple-ratio and indicated that it should be written as the ratio of two projected cross-ratios. Siddiqui [7, 8] introduced variant of Cathelineau’s complexes and describe their relations through morphisms with Grassmannian complexes of configurations. Siddiqui also found morphisms between Grassmannian complexes and Cathelineau’s infinitesimal complexes up to weight 2 and 3 and showed that the associated diagrams are commutative and bi-complex.

We have tried to find new homomorphisms for the geometry of the Grassmannian subcomplex and the variant of Cathelineau’s complex (infinitesimal) which are analogues of Goncharov’s complexes. For weight $n = 2, 3$ and 4, we found that by defining these new morphisms, the associated diagrams become commutative and bi-complex. We also observed that the dimension of points will remain unchanged during the defining of our geometry between the subcomplexes of Grassmannian and infinitesimal complexes.

2 Preliminary and Background

2.1 Grassmannian Complex

Consider the following Grassmannian complex

\[
\begin{array}{cccccc}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\downarrow p & \downarrow p & \downarrow p & \downarrow p & \downarrow p & \downarrow p \\
G_{n+5}(n + 2) & G_{n+4}(n + 2) & G_{n+3}(n + 2) & G_{n+2}(n + 1) & G_{n+1}(n) \\
\downarrow p & \downarrow p & \downarrow p & \downarrow p & \downarrow p \\
G_{n+4}(n + 1) & G_{n+3}(n + 1) & G_{n+2}(n + 1) & G_{n+1}(n) \\
\downarrow p & \downarrow p & \downarrow p & \downarrow p \\
G_{n+3}(n) & G_{n+2}(n) & G_{n+1}(n) \\
\end{array}
\]
where $G_{n+1}(n)$ is a free abelian group generated by $(n+1)$-vectors of dimension $n$ and $d$ is called differential map and is given as

$$d : (q_0, \ldots, q_n) \mapsto \sum_{i=0}^{n} (-1)^i(q_0, \ldots, \hat{q}_i, \ldots, q_n)$$  \hspace{1cm} (1)$$

(see [1]) and $p$ is another differential morphism called projection map given by

$$p : (q_0, \ldots, q_n) \mapsto \sum_{i=0}^{n} (-1)^i(q_i/q_0, \ldots, \hat{q}_i, \ldots, q_n)$$  \hspace{1cm} (2)$$

**Lemma 2.1.** The diagram (A) is bi-complex, i.e. $d \circ d = p \circ p = 0$

**Proof:** For proof (see [1])

**Lemma 2.2.** The diagram (A) is commutative, i.e. $d \circ p = p \circ d$

**Proof:** For proof (see [1])

### 2.2 Polylogarithmic Groups

Let $Z[P^1_F]$ is $Z-$ module generated by $[x] \in P^1_F$ [10], now onwards we will use $F$ as a field and $F^{**} = F - \{0, 1\}$. The group $B_1(F)$ is called scissor congruence group [11] which is a quotient of $Z[F^{**}]$ and the subgroup generated by Abel’s famous five term relation, $[x] - [y] + [\frac{y}{x}] - [\frac{1-y}{1-x}] + [\frac{1-y^{-1}}{1-x^{-1}}]$ where $x \neq y, x, y \neq 0, 1$ (see[3])

#### 2.2.1 Bloch-Suslin and Goncharov’s Polylog Complexes

**Weight 1** We define subgroup $R_1(F) \subset Z[P^1_F]$ generated by $[xy] - [x] - [y]$ where $x, y \in F^\times$ such that $R_1(F) = B_1(F)$ the function $\delta : B_1(F) \rightarrow F^\times, [x] \rightarrow x$ is an isomorphism (see[3]): such that $B_1(F) = F^\times$

**Weight 2** Let us introduce subgroup $R_2(F) \subset Z[P^1_F]$ (see[3]) generated by

$$\sum_{i=0}^{4} (-1)^i[cr[q_0, \ldots, \hat{q}_i, \ldots, q_4]$$  \hspace{1cm} (3)$$

where $cr = \frac{(q_0,q_3)(q_1,q_2)}{(q_0,q_2)(q_1,q_3)}$ is a five term relation of the cross ratio. We introduce a map $\delta_2 : Z[P^1_F/\{0, 1, \infty\}] \rightarrow \wedge^2 F^\times$, defined as $[x] \rightarrow (1 - x) \wedge x$, it has been proved that $\delta_2(R_2(F)) = 0$ (see[3]). Now we define group $B_2(F)$ the factor group of $Z[P^1_F/\{0, 1, \infty\}]//R_2(F)$, now we will introduce Bloch-Suslin complexes given by

$$0 \xrightarrow{\delta} B_2(F) \xrightarrow{\delta} \wedge^2 F^\times \xrightarrow{\delta} 0$$

where $\delta$ is an induced map and is defined as $\delta : [x] \rightarrow (1 - x) \wedge x$
Weight 3  As defined in [3]

\[ r_3(q_0, ..., q_6) = \text{Alt}_6 \frac{\Delta(q_0, q_1, q_3) \Delta(q_1, q_2, q_4) \Delta(q_2, q_0, q_3)}{\Delta(q_0, q_1, q_4) \Delta(q_1, q_2, q_3) \Delta(q_2, q_0, q_3)} \]

it is a triple cross ratio \( \in Z[P^1_F] \). Now we take subgroup \( R_3(F) \in Z[P^1_F] \) [3] is defined as

\[ R_3(F) = \sum_{i=0}^{6} (-1)^i r_3(q_0, ..., \hat{q}_i, ..., q_6)/(q_0, ..., \hat{q}_i, ..., q_6) \in C_0(P^2_F) \]  

