Direct Product of B-algebras

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Abstract

In this paper, we introduce the direct product of B-algebras and we obtain some of its properties.

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1 Introduction

In 2002, the concept of B-algebras [6] was introduced by J. Neggers and H.S. Kim. A B-algebra \( A = (A; \ast, 0) \) is an algebra of type \((2, 0)\), that is, a nonempty set \( A \) together with a binary operation \( \ast \) and a constant 0 satisfying the following axioms for all \( x, y, z \in A \): (I) \( x \ast x = 0 \), (II) \( x \ast 0 = x \), and (III) \( (x \ast y) \ast z = x \ast (z \ast (0 \ast y)) \). In the same paper, the concept of commutative B-algebras was also introduced. A B-algebra \( A \) is commutative if \( x \ast (0 \ast y) = y \ast (0 \ast x) \) for all \( x, y \in A \). H.S. Kim and H.G. Park [4] characterized commutativity of B-algebras. In [7], J. Neggers and H.S. Kim introduced the notions of subalgebras and normality in B-algebras, and established their properties. A nonempty subset \( N \) of \( A \) is called a subalgebra of \( A \) if \( x \ast y \in N \)
for any $x, y \in N$. By (I), 0 is always an element of a subalgebra. A nonempty subset $N$ of $A$ is called a normal subalgebra of $A$ if $(x \ast a) \ast (y \ast b) \in N$ for any $x \ast y, a \ast b \in N$. A. Walendziak [9] characterized normality in B-algebras. J. Neggers and H.S. Kim used the concept of normality in B-algebras to construct quotient B-algebras. That is, given a normal subalgebra $N$ of a B-algebra $A$, the relation $\sim_N$ is defined by $x \sim_N y$ if and only if $x \ast y \in N$ for any $x, y \in A$. Then $\sim_N$ is a congruence relation of $A$. For $x \in A$, we write $xN$ for the congruence class containing $x$, that is, $xN = \{y \in A : x \sim_N y\}$. Denote $A/N = \{xN : x \in A\}$ and define $\ast'$ on $A/N$ by $xN \ast' yN = (x \ast y)N$. Note that $xN = yN$ if and only if $x \sim_N y$. The algebra $A/N = (A/N; \ast', N)$ is a B-algebra, and is called the quotient B-algebra of $A$ modulo $N$. The concept of B-homomorphism was also introduced by J. Neggers and H.S. Kim. A map $\varphi : A \to B$ is called a B-homomorphism if $\varphi(x \ast y) = \varphi(x) \ast \varphi(y)$ for any $x, y \in A$. The kernel of $\varphi$, denoted by ker $\varphi$, is defined to be the set $\{x \in A : \varphi(x) = 0_B\}$. The ker $\varphi$ is a normal subalgebra of $A$, and ker $\varphi = \{0_A\}$ if and only if $\varphi$ is one-one. A B-homomorphism $\varphi$ is called a B-monomorphism, B-epimorphism, or B-isomorphism if $\varphi$ is one-one, onto, or a bijection, respectively. In [7], the first and third isomorphism theorems for B-algebras are established. In [3], J.C. Endam and J.P. Vilela established the second isomorphism theorem for B-algebras. In this paper, we introduced the direct product of B-algebras and established some of its properties.

2 Direct Product of B-algebras

We begin with some examples of B-algebras.

**Example 2.1** The algebra $(\mathbb{Z}; \ast, 0)$ is a B-algebra, where $\ast$ is defined by $x \ast y = x - y$ for all $x, y \in \mathbb{Z}$.

**Example 2.2** [6] Let $A = \{0, 1, 2, 3, 4, 5\}$ be a set with the following table:

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<th>2</th>
<th>3</th>
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Then $(A; \ast, 0)$ is a B-algebra.

Let $A = (A; \ast, 0_A)$ and $B = (B; \ast, 0_B)$ be B-algebras. Define the direct product of $A$ and $B$ to be the structure $A \times B = (A \times B; \circledast, (0_A, 0_B))$, where
A × B is the set \{ (a, b) : a ∈ A and b ∈ B \} and whose binary operation ⊕ is given by \((a_1, b_1) ⊕ (a_2, b_2) = (a_1 * a_2, b_1 * b_2)\). Note that the binary operation ⊕ is componentwise. Thus, the properties (I), (II), and (III) of A × B follow from those of A and B. Hence, the following theorem easily follows.

