A Bound on a Class of Minimal POS Groups

Michael C. Fulkerson

Department of Mathematics and Statistics
University of Central Oklahoma
Edmond, Oklahoma 73034, USA

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Abstract

A finite group $G$ is said to be a POS group if the number of elements of any given order is either zero or divides $|G|$. There are only three known minimal abelian POS groups having order not divisible by 5. We show that these are the only such groups $G$ for which $|G| < 6.62 \times 10^{36}$.

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1 Introduction

In 2002, C. Finch and L. Jones [2] introduced the notion of a POS group: a finite group $G$ is said to be a POS group (or to have perfect order subsets) if the number of elements of any given order is either zero or divides $|G|$.

For example, the group $\mathbb{Z}_3$ is not a POS group since it has 2 elements of order 3, and $2 \nmid 3 = |\mathbb{Z}_3|$. On the other hand, for the group $G \simeq (\mathbb{Z}_2)^2 \times \mathbb{Z}_9$, the number of elements of order 1, 2, 3, 6, 9, and 18 is, respectively, 1, 3, 2, 6, 6, and 18. Since 1, 3, 2, 6, 6, and 18 each divide 36 = $|G|$, $G$ is a POS group.

A fundamental property of POS groups is that if $G$ is a POS group and $p$ is a prime dividing $|G|$, then $p - 1$ divides $|G|$. A proof of this, assuming that $G$ is abelian, can be found in [2]. A nontrivial POS group must thus have even order. In this paper, we focus on abelian POS groups. For the nonabelian case see [3], [1], and [6].

The following important lemmas are proved in [2]:
Lemma 1 Let $G \simeq (\mathbb{Z}_p^a)^t \times M$ and $\hat{G} \simeq (\mathbb{Z}_p^{a+1})^t \times M$, where $M$ is a finite abelian group and $p$ is a prime not dividing $|M|$. If $G$ is a POS group, then so is $\hat{G}$.

Lemma 2 Suppose $G \simeq \mathbb{Z}_p^{a_1} \times \mathbb{Z}_p^{a_2} \times \cdots \times \mathbb{Z}_p^{a_{s-1}} \times (\mathbb{Z}_p^{a_s})^t \times M$, where $M$ is a finite abelian group, $p$ is a prime not dividing $|M|$, and $a_1 \leq a_2 \leq \cdots \leq a_{s-1} < a_s$. If $G$ is a POS group, then so is $\hat{G} \simeq (\mathbb{Z}_p^{a_s})^t \times M$.

Lemma 3 Suppose $G \simeq (\mathbb{Z}_p^a)^t \times M$, where $M$ is a finite abelian group and $p$ is a prime not dividing $|M|$. If $G$ is a POS group, then so is $\hat{G} \simeq (\mathbb{Z}_p^a)^t \times M$.

Lemmas 2 and 3 motivate the notion of a minimal POS group:

Definition An abelian POS group $G \simeq (\mathbb{Z}_2^t \times M$, where $|M|$ is odd, is said to be minimal if there is no proper subgroup $\hat{M}$ of $M$ such that $(\mathbb{Z}_2^t \times \hat{M}$ is a POS group.

By Lemmas 2 and 3, a minimal POS group $G$ (with $|G| > 2$) can be written in the form:

$$G = (\mathbb{Z}_2^t \times (\mathbb{Z}_{p_1}^{t_1} \times \cdots \times (\mathbb{Z}_{p_m}^{t_m})^{t_m},$$

where $p_1 < p_2 < \cdots < p_m$ are odd primes and $m \geq 1$. In the sequel we assume that groups $G$ have this form. We then let

$$n = |G| = 2^t \prod_{i=1}^{m} p_i^{t_i},$$

and

$$f(n) = (2^t - 1) \prod_{i=1}^{m} (p_i^{t_i} - 1).$$

A basic important result, which follows directly from the definition of POS group and from counting elements in abelian groups (see, for example, the first lemma in [2]), is that a group $G$ of the above form is a POS group if and only if $f(n)|n$. This implies, since $f(n) < n$, that $n/f(n) \geq 2$, a fact which we will use often. We state this as a lemma.