(4)

Which is a seven term relation of the triple ratio. Gonchrove define \( \mathcal{B}_3(F) \) which is quotient subgroup \( Z[P^1_F]/R_3(F) \), then he defines the Gonchrove’s complex in weight \( n = 3 \) given by

\[ \mathcal{B}_3(F) \xrightarrow{\delta} \mathcal{B}_2(F) \otimes F^x \xrightarrow{\delta} \wedge^3 F^x \]

Weight \( n \geq 3 \) Gonchrove defined complex for \( \mathcal{B}_n(F) = Z[P^1_F]/R_n(F) \) where \( R_n(F) \) is a kernel of the map \( \delta_n : Z[P^1_F] \to \mathcal{B}_{n-1}(F) \otimes F^x \) given as

\[ \mathcal{B}_n(F) \xrightarrow{\delta} \mathcal{B}_{n-1}(F) \otimes F^x \xrightarrow{\delta} \mathcal{B}_{n-2}(F) \otimes \wedge^2(F) \xrightarrow{\delta} \ldots \xrightarrow{\delta} \mathcal{B}_2(F) \otimes \wedge^{n-2}(F) \xrightarrow{\delta} \frac{\wedge^n F^x}{2 - \text{torsion}} \]   

(5)

2.3 Cathelineau’s Complexes

Cathelineau [6] has defined the F- Vector Space, which is an infinitesimal form of Goncharov’s groups \( \mathcal{B}_n(F) \), as follows

1. \( \beta_1(F) = F \)

2. \( \beta_2(F) = F[F**]_{r_2(F)} \), where \( r_2(F) \) is the kernel of \( \partial_2 : F[F**] \to F \otimes F^x \) defined by \([x] \mapsto x \otimes x + (1-x) \otimes (1-x)\). Cathelineau showed that \( r_2(F) \) is a sub-vector space generated by four elements \([x] - [y] + [x] \frac{y}{x} + (1-x) \frac{1-y}{1-x}] \)

and we obtain a sub complex \( \beta_2(F) \xrightarrow{\partial} F \otimes_F F^x \). where \( \partial \) is a induced map defined as

\[ \partial : \langle x \rangle_2 \mapsto x \otimes x + (1-x) \otimes (1-x) \]  

(6)

The functional equation in \( \beta_2(F) \)

1. The two term relation, \( \langle a \rangle_2 = \langle 1-a \rangle_2 \)
2. The inversion relation, \( \langle a \rangle_2 = -r \langle \frac{1}{2} \rangle_2 \)
3. The four term relation relation, \( \langle a \rangle_2 - \langle b \rangle_2 + a \langle \frac{b}{a} \rangle_2 + (1-a) \langle \frac{1-b}{1-a} \rangle_2 = 0 \)
4. The distribution relation relation, \( \langle a \rangle_2^m = \sum_{\zeta^m=1}^{1} \frac{1}{1-\zeta} \langle \zeta a \rangle_2 \)
If \( r_n(F) \) is a kernel of the map defined by \( \delta_n : F[F^{**}] \to \beta_{n-1} \otimes F^x \oplus F \otimes B_{n-1}(F) \) (see [6]). Now we take \( \beta_n(F) \) the factor subgroup defined as

\[
\beta_n(F) = \frac{F[F^{**}]}{r_n(F)} \tag{7}
\]

The Cathelineau chain complex [6] for \( \beta_n(F) \) and Bloch group [9] \( B_n(F) \) is given by

\[
\beta_n(F) \xrightarrow{\partial_n} \beta_{n-1}(F) \otimes F^x \otimes B_{n-1}(F) \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_3} \beta_2(F) \otimes \wedge^{n-2} F^x \xrightarrow{\partial_2} F \otimes \wedge^{n-1} F^x \tag{B}
\]

where \( \partial_n \) is defined as

\[
\partial_n : [r] \mapsto \langle r \rangle_{n-1} \otimes r + (-1)^{n-1}(1 - r) \otimes [r]_{n-1} \tag{8}
\]

**Lemma 2.3.** \( \partial_{m-1} \circ \partial_m = 0 \) (see [6])

### 2.4 Variant of Cathelineau’s Complex

Define an \( F \)-Vector space where \( D \in \text{Der}_Z(F,F) \) is absolute derivation [7]. Consider \( g_D : Z[F] \to F[F^{**}] \) defined by induced map \([x] = \frac{D(x)}{x(1-x)}[x] \). Let \( \beta^D_n(F) \) is vector space generated by \( x^D, x \in F^{**} \) subject to five term relation \( x^D_2 - y^D_2 + \frac{y^D}{x^2} - \frac{1-y^D}{1-x^2} + 1 \) (see[7]). Where \( x^D = \frac{D(x)}{x(1-x)}[x] \). Now we have \( \partial^D : F[F^{**}] \to F \otimes F \) with \( \partial^D : x^D \to -D \log(1-x) \otimes x + D \log(x) \otimes (1-x) \) where \( D \log(1-x) = \frac{D(1-x)}{(1-x)} \). We can write variant of Cathelineau’s Complex with \( \beta^D_n(F) \) it is factor subgroup and is defined as \( F[F^{**}]/(\rho_2) \), where \( \rho_2 \) is subgroup generated by five term relation \( x^D_2 - y^D_2 + \frac{y^D}{x^2} - \frac{1-y^D}{1-x^2} + 1 \) where \( x \neq y, 1-x \neq 0 \) with \( x^D_2 = \frac{D(x)}{x(1-x)}[x] \). We define Complex

\[
\beta^D_2(F) \xrightarrow{\partial^D} F \otimes Z F^x
\]

where \( \partial^D : x^D_2 \to -D \log(1-x) \otimes x + D \log(x) \otimes (1-x) \)

