Abstract

In this paper we introduce the notion of functionals on R-vector spaces and obtain various properties. We also introduce the concept of dual spaces and Inner product in R-vector spaces and study their properties.

Mathematics Subject Classification: 15A03, 34K06, 46Cxx, 46Exx

Keywords: R-vector space, Special homomorphism, Functional, Dual Space, Inner Product

Introduction

The notion of R-vector spaces was introduced by Raja Gopala Rao [4] as a generalization to the concept of Vector spaces over Boolean algebras (or simply Boolean vector spaces) of Subrahmanyam [6]. F.O.Stroup [1] has made a study on functionals, dual spaces, inner product spaces in Boolean vector spaces. In this paper we extend these notions to R-vector spaces and obtain certain properties. In [3] the authors have studied on special homomorphisms of R-vector spaces. This paper is a further continuation on special homomorphisms and we introduce the notion of functionals using special homomorphisms. This paper consists of four sections. In section one, we will collect certain basic definitions and results concerning R-vector spaces and special homomorphisms. In section two, we introduce the notion of functionals on R-vector spaces and obtain some basic results about functionals. Section three is meant for the study of dual...
spaces. we prove the necessary and sufficient condition for two R-vector spaces to be dual in theorem 3.11. Finally, In section four, we introduce the notion of inner product and prove that V is a dual space to itself if V is normed in theorem 4.4.

1 Preliminaries

We collect some definitions and results concerning R-vector spaces from [2,4,5] and special homomorphisms from [3].

We recall the following

Let \((R, +, .)\) be a commutative regular ring with 1. Let \(a \in R\), then \(a^* = yay\) for some \(y\) in \(R\) which is again a regular element and \(|a| = aa^*\). Clearly \(|a|\) is an idempotent element of \(R\) and \(R_B\) denotes the set of all Boolean elements of \(R\).

Definition 1.1. Let \(V = (V, +)\) be any group and \(R = (R, +, .)\) be a commutative regular ring with unity element 1. Then \(V\) is said to be a Vector space over \(R\) (or simply \(R\)-vector space) if and only if there exists a mapping \(R \times V \rightarrow V\) (the image of any \((a, x) \in R \times V\) will be denoted by \(ax\)) such that for all \(x, y \in V\) and \(a, b \in R\), all the following properties hold.

1. \(a^2(x + y) = ax + ay\)
2. \(a(bx) = (ab)x\) if \(a^2 = a\)
3. \(1x = x\)
4. \((a + b)x = ax + bx\) if \(ab = 0\)
5. \(r(sx) = (rs)x\) if \(r\) and \(s\) are invertible elements of \(R\)

Definition 1.2. An R-vector space \(V\) is said to be normed if and only if there exists a mapping \(\|: V \rightarrow B\) satisfying the following properties.

(1) \(\|x\| = 0 \iff x = 0\) and (2) \(\|ax\| = |a| |x|\) for all \(x \in V, a \in B\)

Definition 1.3. If \(G^*\) is a basis of \(V\) and \(g \in G^*\), then \(|g| = 1\).

Definition 1.4. A finite subset of non zero elements \(x_1, ..., x_n\) of an R-vector space \(V\) is called (linearly) independent over \(R\) if and only if \(a_1x_1 + ... + a_n x_n = 0\) and \(a_1...a_n \neq 0\) imply \(x_1 + ... + x_n = 0\) and a subset \(S\) (of non zero elements) of \(V\) is said to be an independent subset of \(V\) if and only if every finite subset of \(S\) is linearly independent over \(R\).
Definition 1.5. Let $V, W$ be $R$-vector spaces over a regular ring $R$. A mapping $T : V \to W$ is a special homomorphism of $V$ to $W$ provided

$$T(ax + by) = |a|Tx + |b|Ty \text{ if } ab = 0, \forall x, y \in V \text{ and } a, b \in R.$$ 

The set of special homomorphisms from $V$ to $W$ will be denoted by $\text{SHom}(V, W)$.