**Theorem 2.3** The direct product of two B-algebras is also a B-algebra.

Now, we extend this direct product to any finite family of B-algebras and obtain some of its properties. Let \( I_n = \{1, 2, \ldots , n\} \) and let \( \{ A_i = (A_i; *, 0_i) : i ∈ I_n \} \) be a finite family of B-algebras. Define the direct product of B-algebras \( A_1, \ldots , A_n \) to be the structure \( \prod_{i=1}^{n} A_i = \left( \prod_{i=1}^{n} A_i; ⊕, (0_1, \ldots , 0_n) \right) \), where

\[ \prod_{i=1}^{n} A_i = A_1 × \cdots × A_n = \{ (a_1, \ldots , a_n) : a_i ∈ A_i, i ∈ I_n \} \]

and whose operation ⊕ is given by

\[ (a_1, \ldots , a_n) ⊕ (b_1, \ldots , b_n) = (a_1 * b_1, \ldots , a_n * b_n). \]

Obviously, ⊕ is a binary operation on \( \prod_{i=1}^{n} A_i \).

**Corollary 2.4** If \( \{ A_i = (A_i; *, 0_i) : i ∈ I_n \} \) is a family of B-algebras, then \( \prod_{i=1}^{n} A_i \) is a B-algebra.

**Theorem 2.5** Let \( \{ A_i = (A_i; *, 0_i) : i ∈ I_n \} \) be a family of B-algebras. Then each \( A_i \) is commutative if and only if \( \prod_{i=1}^{n} A_i \) is commutative.

**Proof:** Let each \( A_i \) be commutative. If \((a_1, \ldots , a_n), (b_1, \ldots , b_n) ∈ \prod_{i=1}^{n} A_i\), then \( a_i, b_i ∈ A_i \) and \( a_i * (0_i * b_i) = b_i * (0_i * a_i) \) for all \( i ∈ I_n \). Thus,

\[
(a_1, \ldots , a_n) ⊕ ((0_1, \ldots , 0_n) ⊕ (b_1, \ldots , b_n)) = (a_1, \ldots , a_n) ⊕ (0_1 * b_1, \ldots , 0_n * b_n)
\]

\[ = (a_1 * (0_1 * b_1), \ldots , a_n * (0_n * b_n))\]

\[ = (b_1 * (0_1 * a_1), \ldots , b_n * (0_n * a_n))\]

\[ = (b_1, \ldots , b_n) ⊕ (0_1 * a_1, \ldots , 0_n * a_n)\]

\[ = (b_1, \ldots , b_n) ⊕ ((0_1, \ldots , 0_n) ⊕ (a_1, \ldots , a_n)).\]
Therefore, $\prod_{i=1}^{n} A_i$ is commutative.

Conversely, let $\prod_{i=1}^{n} A_i$ be commutative. If $a_i, b_i \in A_i$ for all $i \in I_n$, then

$$(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \prod_{i=1}^{n} A_i \text{ and } (a_1, \ldots, a_n) \ast ((0_1, \ldots, 0_n) \ast (b_1, \ldots, b_n)) = (b_1, \ldots, b_n) \ast ((0_1, \ldots, 0_n) \ast (a_1, \ldots, a_n)).$$

Thus,

$$(a_1 \ast (0_1 \ast b_1), \ldots, a_n \ast (0_n \ast b_n)) = (a_1, \ldots, a_n) \ast (0_1 \ast b_1, \ldots, 0_n \ast b_n) = (a_1, \ldots, a_n) \ast ((0_1, \ldots, 0_n) \ast (b_1, \ldots, b_n)) = (b_1, \ldots, b_n) \ast ((0_1, \ldots, 0_n) \ast (a_1, \ldots, a_n)) = (b_1, \ldots, b_n) \ast (0_1 \ast a_1, \ldots, 0_n \ast a_n) = (b_1 \ast (0_1 \ast a_1), \ldots, b_n \ast (0_n \ast a_n)).$$

This implies that $a_i \ast (0_i \ast b_i) = b_i \ast (0_i \ast a_i)$ for all $i \in I_n$. Therefore, each $A_i$ is commutative. □

**Theorem 2.6** Let $\{\varphi_i: A_i \rightarrow B_i : i \in I_n\}$ be a family of $B$-homomorphisms.

If $\varphi$ is the map $\prod_{i=1}^{n} A_i \rightarrow \prod_{i=1}^{n} B_i$ given by $(a_1, \ldots, a_n) \mapsto (\varphi_1(a_1), \ldots, \varphi_n(a_n)),$

then $\varphi$ is a $B$-homomorphism with $\ker \varphi = \prod_{i=1}^{n} \ker \varphi_i$, $\varphi(\prod_{i=1}^{n} A_i) = \prod_{i=1}^{n} \varphi_i(A_i)$.