The functional equation in \( \beta^D_2(F) \)

1. The two term relation, \( x^D_2 = -1 - x^D_2 \)
2. The inversion relation, \( x^D_2 = -\frac{1}{x^2} \)
3. The five term relation relation, \( x^D_2 - y^D_2 + \frac{y_D}{x^2} - \frac{1-y_D}{1-x^2} + \frac{1-y^{-1}D}{1-x^{-1}2} = 0 \)

We take subgroup \( \beta^D_3(F) = F[F^{**}]/(\rho_3) \), \( \rho_3 \) is subgroup generated by

\[
\sum_{i=0}^{6} (-1)^i r_3(q_0, \ldots, q_i, \ldots, q_6)_2^D \tag{9}
\]
seven term relation of triple ratio. Similarly subgroup \( \beta_n^D(F) = F[F^{**}]/(\rho_n) \), \( \rho_n \) is a kernel of the map defined by \( \partial_n^D : F[F^{**} \to \beta_{n-1}^D \otimes F^x \oplus F \otimes B_{n-1}(F) \). Finally Siddiqui [7] defines the variant of Cathelineau’s Complex for \( \beta_n^D(F) \) and \( B_n(F) \) given by

\[
\beta_n^D(F) \xrightarrow{\partial_n^D} \beta_{n-1}^D(F) \otimes F^x \otimes B_{n-1}(F) \xrightarrow{\partial_{n-1}^D} \beta_{n-2}^D(F) \otimes \cdots \otimes B_1(F) \otimes F^{x-n} \otimes B_0(F) \xrightarrow{\partial_0^D} F \otimes F^x \otimes B_0(F) \xrightarrow{\cdots} \ (C)
\]

\( \partial_n^D \) is a map given by

\[
\partial_n^D : [a]_n^D \mapsto a_{n-1} \otimes a + (-1)^{n-1} D \log(a) \otimes [a]_{n-1} \quad (10)
\]

**Lemma 2.4.** \( \partial_{n-1}^D \circ \partial_n^D = 0 \) (see[7])

### 3 Geometry between Grassmannian and Infinitesimal Complexes

#### 3.1 Weight 2 (Dilogarithm)

Let us connect the subcomplexes of Grassmannian and infinitesimal in weight-2 given below

\[
\begin{array}{ccc}
G_6(3) & \xrightarrow{p} & G_5(2) \\
\downarrow d & & \downarrow d & & \downarrow d \\
G_5(3) & \xrightarrow{p} & G_4(2) & \xrightarrow{p} & G_3(1) \\
\downarrow h_1^2 & & \downarrow h_2^1 \\
\beta_2^D(F) & \xrightarrow{\partial^D} & F \otimes F^x \\
\end{array}
\]

where

\[
h_2^2 : (q_0, q_1, q_2) \rightarrow \sum_{i=0}^{2} (-1)^i \frac{D \triangle(q_i)}{\triangle(q_i)} \otimes \frac{\triangle(q_{i+1})}{\triangle(q_{i+2})} \quad (mod \ 3) \quad (11)
\]

and

\[
h_1^2(q_0, q_1, q_2, q_3) = \sum_{i=0}^{3} (-1)^i cr(q_0, \ldots, q_3)_2^D \quad (12)
\]

where \( cr(q_0, \ldots, q_3) = \frac{\triangle(q_0, q_3) \triangle(q_1, q_2)}{\triangle(q_0, q_2, q_3) \triangle(q_1, q_3)} \). The morphism \( h_2^2 \) is well defined (see[7]) so first we prove that the morphism \( h_0^2 \) is well defined

**Lemma 3.1.** \( h_0^2 \) is independent of volume form by vectors in \( V_2 \).

**Proof:**

The proof of this is the same as Lemma (3.1.1) of [7]. Let \( h_0^2(q_0, q_1, q_2) \) can be written as

\[
h_0^2(q_0, q_1, q_2) = \frac{D \triangle(q_1)}{\triangle(q_0)} \otimes \frac{\triangle(q_2)}{\triangle(q_1)} - \frac{D \triangle(q_2)}{\triangle(q_2)} \otimes \frac{\triangle(q_0)}{\triangle(q_1)} + \frac{D \triangle(q_0)}{\triangle(q_0)} \otimes \frac{\triangle(q_1)}{\triangle(q_2)}
\]
so if we change volume \( V = \alpha V \) where \( \alpha \in \text{field } F \) then the right side will remain unchanged so \( h_0^2 \) is independent of volume form by vectors in \( V_2 \).