Lemma 1.6. Let $V$ and $W$ be $R$-vector spaces and $T$ be a mapping from $V$ to $W$. $T \in \text{SHom}(V, W) \iff T(ax) = |a|Tx, \forall x \in V \text{ and } a \in R$.

Theorem 1.7. Let $V$ and $W$ be $R$-vector spaces. Then $\text{SHom}(V, W)$ is an $R$-vector space if the scalar multiplication is defined by $(aT)(x) = |a|Tx, \forall x \in V$, $T \in \text{SHom}(V, W)$ and $a \in R$.

Theorem 1.8. Let $V$ and $W$ be $R$-vector spaces. If $V$ has a finite basis $G^* = \{x_1, ..., x_n\}$, then $\text{SHom}(V, W)$ is isomorphic to $\sum_{i=1}^{n} W_i$, $W_i = W, \forall 1 \leq i \leq n$.

2 Functionals

In this section we introduce functionals on $R$-vector spaces and obtain certain properties.

Let $(R, +, .)$ be a commutative regular ring with 1 which is not a Boolean ring and $W = (R, +)$ be any additive abelian group of $R$. Define $\otimes : R \times W \to W$ by $a \otimes x = |a| x, \forall a \in R, x \in W$, which is the ring product of $|a|$ and $x$.

We have the following simple

Remark 2.1. $W$ is an $R$-vector space.

Lemma 2.2. If $R = (R, +, .)$ is a regular ring and $W = (R, +)$ be an $R$-vector space. Then

(i) $1$ is a basis of $W = (R, +)$

(ii) $|x| = x$ for each $x \in R_B$.

Proof. Trivial

Definition 2.3. Let $V$ and $W = (R, +)$ be $R$-vector spaces. A mapping $T$ from $V$ to $W$ is called a linear functional on $V$ provided:

$$T(a \otimes x + b \otimes y) = a \otimes Tx + b \otimes Ty, ab = 0, x, y \in V \text{ and } a, b \in R.$$ 

Remark 2.4. The set of all linear functionals on $V$ will be denoted by $\mathcal{V} := \text{SHom}(V, W), \text{where } W = (R, +)$.
Definition 2.5. Let $V$ and $W = (R, +)$ be $R$-vector spaces. A mapping $T$ from $V$ to $W$ is called a strong linear functional on $V$ provided:

$T(a \otimes x + b \otimes y) = a \otimes Tx + b \otimes Ty, \forall x, y \in V$ and $a, b \in R.$

Remark 2.6. Let $\overline{V} = \{T | T : V \to W$ is a $SHom$ $\}$ where $V, W$ are $R$-vector spaces. Clearly $(\overline{V}, +)$ is an abelian group. If we define $\otimes : R \times \overline{V} \to \overline{V}$ by $(a, T)(x) = (a \otimes T)(x) = a |Tx|,$ where $(T + S)(x) = Tx + Sx$ and $(a \otimes T)(x) = a |Tx|$ for all $a \in R, x \in V, T, S \in \overline{V}, \overline{V}$ is an $R$-vector space.

We furnish the following as a special case of theorem 1.8

Theorem 2.7. If $V$ is an $R$-vector space of finite dimension $n$ and $W_i = (R, +)$ for $i = 1, 2, ..., n$, then $\overline{V}$ is isomorphic to $\sum_{i=1}^{n} W_i,$ where $(R, +)$ is considered as an $R$-vector space.

Lemma 2.8. If $V$ is a normed $R$-vector space and $Nx = |x|$ for some $x \in V.$ Then $N \in \overline{V}.$

Proof. $N(a \otimes x) = N(|a|x) = |a||x| = |a|Nx = a \otimes Nx, x \in V$ and $a \in R.$

Remark 2.9. Let $V$ be a normed $R$-vector space. For each $a \in R,$ $\overline{a}$ be a mapping of $V$ to $B$ defined by $\overline{a}(x) = |a||x| = a \otimes |x|$ for each $x \in V.$

Lemma 2.10. If $V$ is a normed $R$-vector space and $a \in R,$ then $\overline{a} \in \overline{V} = SHom(V, W)$.