Furthermore, $\varphi$ is a $B$-monomorphism (respectively, $B$-epimorphism) if and only if each $\varphi_i$ is a $B$-monomorphism (respectively, $B$-epimorphism).

**Proof:** Let $\{\varphi_i: A_i \rightarrow B_i : i \in I_n\}$ be a family of $B$-homomorphisms and let $\varphi$ be the map $\prod_{i=1}^{n} A_i \rightarrow \prod_{i=1}^{n} B_i$ given by $(a_1, \ldots, a_n) \mapsto (\varphi_1(a_1), \ldots, \varphi_n(a_n)).$

If $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \prod_{i=1}^{n} A_i$, then

$$\varphi((a_1, \ldots, a_n) \ast (b_1, \ldots, b_n)) = \varphi((a_1 \ast b_1, \ldots, a_n \ast b_n)) = (\varphi_1(a_1 \ast b_1), \ldots, \varphi_n(a_n \ast b_n)) = (\varphi_1(a_1) \ast \varphi_1(b_1), \ldots, \varphi_n(a_n) \ast \varphi_n(b_n)) = (\varphi_1(a_1), \ldots, \varphi_n(a_n)) \ast (\varphi(b_1), \ldots, \varphi(b_n)) = \varphi((a_1, \ldots, a_n)) \ast \varphi((b_1, \ldots, b_n)).$$
This shows that $\varphi$ is a $B$-homomorphism. Moreover, if $\varphi$ is a $B$-homomorphism, then each $\varphi_i$ is also a $B$-homomorphism. Now,

\[(a_1, \ldots, a_n) \in \ker \varphi \iff \varphi((a_1, \ldots, a_n)) = (0_1, \ldots, 0_n)\]
\[\iff (\varphi_1(a_1), \ldots, \varphi_n(a_n)) = (0_1, \ldots, 0_n)\]
\[\iff \varphi_i(a_i) = 0_i \text{ for each } i \in I_n\]
\[\iff a_i \in \ker \varphi_i \text{ for each } i \in I_n\]
\[\iff (a_1, \ldots, a_n) \in \prod_{i=1}^n \ker \varphi_i.\]

Thus, $\ker \varphi = \prod_{i=1}^n \ker \varphi_i$. Let $A = \prod_{i=1}^n A_i$. Then

\[(b_1, \ldots, b_n) \in \varphi(A) \iff \exists (a_1, \ldots, a_n) \in A \ni (b_1, \ldots, b_n) = \varphi((a_1, \ldots, a_n))\]
\[\iff \exists (a_1, \ldots, a_n) \in A \ni (b_1, \ldots, b_n) = (\varphi_1(a_1), \ldots, \varphi_n(a_n))\]
\[\iff \exists a_i \in A_i \ni b_i = \varphi_i(a_i) \in \varphi(A_i) \text{ for each } i \in I_n\]
\[\iff (b_1, \ldots, b_n) \in \prod_{i=1}^n \varphi_i(A_i).\]

Thus, $\varphi(\prod_{i=1}^n A_i) = \prod_{i=1}^n \varphi_i(A_i)$.