**Lemma 3.2.** \( h_0^2 \circ p \) is independent of length of vectors in \( V_2 \).

*Proof:* The proof of this is the same as Lemma (3.1.2) of [7]. Let \( h_0^2 \circ p(q_0, q_1, q_2, q_3) \) can be written as

\[
h_0^2 \circ p(q_0, q_1, q_2, q_3) = \frac{D \triangle(q_0/q_1)}{\triangle(q_0/q_1)} \otimes \frac{\triangle(q_0/q_2)}{\triangle(q_0/q_3)} - \frac{D \triangle(q_0/q_2)}{\triangle(q_0/q_3)} \otimes \frac{\triangle(q_0/q_1)}{\triangle(q_0/q_3)} + \frac{D \triangle(q_0/q_3)}{\triangle(q_0/q_3)} \otimes \frac{\triangle(q_0/q_1)}{\triangle(q_0/q_3)} + \frac{D \triangle(q_0/q_3)}{\triangle(q_0/q_3)} \otimes \frac{\triangle(q_0/q_1)}{\triangle(q_0/q_3)} + \frac{D \triangle(q_0/q_3)}{\triangle(q_0/q_3)} \otimes \frac{\triangle(q_0/q_1)}{\triangle(q_0/q_3)} + \frac{D \triangle(q_0/q_3)}{\triangle(q_0/q_3)} \otimes \frac{\triangle(q_0/q_1)}{\triangle(q_0/q_3)} + \frac{D \triangle(q_0/q_3)}{\triangle(q_0/q_3)} \otimes \frac{\triangle(q_0/q_1)}{\triangle(q_0/q_3)} \]

(13)

therefore, if we change the length of vector like \( (q_0, q_1, q_2, q_3) = \gamma(q_0, q_1, q_2, q_3) \) where \( \gamma \in \text{field } F \) and use the homomorphism property \( D \log(\gamma q_0) = \gamma D \log(q_0) \) then the difference will be zero. Therefore \( h_0^2 \) is independent of the length of vectors in \( V_2 \).

**Lemma 3.3.** The diagrams given below are bi-complex.

\[
(i) \quad G_4(1) \xrightarrow{d} G_3(1) \xrightarrow{h_0^2} F \otimes F^\times
\]

\[
(ii) \quad G_5(2) \xrightarrow{d} G_4(2) \xrightarrow{h_0^2} \beta_2^D(F)
\]

*(i)Proof:* let \( (q_0, q_1, q_2, q_3) \) are four points of dimension one \( \in G_4(1) \), apply map \( d \) we get

\[
d(q_0, q_1, q_2, q_3) = (q_1, q_2, q_3) - (q_0, q_2, q_3) + (q_0, q_1, q_3) - (q_0, q_1, q_2)
\]
now apply \( h_0^2 \)

\[
h_0^2 \circ d(q_0, q_1, q_2, q_3) = \frac{D\Delta(q_1)}{\Delta(q_1)} \otimes \frac{\Delta(q_2)}{\Delta(q_2)} \frac{D\Delta(q_2)}{\Delta(q_2)} \otimes \frac{\Delta(q_1)}{\Delta(q_1)} + \frac{D\Delta(q_3)}{\Delta(q_3)} \otimes \frac{\Delta(q_1)}{\Delta(q_1)} - \frac{D\Delta(q_0)}{\Delta(q_0)} \otimes \frac{\Delta(q_2)}{\Delta(q_2)} \frac{D\Delta(q_2)}{\Delta(q_2)} \otimes \frac{\Delta(q_0)}{\Delta(q_0)} + \frac{D\Delta(q_0)}{\Delta(q_0)} \otimes \frac{\Delta(q_1)}{\Delta(q_1)} - \frac{D\Delta(q_0)}{\Delta(q_0)} \otimes \frac{\Delta(q_1)}{\Delta(q_1)} + \frac{D\Delta(q_0)}{\Delta(q_0)} \otimes \frac{\Delta(q_1)}{\Delta(q_1)} - \frac{D\Delta(q_0)}{\Delta(q_0)} \otimes \frac{\Delta(q_1)}{\Delta(q_1)}
\]

\[= 0\]

(ii) Proof

let \((q_0, q_1, q_2, q_3, q_4) \in G_5(2)\) apply map \(d\) we get

\[
d(q_0, q_1, q_2, q_3, q_4) = (q_1, q_2, q_3, q_4) - (q_0, q_2, q_3, q_4) + (q_0, q_1, q_3, q_4)
\]

\[- (q_0, q_1, q_2, q_4) + (q_0, q_1, q_2, q_3)\]

now apply \( h_1^2 \)

\[
h_1^2 \circ d(q_0, q_1, q_2, q_3, q_4) = h_1^2[(q_1, q_2, q_3, q_4) - (q_0, q_2, q_3, q_4) + (q_0, q_1, q_3, q_4)
\]

\[- (q_0, q_1, q_2, q_4) + (q_0, q_1, q_2, q_3)]\]

\[= cr(q_1, q_2, q_3, q_4)_2^D - cr(q_0, q_2, q_3, q_4)_2^D
\]

\[+ cr(q_0, q_1, q_3, q_4)_2^D - cr(q_0, q_1, q_2, q_4)_2^D
\]

\[+ cr(q_0, q_1, q_2, q_3)_2^D\]

which is a five term relation in \( \beta_2^D(F) \) and equal to zero. Therefore \( h_1^2 \circ d = 0 \)

**Lemma 3.4.** The lower square of the diagram \( D \) is commutative.

Proof:

As we know \( \frac{D\Delta(q_0)}{\Delta(q_0)} = D\log(q_0) \) and we have \((q_0, q_1, q_2, q_3) \in G_4(2)\) now apply morphism \( p \) we get

\[
p(q_0, q_1, q_2, q_3) = (q_0/q_1, q_2, q_3) - (q_1/q_0, q_2, q_3) + (q_2/q_0, q_1, q_3) - (q_3/q_0, q_1, q_2)
\]
now apply $h^2_0$