Proof. $\overline{a}(b \otimes x) = a \otimes |b \otimes x| = |a||b||x| = |b|(a \otimes |x|) = |b|\overline{a}(x) = b \otimes \overline{a}(x).$ Thus the result holds by lemma 1.6.

Lemma 2.11. If $V$ is a normed $R$-vector space and $a, b \in R_B,$ then

(i). $\overline{a + b} = \overline{a} + \overline{b}$

(ii). $\overline{ab} = \overline{a} \overline{b}$

(iii). $\overline{ab} = \overline{a} \overline{b}$

Proof. From the above lemma, $\overline{a}, \overline{b}, \overline{a + b}, \overline{ab} \in \overline{V}.$ The expressions $\overline{a + b}, \overline{ab}$ and $\overline{a} \overline{b}$ are well defined according to $(T + S)x = Tx + Sx, (a \otimes T)x = |a|Tx$ and $(TS)x = |Tx|(Sx)$ respectively.

(i). $(\overline{a + b})(x) = |a + b||x| = (a + b)|x| = a|x| + b|x| = \overline{a}(x) + \overline{b}(x) = (\overline{a} + \overline{b})(x)$.

(ii). $(\overline{ab})(x) = a(\overline{b})(x) = a(|b||x|) = a(|b||x|) = (ab)|x| = |ab| |x| = \overline{a|b|}(x).

(iii). $\overline{ab}(x) = |ab||x| = (|a||b|)|x| = (|a||b|)|x| = \overline{|a||b|}(x) = \overline{a}(\overline{b})(x) = (\overline{a} \overline{b})(x).$

Lemma 2.12. If $V$ is a normed $R$-vector space and $[V] = R_B,$ then $\overline{a} = \overline{b} \Rightarrow a = b.$
Proof. Suppose $\alpha = \beta$ for some $a, b \in R_B$. Since $[V] = R_B$, there exist $x, y \in V$ such that $|x| = |a| = a$ and $|y| = |b| = b$.
By the remark 2.9, $\alpha(x) = |a||x| = aa = a$ and $\beta(x) = |b||x| = |b||a| = ba$.
Since $\alpha = \beta$, we have $a = ba$ and $b = ab$.

Theorem 2.13. If $V$ is a normed $R$-vector spaces and $[V] = R_B$, then
(i) The $R$-vector spaces $W = (R, +)$ is isomorphically contained in the $R$-vector space $V$.
(ii) The regular ring $(R, +, \cdot)$ is isomorphically contained in the regular ring $(V, +, \cdot)$.

Proof. Consider the mapping $\gamma : R_B \to V$ defined by $\gamma(a) = \alpha$ for each $a \in R_B$. By lemma 2.12, $\gamma$ is one to one. To prove (i), let $x, y \in W = (R, +)$ and $a, b \in R_B$. Then by lemma 2.11, we have $\gamma(x + y) = \alpha + \beta = \alpha(x) + \alpha(y)$ and $\gamma(a \otimes x) = a \otimes \gamma(x)$. Hence, $\gamma$ is an isomorphism. For proving (ii), let $a, b \in R_B$, then $\gamma(ab) = \alpha \beta = \alpha \beta = \gamma(a) \gamma(b)$. Similarly, $\gamma(a + b) = \alpha + \beta = \gamma(a) + \gamma(b)$. Hence, $\gamma$ is an isomorphism of the regular ring $(R, +, \cdot)$ in to the regular ring $(V, +, \cdot)$.

3 Dual spaces

In this section we introduce dual spaces and study certain properties.

Definition 3.1. Let $V$ and $W$ be $R$-vector spaces and $V \times W = \{(x, z) : x \in V, z \in W\}$. A mapping $U : V \times W \to R$ is called a bilinear function on $V \times W$ provided:
(3.1.1) $U(a \otimes x + b \otimes y, z) = a \otimes U(x, z) + b \otimes U(y, z)$
(3.1.2) $U(x, a \otimes w + b \otimes z) = a \otimes U(x, w) + b \otimes U(x, z)$, $\forall x, y \in V$ and $w, z \in W$ whenever $a, b \in R, ab = 0$.

