On a Conjecture about
Stable Partitions\textsuperscript{1}

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Abstract

We provide a counterexample to a conjecture put forward in [10]. This conjecture assumed the existence and uniqueness of an array of 2-regular partitions with certain properties. We prove that if such an array existed, it would not be unique (although we cannot say anything about its existence). We used computer software written in GAP to obtain this and other counterexamples, and to test for the existence of such an array of 2-regular partitions.

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1 Introduction

In the modern representation theory of groups, Alperin’s Weight Conjecture remains as one of the most important and difficult open problems. In essence,

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Alperin’s Weight Conjecture claims that the number of weights for a group algebra equals its number of irreducible modules. This conjecture has been proved for a large number of groups, including the symmetric groups (see [2]); however, in the case of the symmetric groups, no explicit bijection between weights and irreducibles is known. In the case of the symmetric groups in characteristic $p$, the irreducible modules are parameterized by $p$-regular partitions, so it is possible to formulate this conjecture in purely combinatorial terms. In [10], the authors put forward a combinatorial conjecture that, if true, would provide an explicit bijection between weights and irreducibles for the symmetric groups in characteristic two. The key ingredient in this combinatorial proof of Alperin’s Weight Conjecture is the existence and uniqueness of an array of 2-regular partitions with certain properties, and the authors of [10] proved some results towards the uniqueness of such an array. In this paper we prove that even if such an array of 2-regular partitions existed, it would not be unique.

In Section 3 we give the basic definitions from the theory of partitions and state the combinatorial conjecture from [10]. In Section 4 we present our counterexample.

2 Alperin’s Weight Conjecture

In this section we provide the definition of weight, and also formulate Alperin’s Conjecture in its most general form. We mention some classes of groups for which it is known to be valid (including the symmetric groups) and we note the possible advantages of a combinatorial proof, that is, an explicit bijection between weights and irreducible modules.

Let $G$ denote a finite group, $p$ a prime number, and $k$ a splitting field for $G$ in characteristic $p$. All our modules will be finite dimensional over $k$.

**Definition 2.1.** A weight for $G$ is a pair $(Q,S)$ where $Q$ is a $p$-subgroup and $S$ is a simple module for $k[N_G(Q)]$ which is projective when regarded as a module for $k[N_G(Q)/Q]$.

**Remark 2.2.** Since $S$ is $k[N(Q)]$-simple and $Q$ is a $p$-subgroup of $N_G(Q)$, it follows that $Q$ acts trivially on $S$, so $S$ is also a $k[N_G(Q)/Q]$-module and the definition makes sense. Moreover, $S$ is $k[N_G(Q)/Q]$-simple as well.

**Remark 2.3.** If we replace $S$ by an isomorphic $k[N_G(Q)]$-module we consider this the same weight, and we make the same identification when we replace $Q$ by a conjugate subgroup (so that the normalizers will be conjugate, too).

**Conjecture 2.4.** (Alperin’s Conjecture) The number of weights for $G$ equals the number of simple $kG$-modules.
A stronger version of the preceding statement is that there is a bijection within each block of the group algebra.

**Definition 2.5.** If \((Q, S)\) is a weight for \(G\), then \(S\) belongs to a block \(b\) of \(N_G(Q)\) and this block corresponds with a block \(B\) of \(G\) via the Brauer correspondence; hence we can say that the weight \((Q, S)\) belongs to the block \(B\) of \(G\) so the weights are partitioned into blocks.

**Conjecture 2.6.** (Alperin’s Conjecture, Block Form) The number of weights in a block of \(G\) equals the number of simple modules in the block.

This version of the conjecture implies the original one, as it can be obtained by summing the equalities from the stronger conjecture over the blocks. This stronger conjecture has been proved when \(G\) is a:

- Finite group of Lie type and characteristic \(p\) (Cabanes, [8]).
- Soluble group (Okuyama, [12]).
- Symmetric group (Alperin and Fong, [2]).
- \(GL(n, q)\), \(p\) odd and \(p\) does not divide \(q\) (Alperin and Fong, [2]).
- \(GL(n, q)\), \(p = 2\) and \(q\) odd (An, [3]).

The conjecture has also been checked in a variety of other cases (see [4], [5], [6], [7]).

Alperin and Fong proved the conjecture in the case of the symmetric groups by establishing a numerical equality. This is enough to prove that there is a bijection, but it did not suggest a deeper reason for the relationship. For finite groups in general one does not expect to have any canonical bijection between weights and simple modules; as a matter of fact, Alperin himself says this is unlikely (see [1], p 369). For groups of Lie type in their defining characteristic there is a canonical bijection (described in [1]). Since symmetric groups and groups of Lie type have such strong connections in their representation theory, it is reasonable to ask whether there is some canonical bijection in the case of symmetric groups.