\[
\begin{align*}
\ h^2_0 \circ p(q_0, q_1, q_2, q_3) &= D \log \Delta(q_0, q_1) \otimes \frac{\Delta(q_0, q_2)}{\Delta(q_0, q_3)} - D \log \Delta(q_0, q_2) \otimes \frac{\Delta(q_0, q_1)}{\Delta(q_0, q_3)} \\
&\quad + D \log \Delta(q_0, q_3) \otimes \frac{\Delta(q_0, q_1)}{\Delta(q_0, q_2)} - D \log \Delta(q_1, q_0) \otimes \frac{\Delta(q_1, q_2)}{\Delta(q_1, q_3)} \\
&\quad + D \log \Delta(q_1, q_2) \otimes \frac{\Delta(q_1, q_0)}{\Delta(q_1, q_3)} - D \log \Delta(q_1, q_3) \otimes \frac{\Delta(q_1, q_0)}{\Delta(q_1, q_2)} \\
&\quad + D \log \Delta(q_2, q_0) \otimes \frac{\Delta(q_2, q_1)}{\Delta(q_2, q_3)} - D \log \Delta(q_2, q_1) \otimes \frac{\Delta(q_2, q_0)}{\Delta(q_2, q_3)} \\
&\quad + D \log \Delta(q_2, q_3) \otimes \frac{\Delta(q_2, q_0)}{\Delta(q_2, q_1)} - D \log \Delta(q_3, q_0) \otimes \frac{\Delta(q_3, q_1)}{\Delta(q_3, q_2)} \\
&\quad + D \log \Delta(q_3, q_1) \otimes \frac{\Delta(q_3, q_0)}{\Delta(q_3, q_2)} - D \log \Delta(q_3, q_2) \otimes \frac{\Delta(q_3, q_0)}{\Delta(q_3, q_1)} \\
&= \Delta(q_0, q_3) \Delta(q_1, q_3) D \\
&\quad \div (\Delta(q_0, q_2) \Delta(q_1, q_3))^2
\end{align*}
\]

now, let take again $(q_0, q_1, q_2, q_3) \in G_4(2)$ and apply $h^2_1$

\[
\begin{align*}
\ h^2_1(q_0, q_1, q_2, q_3) &= \frac{\Delta(q_0, q_3) \Delta(q_1, q_3) D}{(\Delta(q_0, q_2) \Delta(q_1, q_3))^2}
\end{align*}
\]

now apply $\partial D$

\[
\begin{align*}
\partial D \circ h^2_1(q_0, q_1, q_2, q_3) &= -D \log \left(1 - \frac{\Delta(q_0, q_3) \Delta(q_1, q_3)}{\Delta(q_0, q_2) \Delta(q_1, q_3)} \otimes \frac{\Delta(q_0, q_3) \Delta(q_1, q_3)}{\Delta(q_0, q_2) \Delta(q_1, q_3)} \right) \\
&\quad + D \log \left(\frac{\Delta(q_0, q_3) \Delta(q_1, q_3)}{\Delta(q_0, q_2) \Delta(q_1, q_3)} \otimes \left(1 - \frac{\Delta(q_0, q_3) \Delta(q_1, q_3)}{\Delta(q_0, q_2) \Delta(q_1, q_3)} \right) \right)
\end{align*}
\]

using Siegal cross ratio properties [12], we get

\[
\begin{align*}
\partial D \circ h^2_1(q_0, q_1, q_2, q_3) &= -D \log \left(\frac{\Delta(q_0, q_1) \Delta(q_2, q_3)}{\Delta(q_0, q_2) \Delta(q_1, q_3)} \otimes \frac{\Delta(q_0, q_3) \Delta(q_1, q_3)}{\Delta(q_0, q_2) \Delta(q_1, q_3)} \right) \\
&\quad + D \log \left(\frac{\Delta(q_0, q_3) \Delta(q_1, q_3)}{\Delta(q_0, q_2) \Delta(q_1, q_3)} \otimes \left(\frac{\Delta(q_0, q_3) \Delta(q_2, q_3)}{\Delta(q_0, q_2) \Delta(q_1, q_3)} \right) \right)
\end{align*}
\]
After simplifications we get

\[
\partial^D \circ h^2_1(q_0, q_1, q_2, q_3) = D\log \triangle(q_0, q_1) \otimes \frac{\triangle(q_0, q_2)}{\triangle(q_0, q_3)} + D\log \triangle(q_0, q_2) \otimes \frac{\triangle(q_0, q_1)}{\triangle(q_0, q_3)} - D\log \triangle(q_1, q_0) \otimes \frac{\triangle(q_1, q_2)}{\triangle(q_1, q_3)} - D\log \triangle(q_1, q_0) \otimes \frac{\triangle(q_1, q_3)}{\triangle(q_1, q_2)} + D\log \triangle(q_1, q_2) \otimes \frac{\triangle(q_1, q_0)}{\triangle(q_1, q_3)} + D\log \triangle(q_2, q_0) \otimes \frac{\triangle(q_2, q_1)}{\triangle(q_2, q_3)} + D\log \triangle(q_2, q_0) \otimes \frac{\triangle(q_2, q_3)}{\triangle(q_2, q_1)} + D\log \triangle(q_2, q_3) \otimes \frac{\triangle(q_2, q_0)}{\triangle(q_2, q_1)} + D\log \triangle(q_3, q_1) \otimes \frac{\triangle(q_3, q_0)}{\triangle(q_3, q_1)} + D\log \triangle(q_3, q_1) \otimes \frac{\triangle(q_3, q_0)}{\triangle(q_3, q_1)} - D\log \triangle(q_3, q_0) \otimes \frac{\triangle(q_3, q_1)}{\triangle(q_3, q_0)} - D\log \triangle(q_3, q_0) \otimes \frac{\triangle(q_3, q_1)}{\triangle(q_3, q_0)}
\]

Using Eq. (14) and Eq. (18) we say that the diagram is commutative.