To prove the last statement, let $\varphi$ be one-to-one. If $\varphi_i(a_i) = \varphi_i(b_i)$ for each $i \in I_n$, then

\[\varphi((a_1, \ldots, a_n)) = (\varphi_1(a_1), \ldots, \varphi_n(a_n))\]
\[= (\varphi_1(b_1), \ldots, \varphi_n(b_n))\]
\[= \varphi((b_1, \ldots, b_n)).\]

Since $\varphi$ is one-to-one, $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$, that is, $a_i = b_i$ for each $i \in I_n$. Therefore, $\varphi_i$ is one-to-one for each $i \in I_n$. Conversely, let $\varphi_i$ be one-to-one for each $i \in I_n$. If $\varphi((a_1, \ldots, a_n)) = \varphi((b_1, \ldots, b_n))$, then

\[(\varphi_1(a_1), \ldots, \varphi_n(a_n)) = \varphi((a_1, \ldots, a_n))\]
\[= \varphi((b_1, \ldots, b_n))\]
\[= (\varphi_1(b_1), \ldots, \varphi_n(b_n)).\]

Thus, $\varphi_i(a_i) = \varphi_i(b_i)$ for each $i \in I_n$. Since each $\varphi_i$ is one-to-one, $a_i = b_i$ for each $i \in I_n$ and so $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$. Therefore, $\varphi$ is one-to-one.

Finally, we show that $\varphi$ is onto if and only if each $\varphi_i$ is. Let $\varphi$ be onto. If $b_i \in B_i$ for each $i \in I_n$, then $(b_1, \ldots, b_n) \in \prod_{i=1}^n B_i$. Since $\varphi$ is onto,
there exists \((a_1, \ldots, a_n) \in \prod_{i=1}^{n} A_i\) such that \((b_1, \ldots, b_n) = \varphi((a_1, \ldots, a_n)) = (\varphi_1(a_1), \ldots, \varphi_n(a_n))\), that is, \(b_i = \varphi_i(a_i)\) for each \(i \in I_n\). Therefore, \(\varphi_i\) is onto for each \(i \in I_n\). Conversely, let \(\varphi_i\) be onto for each \(i \in I_n\). If \((b_1, \ldots, b_n) \in \prod_{i=1}^{n} B_i\), then \(b_i \in B_i\) for each \(i \in I_n\). Since each \(\varphi_i\) is onto, there exists \(a_i \in A_i\) such that \(b_i = \varphi_i(a_i)\) for each \(i \in I_n\) so that \((b_1, \ldots, b_n) = (\varphi_1(a_1), \ldots, \varphi_n(a_n)) = \varphi((a_1, \ldots, a_n))\). Therefore, \(\varphi\) is onto and so the theorem is finally proved. \(\square\)

**Remark 2.7** Let \(\{A_i = (A_i; *, 0_i): i \in I_n\}\) and \(\{B_i = (B_i; *, 0_i): i \in I_n\}\) be any two families of B-algebras such that \(A_i \cong B_i\) for each \(i \in I_n\). Then \(\prod_{i=1}^{n} A_i \cong \prod_{i=1}^{n} B_i\).

**Theorem 2.8** Let \(\{A_i = (A_i; *, 0_i): i \in I_n\}\) be a family of B-algebras and let \(J_i\) be a normal subalgebra of \(A_i\) for each \(i \in I_n\). Then \(\prod_{i=1}^{n} J_i\) is a normal subalgebra of \(\prod_{i=1}^{n} A_i\) and \(\prod_{i=1}^{n} A_i / \prod_{i=1}^{n} J_i \cong \prod_{i=1}^{n} (A_i/J_i)\).

**Proof:** Let \(\{A_i = (A_i; *, 0_i) : i \in I_n\}\) be a family of B-algebras and let \(J_i\) be a normal subalgebra of \(A_i\) for each \(i \in I_n\). Then \((0_1, \ldots, 0_n) \in \prod_{i=1}^{n} J_i\) since \(0_i \in J_i\) for each \(i \in I_n\) and so \(\prod_{i=1}^{n} J_i\) is not empty. Let \((x_1, \ldots, x_n) \circ (y_1, \ldots, y_n), (a_1, \ldots, a_n) \circ (b_1, \ldots, b_n) \in \prod_{i=1}^{n} J_i\). Then \((x_1 \circ y_1, \ldots, x_n \circ y_n), (a_1 \circ b_1, \ldots, a_n \circ b_n) \in \prod_{i=1}^{n} J_i\). This means that \(x_i \circ y_i, a_i \circ b_i \in J_i\) for each \(i \in I_n\).

Since each \(J_i\) is a normal subalgebra of \(A_i\), \((x_i \circ a_i) \circ (y_i \circ b_i) \in J_i\). Hence, \(((x_1, \ldots, x_n) \circ (a_1, \ldots, a_n)) \circ ((y_1, \ldots, y_n) \circ (b_1, \ldots, b_n)) = (x_1 \circ a_1, \ldots, x_n \circ a_n) \circ (y_1 \circ b_1, \ldots, y_n \circ b_n) \in \prod_{i=1}^{n} J_i\).