Definition 3.2. If $V$ and $W$ are $R$-vector spaces, then a bilinear function $U$ on $V \times W$ is called non-degenerate provided:
(3.2.1) $U(x, z) = 0$ for each $z \in W \Rightarrow x = 0$
(3.2.2) $U(x, z) = 0$ for each $x \in V \Rightarrow z = 0$

Definition 3.3. If $V$ and $W$ are $R$-vector spaces and $U$ is a non-degenerate bilinear function on $V \times W$, then $V$ and $W$ are said to be dual spaces with respect to $U$.

Remark 3.4. In general, two $R$-vector spaces will be called dual spaces if they are dual with respect to at least one bilinear function.
Remark 3.5. Let $V$ and $W$ be $R$-vector spaces. If $U : V \times W \rightarrow R$ is a bilinear function on $V \times W$, then $U^\prime : W \times V \rightarrow R$ is also a bilinear function on $W \times V$ when $U(x, z) = U^\prime(z, x)$.

Corollary 3.6. If $U$ is non-degenerate, then $U^\prime$ is also non-degenerate.

Proof. follows immediately from the definition.

Lemma 3.7. Let $V$ and $W$ be $R$-vector spaces. If $U$ is a bilinear function on $V \times W$, then

(i). $U(a \otimes x, z) = a \otimes U(x, z)$

(ii). $U(x, b \otimes z) = b \otimes U(x, z)$, $\forall x \in V, z \in W$ and $a, b \in R$.

Proof. Letting $b = 0$ in (3.1.1) and $a = 0$ in (3.1.2) respectively yields the desired conclusions.

Theorem 3.8. If $V$ is a normed $R$-vector space and $\bar{V}$ is a space of linear functionals on $V$, then $V$ and $\bar{V}$ are dual spaces.

Proof. Let $U : V \times \bar{V} \rightarrow (R, +)$ defined by $U(x, T) = Tx$ for all $x \in V$ and $T \in \bar{V}$. Suppose $x, y \in V$, $T, S \in \bar{V}$ and $a, b \in R$ with $ab = 0$. By definition 2.3, we have

$U(a \otimes x + b \otimes y, T) = a \otimes U(x, T) + b \otimes U(y, T)$ and by remark 2.6 $U(x, a \otimes T + b \otimes S) = a \otimes U(x, T) + b \otimes U(x, S)$. Thus, by definition 3.1, $U$ is a bilinear functional on $V \times \bar{V}$. Suppose $x \in V^*$ and $T \in \bar{V}$. To show $U$ is non degenerate, it will suffice to establish the existence of elements $N \in \bar{V}$ and $y \in V$ such that $U(x, N) \neq 0$ and $U(y, T) \neq 0$. Since $T \in \bar{V}$, $T$ is not the zero mapping. Hence, there exists at least one element $y \in V$ for which $Ty \neq 0$. Let $N$ denote the norm mapping on $V$. Then by lemma 2.8, it is clear that $N \in \bar{V}$. Since $x \neq 0$, $N x = |x| \neq 0$. We have $\Rightarrow U(x, N) = N x \neq 0$ and $U(y, T) = Ty \neq 0$. Hence by definition 3.2, $U$ is a non degenerate bilinear functional on $V \times \bar{V}$. Thus, $V$ and $\bar{V}$ are dual spaces.

Definition 3.9. Let $V$ be an $R$-vector space. A subset $M$ of $\bar{V}$ is called a total subset of $\bar{V}$ provided that for each $x \in V^*$ there exists an element $T$ of $M$ such that $Tx \neq 0$.

Lemma 3.10. If $V$ is a normed $R$-vector space, then $\bar{V}$ is a total set.

Proof. Let $N x = |x|$ for each $x \in V$. Then $N \in \bar{V}$ by lemma 2.8. If $x \neq 0$, then $N x = |x| \neq 0$. Thus, $\bar{V}$ is a total set.