### 3.2 Weight 3 (Trilogarithm)

For this weight we define geometry between the two sub-complexes of Grassmannian and infinitesimal for weight \( n = 3 \) like given below.

\[
\begin{array}{ccc}
G_7(3) & \xrightarrow{p} & G_6(2) \\
\downarrow d & & \downarrow d \\
G_6(3) & \xrightarrow{p} & G_5(2) \\
\downarrow h^3_1 & & \downarrow h^3_0 \\
G_5(3) & \xrightarrow{p} & G_4(1)
\end{array}
\]  

(E)

where

\[
h^3_0 : (q_0, q_1, q_2, q_3) \rightarrow \sum_{i=0}^{3} (-1)^{i} D\log(q_i) \otimes \frac{\triangle(q_{i+1})}{\triangle(q_{i+2})} \wedge \frac{\triangle(q_{i+2})}{\triangle(q_{i+3})} \pmod{4} \tag{19}
\]

and

\[
h^3_1(q_0, ..., q_4) = \frac{-1}{3} \left( \sum_{i=0}^{4} (-1)^i (c^r(q_0, ..., \hat{q}_i, ..., q_4))D \otimes \prod_{j \neq i} \triangle(q_i, q_j) - D\log(\prod_{j \neq i} \triangle(q_i, q_j)) \otimes [q_0, ..., \hat{q}_i, ..., q_4]_2 \right) \tag{20}
\]

First we show that the morphism \( h^3_0 \) and \( h^3_1 \) are well defined.
Lemma 3.5. $h_0^3$ is independent of volume form by vectors in $V_3$. Proof: Let $h_0^3(q_0, q_1, q_2, q_3)$ be written as
\[
\begin{align*}
    h_0^3(q_0, q_1, q_2, q_3) &= \frac{D \Delta(q_1)}{\Delta(q_1)} \otimes \frac{\Delta(q_0)}{\Delta(q_2)} \wedge \frac{\Delta(q_2)}{\Delta(q_3)} - \frac{D \Delta(q_2)}{\Delta(q_2)} \otimes \frac{\Delta(q_0)}{\Delta(q_1)} \wedge \frac{\Delta(q_1)}{\Delta(q_3)} \\
    &+ \frac{D \Delta(q_0)}{\Delta(q_0)} \otimes \frac{\Delta(q_1)}{\Delta(q_2)} \wedge \frac{\Delta(q_2)}{\Delta(q_3)} - \frac{D \Delta(q_3)}{\Delta(q_3)} \otimes \frac{\Delta(q_0)}{\Delta(q_1)} \wedge \frac{\Delta(q_1)}{\Delta(q_2)}
\end{align*}
\]
(21)
so if we change volume $V = \lambda V$ where $\lambda \in F$ then the right side will remain unchanged, therefore, $h_0^3$ is independent of volume form by vectors in $V_3$.

Lemma 3.6. $h_0^3 \circ p$ is independent of length of vectors in $V_3$.

Proof: The proof is the same as Lemma (2.3)

Lemma 3.7. $h_1^3$ is independent of volume form.

Proof:
\[
h_1^3(q_0, q_1, q_2, q_3, q_4) = -\frac{1}{3} \left( \sum_{i=0}^{4} (-1)^i (q_0, ..., \hat{q}_i, ..., q_4) \otimes \prod_{j \neq i} \Delta(q_i, q_j) \right)
\]
(22)
if we change volume $V = \lambda V$ where $\lambda \in F$, the difference give us
\[
h_1^3(q_0, q_1, q_2, q_3, q_4) = -\frac{1}{3} \sum_{i=0}^{4} (-1)^i (cr(\lambda q_0, ..., \hat{q}_i, ..., q_4) \otimes \lambda^4
\]
which is a five term relation in $\beta_2^D(F)$ and equal to zero. $h_1^3$ is independent of volume form.

Lemma 3.8. $h_1^3 \circ p$ is independent of length of vectors in $V_3$.

Proof: Let suppose $(q_0, q_1, q_2, q_3, q_4, q_5) - (\lambda q_0, q_1, q_2, q_3, q_4, q_5) = 0$ ($\lambda \in F$), $h_1^3 \circ p[(q_0, q_1, q_2, q_3, q_4) - (\lambda q_0, q_1, q_2, q_3, q_4, q_5)] = 0$ the difference give us
\[
= -\frac{1}{3} \left( \sum_{i=0}^{4} (-1)^i (cr(q_0, ..., \hat{q}_i, ..., q_4) \otimes \lambda^4
\]
(24)
which is a five term relation in $\beta_2^D(F)$ and equal to zero. $h_1^3$ is independent of length of vectors.
Lemma 3.9. The lower square of the above diagram $E$ is commutative.

Proof:

Let us consider five points $(q_0, q_1, q_2, q_3, q_4) \in G_5(2)$ now apply projection map $p$ we get

$$p(q_0, ..., q_4) = \sum_{j=0}^{4} (-1)^j (q_j/q_0, ..., \hat{q}_j, ..., q_4)$$  \hspace{1cm} (25)

Now apply $h^3_0$

$$h^3_0 \circ p(q_0, ..., q_4) = \sum_{j=0}^{4} (-1)^j \sum_{i \neq j}^{3} (-1)^i D \log(\triangle(q_j, q_i) \otimes \frac{\triangle(q_j, q_{i+1})}{\triangle(q_j, q_{i+2})} \wedge \frac{\triangle(q_j, q_{i+3})}{\triangle(q_j, q_{i+3})})$$  \hspace{1cm} (26)