Therefore, \(\prod_{i=1}^{n} J_i\) is a normal subalgebra of \(\prod_{i=1}^{n} A_i\).

Let \(J = \prod_{i=1}^{n} J_i\) and \(A = \prod_{i=1}^{n} A_i\). Define \(\varphi : A/J \to \prod_{i=1}^{n} (A_i/J_i)\) given by
\[ \varphi((a_1, \ldots, a_n)J) = (a_1J_1, \ldots, a_nJ_n) \text{ for all } (a_1, \ldots, a_n)J \subseteq A/J. \] Let \((a_1, \ldots, a_n)J, (b_1, \ldots, b_n)J \subseteq A/J. \] If \((a_1, \ldots, a_n)J = (b_1, \ldots, b_n)J, \) then \((a_1, \ldots, a_n) \sim_J (b_1, \ldots, b_n), \) that is, \((a_1 * b_1, \ldots, a_n * b_n) = (a_1, \ldots, a_n) \odot (b_1, \ldots, b_n) \in J. \] Thus, \(a_i * b_i \in J_i \) for all \(i \in I_n, \) that is, \(a_i \sim_J b_i \) so that \(a_iJ_i = b_iJ_i. \) It follows that 

\[ \varphi((a_1, \ldots, a_n)J) = (a_1J_1, \ldots, a_nJ_n) = (b_1J_1, \ldots, b_nJ_n) = \varphi((b_1, \ldots, b_n)J). \]

This shows that \(\varphi\) is well-defined. If \((a_1, \ldots, a_n)J, (b_1, \ldots, b_n)J \subseteq A/J, \) then

\[
\varphi((a_1, \ldots, a_n)J *^\ast (b_1, \ldots, b_n)J) = \varphi(((a_1, \ldots, a_n) \odot (b_1, \ldots, b_n))J)\\ = \varphi((a_1 * b_1, \ldots, a_n * b_n)J)\\ = ((a_1J_1, \ldots, a_nJ_n) \odot (b_1J_1, \ldots, b_nJ_n)\\ = (a_1J_1, \ldots, a_nJ_n) \odot (b_1J_1, \ldots, b_nJ_n)\\ = \varphi((a_1, \ldots, a_n)J) \odot \varphi((b_1, \ldots, b_n)J).
\]

This shows that \(\varphi\) is a homomorphism.

If \(\varphi((a_1, \ldots, a_n)J) = \varphi((b_1, \ldots, b_n)J), \) then

\[
(a_1J_1, \ldots, a_nJ_n) = \varphi((a_1, \ldots, a_n)J)\\ = \varphi((b_1, \ldots, b_n)J)\\ = (b_1J_1, \ldots, b_nJ_n).
\]

Thus, \(a_iJ_i = b_i/J_i \) for all \(i \in I_n. \) Hence, \(a_i \sim_J b_i, \) that is, \(a_i * b_i \in J_i \) for all \(i \in I_n \) so that \((a_1, \ldots, a_n) \odot (b_1, \ldots, b_n) = (a_1 * b_1, \ldots, a_n * b_n) \subseteq J. \) Thus, \((a_1, \ldots, a_n) \sim_J (b_1, \ldots, b_n) \) and so \((a_1, \ldots, a_n)J = (b_1, \ldots, b_n)J. \) This shows that \(\varphi\) is one-to-one.

If \((a_1J_1, \ldots, a_nJ_n) \subseteq \prod_{i=1}^{n}(A_i/J_i), \) then \(a_i \in A_i \) for all \(i \in I_n, \) that is, \((a_1, \ldots, a_n) \subseteq A. \) It follows that \((a_1J_1, \ldots, a_nJ_n) = \varphi((a_1, \ldots, a_n)J), \) where \((a_1, \ldots, a_n)J \subseteq A/J. \) This shows that \(\varphi\) is onto. Therefore, \(\varphi\) is a B-isomorphism, that is, \(\prod_{i=1}^{n}A_i/\prod_{i=1}^{n}J_i \cong \prod_{i=1}^{n}(A_i/J_i).\]

\[ \square \]

References


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