Theorem 3.11. Let $V$ and $W$ be $R$-vector spaces. A necessary and sufficient condition for $V$ and $W$ to be dual spaces is the existence of a special homomorphism $T$ of $V$ into $\bar{W}$ such that $T(V)$ is a total subset of $\bar{W}$ and $T^{-1}\{0\} = \{0\}$. 
4 Inner Product

Subrahmanyam established that any normed boolean vector space $V$ admits a unique "inner product" mapping, $[x,y]$, of $V \times V$ in to $B$.

In this section we introduce the notion of inner product on R-vector spaces and study its properties.

**Definition 4.1.** Let $V$ be a normed R-Vector space, $[ ] : V \times V \to R_B$, is an inner product mapping such that

(i). $[x,y]|x-y| = 0$
(ii). $[x,y] + [x-y] = |x| + |y|
(iii). $[x,y] = [y,x]
(iv). [a \otimes x + b \otimes z,y] = a \otimes [x,y] + b \otimes [z,y], ab = 0

Let $V$ be a normed R-vector space. For each $x \epsilon V$, let $\bar{x}$ denote the mapping of $V$ in to $R_B$ defined by $\bar{x}(y) = [x,y]$ for each $y \epsilon V$.

**Lemma 4.2.** If $V$ is a normed R-vector space, then $\{\bar{x} : x \epsilon V\}$ is a total subset of $V$.

**Proof.** Let $x, y, z \epsilon V$ and $a, b \epsilon R_B$ with $ab = 0$. Then $\bar{x}(a \otimes y + b \otimes z)$

$= [x, a \otimes y + b \otimes z] = [a \otimes y + b \otimes z, x] = a \otimes [y, x] + b \otimes [z, x] = a \otimes [x, y] + b \otimes [x, z]

= a \otimes \bar{x}(y) + b \otimes \bar{x}(z).$ Thus $\bar{x}V$. Suppose $x \epsilon V^*$. By 4.1 (ii), $[x,x] = |x| + |x| = |x|$. Since $x \neq 0$, it is clear that $|x| \neq 0$. Hence $\bar{x}(x) = [x,x] \neq 0$. Thus, the result holds.

**Remark 4.3.** Let $V$ be a normed R-vector space. If $[x,y] = [z,y]$ for each $y \epsilon V$ then $x = z$.

**Theorem 4.4.** If $V$ is a normed R-vector space, then $V$ is a dual space to itself.

**Proof.** Consider the mapping $T : V \to \overline{V}$ defined by $Tx = \bar{x}$ for each $x \epsilon V$, where $\bar{x}(y) = [x,y]$ for each $y \epsilon V$. By lemma 4.2, $T(V)$ is a total subset of $\overline{V}$. Now let $x, y, z \epsilon V$ and $a, b \epsilon R_B$ with $ab = 0$. Then $a \otimes x + b \otimes y(z) = [a \otimes x + b \otimes y, z] = a \otimes [x, z] + b \otimes [y, z] = a \otimes x + b \otimes y = a \otimes \bar{x}(z) + b \otimes \bar{y}(z) = (a \otimes \bar{x} + b \otimes \bar{y})(z)$. Hence, $T(a \otimes x + b \otimes y) = a \otimes x + b \otimes y = a \otimes \bar{x} + b \otimes \bar{y} = a \otimes Tx + b \otimes Ty$. Thus, $TeShom(V,\overline{V})$. Finally suppose $Tx = \bar{x}$ is the zero mapping in $\overline{V}$. Then $\bar{x}(y) = [x,y] = 0$ for each $x \epsilon V$. Taking $a = b = 0$ in (iv) of definition 4.1 yields $[0,y] = 0$ for each $y \epsilon V$. Hence by remark 4.3, $x = 0$. Thus $T^{-1}\{0\} = \{0\}$. Hence the theorem follows from definition 4.1 and Theorem 3.11. 


References


Received: September 2, 2015; Published: October 21, 2015