Now lets take again $(q_0, ..., q_4) \in G_5(2)$ and apply $h^3_1$

$$h^3_1(q_0, ..., q_4) = -\frac{1}{3} \left( \sum_{i=0}^{4} (-1)^i \left( [cr(q_0, ..., \hat{q}_i, ..., q_4)]_2 \otimes \prod_{j \neq i}^{4} \triangle(q_i, q_j) \right) \right.$$  

$$- D \log(\prod_{j \neq i}^{4} \triangle(q_i, q_j)) \otimes [cr(q_0, ..., \hat{q}_i, ..., q_4)]_2 \right)$$  \hspace{1cm} (27)

Now apply $\partial^D$

$$\partial^D \circ h^3_1(q_0, ..., q_4) = -\frac{1}{3} \left( \sum_{i=0}^{4} (-1)^i (-D \log(1 - cr(q_0, ..., \hat{q}_i, ..., q_4)) \otimes cr(q_0, ..., \hat{q}_i, ..., q_4)) \right.$$  

$$\wedge \prod_{j \neq i}^{4} \triangle(q_i, q_j) + D \log cr(q_0, ..., \hat{q}_i, ..., q_4) \otimes (1 - cr(q_0, ..., \hat{q}_i, ..., q_4))$$  

$$\wedge \prod_{j \neq i}^{4} \triangle(q_i, q_j) - D \log(\prod_{j \neq i}^{4} \triangle(q_i, q_j)) \otimes (1 - cr(q_0, ..., \hat{q}_i, ..., q_4))$$  

$$\wedge cr(q_0, ..., \hat{q}_i, ..., q_4) \right)$$  \hspace{1cm} (28)

After using tensor, wedge and Siegel cross ratio properties [12], we get

$$= \sum_{j=0}^{4} (-1)^j \sum_{i \neq j}^{3} (-1)^i D \log(\triangle(q_j, q_i) \otimes \frac{\triangle(q_j, q_{i+1})}{\triangle(q_j, q_{i+2})} \wedge \frac{\triangle(q_j, q_{i+3})}{\triangle(q_j, q_{i+3})})$$  \hspace{1cm} (29)

Using Eq. (26) and Eq. (29), we observed that

$$h^3_0 \circ p = \partial^D \circ h^3_1$$


### 3.3 Weight 4

For this weight we connect sub complexes of Grassmannian and infinitesimal in weight 4, like given below:

\[
\begin{array}{ccc}
G_8(3) & \xrightarrow{p} & G_7(2) \\
\downarrow d & & \downarrow d \\
G_7(2) & \xrightarrow{p} & G_6(2) \\
\downarrow h_0^4 & & \downarrow h_0^4 \\
G_6(2) & \xrightarrow{p} & G_5(1) \\
\downarrow \beta_D^2(F) \otimes F^\times \oplus F \otimes B_2(F) \otimes F^\times & \xrightarrow{p} & F \otimes F^\times \otimes F^\times \otimes F^\times \\
\end{array}
\]

where

\[
h_0^4 : (q_0, ..., q_4) \to \sum_{i=0}^4 (-1)^i D \log(q_i) \otimes \frac{\triangle(q_{i+1})}{\triangle(q_{i+2})} \wedge \frac{\triangle(q_{i+2})}{\triangle(q_{i+3})} \wedge \frac{\triangle(q_{i+3})}{\triangle(q_{i+4})} \quad (\text{mod } 5)
\]

and \(h_1^4\) as given below:

\[
h_1^4(q_0, ..., q_5) = \frac{1}{6} \left[ \sum_{j=0}^5 (-1)^j \left( c(q_0, ..., \hat{q}_i, \hat{q}_j, ..., q_5) \right)_2 \otimes \prod_{k \neq j, k=0}^5 \triangle(q_j, q_k) \right. \\
\left. \wedge \prod_{k \neq i, k=0}^5 \triangle(q_i, q_k) - D \log(\prod_{k \neq i, k=0}^5 \triangle(q_i, q_k)) \otimes \left[ c(q_0, ..., \hat{q}_i, \hat{q}_j, ..., q_5) \right]_2 \right. \\
\left. \wedge \prod_{k \neq j, k=0}^5 \triangle(q_j, q_k) + D \log(\prod_{k \neq j, k=0}^5 \triangle(q_j, q_k)) \otimes \left[ c(q_0, ..., \hat{q}_i, \hat{q}_j, ..., q_5) \right]_2 \right. \\
\left. \wedge \prod_{k \neq i, k=0}^5 \triangle(q_i, q_k) \right]
\]

First, we show that the morphisms \(h_0^4\) and \(h_1^4\) are well defined.

**Lemma 3.10.** \(h_0^4\) is independent of volume form by vectors in \(V_4\).

*Proof:* The proof is the same as Lemma (3.5).

**Lemma 3.11.** \(h_0^4 \circ p\) is independent of length of vectors in \(V_4\).

*Proof:* The proof is same as Lemma (3.6).
Lemma 3.12. $h_1^4$ is independent of length of vectors.

Proof:

Let $(q_0, q_1, q_2, q_3, q_4, q_5) = (\lambda q_0, q_1, q_2, q_3, q_4, q_5)$ where $\lambda \in F$. $h_1^3((q_0, q_1, q_2, q_3, q_4, q_5) - (\lambda q_0, q_1, q_2, q_3, q_4, q_5)) = 0$.