Lemma 4 A group $G$ having the above form is a POS group if and only if $n/f(n) \in \mathbb{N} \setminus \{1\}$. In particular, for a POS group $G$, $n/f(n) \geq 2$.

The following ten groups are the only known minimal POS groups:

$$\mathbb{Z}_2$$

$$(\mathbb{Z}_2)^2 \times \mathbb{Z}_3$$

$$(\mathbb{Z}_2)^3 \times \mathbb{Z}_3 \times \mathbb{Z}_7$$
Of the above groups, only the first group, \( \mathbb{Z}_2 \), has order which is not divisible by 3. Similarly, only the first three groups, \( \mathbb{Z}_2 \), \( (\mathbb{Z}_2)^2 \times \mathbb{Z}_3 \), and \( (\mathbb{Z}_2)^3 \times \mathbb{Z}_3 \times \mathbb{Z}_7 \), have orders which are not divisible by 5. This leads to the following questions, the first of which is a small adjustment of a question posed by Jones and Toppin [5]:

**Question 1**: If \( G \) is a minimal POS group and \( G \) is not isomorphic to \( \mathbb{Z}_2 \), then must \(|G|\) be divisible by 3?

**Question 2**: If \( G \) is a minimal POS group and \( G \) is not isomorphic to \( \mathbb{Z}_2 \), \( (\mathbb{Z}_2)^2 \times \mathbb{Z}_3 \), or \( (\mathbb{Z}_2)^3 \times \mathbb{Z}_3 \times \mathbb{Z}_7 \), then must \(|G|\) be divisible by 5?

Jones and Toppin [5] gave an affirmative answer to Question 1 under the additional assumption that \(|G| < 4.58 \times 10^{457008}\). Later, using sieve methods, K. Ford, S. Konyagin, and F. Luca [4] gave a full affirmative answer to Question 1. We will here give an affirmative answer to Question 2 under the additional assumption that \(|G| < 6.62 \times 10^{336}\).

**Theorem 5** Suppose \( G \) is a minimal POS group, with \( n = |G| \). Suppose also that 5 \( \nmid n \) and that \( G \) is not isomorphic to \( \mathbb{Z}_2 \), \( (\mathbb{Z}_2)^2 \times \mathbb{Z}_3 \), or \( (\mathbb{Z}_2)^3 \times \mathbb{Z}_3 \times \mathbb{Z}_7 \). Then \(|G| > 6.62 \times 10^{336}\).

We define a set a primes \( \mathcal{P} \) inductively as follows: (i) \( 2, 3 \in \mathcal{P} \), (ii) \( 5 \notin \mathcal{P} \), (iii) a prime \( p > 5 \) is in \( \mathcal{P} \) if and only if all prime factors of \( p - 1 \) are in \( \mathcal{P} \). We have:

\[
\]

It follows from Lemma 4 that if \( G \) is a POS group with 5 \( \nmid n \), and if \( p \) is a prime dividing \(|G|\), then \( p \in \mathcal{P} \).

## 2 A Preliminary Lemma

We begin by showing that if \( G \) is minimal POS group and \( G \) is not one of the known minimal POS groups (listed above), then \( 2^{14} | n \).
**Definition** A prime power $p^a$ is said to *exactly divide* an integer $n$ (and we write $p^a | n$) if $p^a | n$, but $p^{a+1} \not| n$.

It is straightforward to verify that the only minimal POS groups $G$ for which $2^t | n = |G|$ for $t = 1, 2, 3, 4, 5, 8, 11, 16, 17$, and 32 are the groups listed above (i.e. the ten known minimal POS groups). For example, the only minimal POS group $G$ for which $2^2 | n = |G|$ is $G \simeq (\mathbb{Z}_2)^5 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_{31}$.

To prove this, note by Lemma 4 that if $2^5 | n$, then $2^5 - 1 = 31 | n$. Similarly, since 31 is prime, $31 - 1 = 30 | n$. Thus 3 and 5 must also each divide $n$. Note that, by Lemma 4, $(\mathbb{Z}_2)^5 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_{31}$ is in fact a POS group (i.e. $(32/31)(3/2)(5/4)(31/30) = 2$). Since 3, 5, and 31 must divide $n$, $G$ is in fact a minimal POS group, and the only one for which $2^5 | n$.