If true, Alperin’s conjecture would imply a number of known results, until now unrelated (see [1]). It is also reasonable to expect that if an explicit bijection can be given to prove it, this may reveal new connections between simple \(kG\)-modules and weights; there are many results known about the former, and the latter are related to the blocks of defect zero, which are not as easy to deal with as the simple modules. In fact, this is really the true importance of Alperin’s conjecture in that it provides a connection between the blocks of defect zero and the set of all simple modules. More specifically, Alperin’s
conjecture has been shown by Knörr and Robinson [11] to be equivalent to a statement which expresses the number of blocks of defect zero of a group in terms of the number of \( p \)-modular irreducibles of sections of the group of the form \( N_G(P)/P \), \( P \leq G \) a \( p \)-subgroup. These latter numbers are easy to compute, since by a theorem of Brauer the number of \( p \)-modular irreducibles of a group equals the number of \( p \)-regular conjugacy classes.

3 Partitions and 2-cores.

We state the basic concepts and facts from the theory of partitions. For a more detailed description of partitions and cores, we refer the reader to [9].

**Definition 3.1.** A partition \( \lambda \) is a sequence of non-increasing non-negative integers \( \lambda_1 \geq \lambda_2 \geq \ldots \) such that there exists \( N \) with \( \lambda_m = 0 \) for all \( m \geq N \). If \( n = \sum \lambda_i \), we say that \( \lambda \) is a partition of \( n \). We sometimes write the partition as \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) where \( \lambda_k \) is the last entry greater than zero. The partition \( \lambda \) is \( p \)-singular if it has at least \( p \) rows of the same size; otherwise, \( \lambda \) is \( p \)-regular. In particular, the partition \( \lambda = (\lambda_1, \ldots, \lambda_t) \) is 2-regular if and only if its rows are of different sizes, i.e. \( \lambda_1 > \lambda_2 > \cdots > \lambda_t \). A skew-hook is a connected part of the rim of \( \lambda \) which can be removed to leave a proper diagram. The \( r \)-core of \( \lambda \) is the partition obtained by removing all possible skew-hooks of size \( r \) from \( \lambda \) (this is a well-defined partition, that is, the order in which we remove the skew-hooks does not matter).

The following conjecture is taken from [10] (we omitted all references to weight subgroups):

**Conjecture 3.2.** It is possible to arrange all 2-regular partitions in an infinite table satisfying the following conditions.

1. The first row consists of all triangular partitions (including the triangular partitions of size 0 and 1).

2. The first partition of every row has empty 2-core. The second partition has 2-core of size 1, the third has 2-core of size 3 and the fourth has 2-core of size 6. In other words, the 2-core of every partition along the \( i \)-th column is the \( i \)-th triangular partition (where \( \emptyset \) is the first triangular partition).

3. For every row, all partitions \( \lambda \) in that row are such that the difference of the size of \( \lambda \) minus the size of its 2-core is constant. In other words, the difference in size between the first partition and the \( i \)-th partition on any given row, is the \( i \)-th triangular number.
4. Along every row, each partition is contained in the one to its right.

In [10], the authors proved that if a table of partitions satisfying all the above conditions exists, then most of its data are completely determined by a few entries, which led them to believe that such a table of partitions, if it existed, would be unique (up to a permutation of its rows).

4 Counterexample.

We wrote computer software in GAP([13]) to test this conjecture, and found a large number of counterexamples (we used James’ abacus to calculate 2-cores). The smallest countereexample is described below.

**Theorem 4.1.** The array of partitions described in the Conjecture from Section 3 is not unique.

*Proof.* Consider the following two ways to assign 2-regular partitions of 11 with 2-core (1) to 2-regular partitions of 10 with empty 2-core.

\[
\begin{align*}
(10) & \mapsto (11) \\
(9, 1) & \mapsto (9, 2) \\
(8, 2) & \mapsto (8, 2, 1) \\
(5, 3, 2) & \mapsto (5, 4, 2) \\
(6, 3, 1) & \mapsto (6, 4, 1) \\
(6, 4) & \mapsto (7, 4) \\
(7, 3) & \mapsto (7, 3, 1)
\end{align*}
\]

\[
\begin{align*}
(10) & \mapsto (11) \\
(9, 1) & \mapsto (9, 2) \\
(8, 2) & \mapsto (8, 2, 1) \\
(5, 3, 2) & \mapsto (5, 4, 2) \\
(6, 3, 1) & \mapsto (7, 3, 1) \\
(6, 4) & \mapsto (6, 4, 1) \\
(7, 3) & \mapsto (7, 4)
\end{align*}
\]

Both of these assignments preserve the properties that the array of partitions must have. If there existed an array satisfying all conditions stated in the Conjecture, we could simply change the positions of the above partitions and obtain at least two different arrays.

\[\square\]
Our computer software also showed that up to the values we searched, it was possible to construct an array of 2-regular partitions with the desired properties. Therefore, in order to have uniqueness, it will be necessary to find more restrictions for the table of partitions.

References


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