$h_1^3((q_0, \ldots, q_5) - (\lambda q_0, \ldots, q_5)) = \frac{1}{6} \left[ \sum_{i,j} (-1)^i \left( cr(q_0, \ldots, \hat{q}_i, \hat{q}_j, \ldots, q_5) D_2 \right) \right] \otimes \lambda^5$

$$= \frac{1}{6} \left( cr(q_1, q_2, q_3, q_4) D_2 - cr(\lambda q_0, q_2, q_3, q_4) D_2 + cr(\lambda q_0, q_1, q_3, q_4) D_2 
- cr(\lambda q_0, q_1, q_2, q_4) D_2 + cr(\lambda q_0, q_1, q_2, q_3) D_2 \right) \otimes \lambda^5$$

which is five term relation in $\beta_2^D(F)$ and equal to zero. $h_1^4$ is independent of length of vectors.

Lemma 3.13. The lower square of figure $(F)$ is commutative i.e $h_0^4 \circ p = \partial^D \circ h_1^4$.

Proof:

Let us take six points of dimension two $(q_0, q_1, q_2, q_3, q_4, q_5) \in G_6(2)$, apply morphism $p$ we get

$$p(q_0, \ldots, q_5) = \sum_{i=0}^5 (q_i/q_0, \ldots, q_i, \ldots, q_5)$$

(33)

then we apply $h_0^4$

$$h_0^4 \circ p(q_0, \ldots, q_5) = \sum_{i=0}^5 (-1)^i \sum_{j=0, j \neq i}^4 (-1)^j D \log \triangle(q_i, q_j) \otimes \frac{\triangle(q_i, q_{j+1})}{\triangle(q_i, q_{j+2})} \wedge \frac{\triangle(q_i, q_{i+2})}{\triangle(q_i, q_{i+3})}$$

(34)

now let's take $(q_0, q_1, q_2, q_3, q_4, q_5) \in G_6(2)$, apply $h_1^4$

$$h_1^4(q_0, \ldots, q_5) = \frac{1}{6} \left[ \sum_{i,j} (-1)^i \left( cr(q_0, \ldots, \hat{q}_i, \hat{q}_j, \ldots, q_5) D_2 \otimes \prod_{k \neq j}^5 \triangle(q_j, q_k) \wedge \prod_{k \neq i}^5 \triangle(q_i, q_k) 
- D \log \left( \prod_{k \neq j}^5 \triangle(q_j, q_k) \otimes \left[ cr(q_0, \ldots, \hat{q}_i, \hat{q}_j, \ldots, q_5) D_2 \wedge \prod_{k \neq j}^5 \triangle(q_j, q_k) \right] \right) 
+ D \log \left( \prod_{k \neq j}^5 \triangle(q_j, q_k) \otimes \left[ cr(q_0, \ldots, \hat{q}_i, \hat{q}_j, \ldots, q_5) D_2 \wedge \prod_{k \neq i}^5 \triangle(q_i, q_k) \right] \right) \right]$$
now apply $\partial^D$

$$\partial^D \circ h_1^4(q_0, \ldots, \hat{q}_i, \hat{q}_j, \ldots, q_5) = \frac{1}{6} \sum_{i \neq j} (-1)^i \left( -D\log(1 - (cr(q_0, \ldots, \hat{q}_i, \hat{q}_j, \ldots, q_5))) \otimes \right.$$

$$(cr(q_0, \ldots, \hat{q}_i, \hat{q}_j, \ldots, q_5)) \wedge \prod_{k \neq j} \Delta(q_j, q_k) \wedge \prod_{k \neq i} \Delta(q_i, q_k)$$

$$+ D\log(cr(q_0, \ldots, \hat{q}_i, \hat{q}_j, \ldots, q_5)) \otimes$$

$$(1 - (cr(q_0, \ldots, \hat{q}_i, \hat{q}_j, \ldots, q_5))) \wedge \prod_{k \neq j} \Delta(q_j, q_k) \wedge \prod_{k \neq i} \Delta(q_i, q_k)$$

$$- D\log(\prod_{k \neq i} \Delta(q_i, q_k)) \otimes (1 - (cr(q_0, \ldots, \hat{q}_i, \hat{q}_j, \ldots, q_5)))$$

$$\wedge (cr(q_0, \ldots, \hat{q}_i, \hat{q}_j, \ldots, q_5)) \wedge \prod_{k \neq j} \Delta(q_j, q_k)$$

$$+ D\log(\prod_{k \neq j} \Delta(q_j, q_k)) \otimes (1 - (cr(q_0, \ldots, \hat{q}_i, \hat{q}_j, \ldots, q_5)))$$

$$\wedge (cr(q_0, \ldots, \hat{q}_i, \hat{q}_j, \ldots, q_5)) \wedge \prod_{k \neq i} \Delta(q_i, q_k) \left) \right]$$

apply Seigal [12] cross ratio property we get

$$= \sum_{i=0}^{5} (-1)^i \sum_{j=0, j \neq i}^{4} (-1)^j D\log \Delta(q_i, q_j) \otimes \frac{\Delta(q_i, q_{j+1})}{\Delta(q_i, q_{j+2})} \wedge \frac{\Delta(q_i, q_{i+2})}{\Delta(q_i, q_{i+3})} \wedge \frac{\Delta(q_i, q_{i+3})}{\Delta(q_i, q_{i+4})}$$

(35)

From Eq. (34) and Eq. (35), we observed that

$$h_0^4 \circ p = \partial^D \circ h_1^4$$

4 Conclusion

In this article, we have defined new morphism for weight 2 and 3 and also, for the first time, weight 4 is investigated. It has been observed that by using new morphism that increasing weight does not effect the dimension of points. Using new morphism, calculation is easy and simple.
References


Received: February 24, 2016; Published: April 2, 2016