It may also be verified that there are no minimal POS groups $G$ for which $2^t | n = |G|$ for $t = 6, 7, 9, 10, 12$, and 13. For example, we now show that there is no minimal POS group $G$ with the property that $2^6 | n$: Suppose there is such a group $G$. Then $2^6 - 1 = 63 | n$, which means that both $3^2$ and 7 divide $n$.

Case 1: Suppose $5 \not| n$. Then $G \simeq (\mathbb{Z}_2)^6 \times (\mathbb{Z}_3)^a \times (\mathbb{Z}_7)^b \times M$ where $a \geq 2$, $b \geq 1$, and 2, 3, 5, and 7 do not divide $|M|$. Furthermore, at most 4 primes divide $|M|$, because otherwise $2^7 | f(n)$ but $2^7 \not| n$, which contradicts the fact that $n/f(n)$ must be an integer (Lemma 4). For such a $G$, the maximum value of $n/f(n)$ will occur for $G \simeq (\mathbb{Z}_2)^6 \times (\mathbb{Z}_3)^2 \times \mathbb{Z}_7 \times \mathbb{Z}_{11} \times \mathbb{Z}_{13} \times \mathbb{Z}_{17} \times \mathbb{Z}_{19}$, but in this case $n/f(n) \approx 1.782 < 2$, a contradiction of Lemma 4.

Case 2: Suppose $5 | n$. Then $G \simeq (\mathbb{Z}_2)^6 \times (\mathbb{Z}_3)^a \times (\mathbb{Z}_5)^b \times (\mathbb{Z}_7)^c \times M$ where $a \geq 2$, $b \geq 1$, $c \geq 1$, and 2, 3, 5, and 7 do not divide $|M|$. Using a similar argument to the one given in Case 1 (and the fact that $2^2 | 5 - 1$), at most 2 primes divide $|M|$. For such a $G$ the maximum value of $n/f(n)$ will occur for $G \simeq (\mathbb{Z}_2)^6 \times (\mathbb{Z}_3)^2 \times \mathbb{Z}_5 \times \mathbb{Z}_7 \times \mathbb{Z}_{11} \times \mathbb{Z}_{13}$, but in this case $n/f(n) \approx 1.986 < 2$, a contradiction.

It can similarly be shown that there are no minimal POS groups $G$ such that $2^t | n$ for $t = 7, 9, 10, 12$, and 13, although the arguments involve more cases for the larger values of $t$. We thus have the following lemma:

**Lemma 6** If $G$ is a minimal POS group that is not listed above, then $2^{14} | n$.

### 3 Proof of Theorem 5

**Proof:** Suppose $G$ is a minimal POS group with $n = |G|$. Suppose also that $5 \not| n$ and that $G$ is not isomorphic to $\mathbb{Z}_2$, $(\mathbb{Z}_2)^2 \times \mathbb{Z}_3$, or $(\mathbb{Z}_2)^3 \times \mathbb{Z}_3 \times \mathbb{Z}_7$. We have by the main result in [4] that $3 | n$. By Lemma 6 we also have that $2^{14} | n$. 

We will now show that $|G| > 6.62 \times 10^{336}$ by dividing the situation into 16 cases.

In Case 1, we assume that $3^1 || n$, $7^1 || n$, $29^1 || n$, and $59^1 || n$. (We do not assume that $59^1$ exactly divides $n$.) We thus have that $13 \nmid n$ because, otherwise, since $7|n$ and $13|n$, then $3^2|n$, contradicting that $3^1||n$. In other words, in this case it is implied that $13 \nmid n$. Similar reasoning can be used to show, for example, that in Case 1 the numbers 19, 37, 43, and 53 also cannot divide $n$.

Table 1 lists all 16 cases. The fact that, in Case 1, under the columns labelled “3”, “7”, and “29” are the numbers 1, 1, and 1, respectively, means that $3^1||n$, $7^1||n$, and $29^1||n$, respectively. The “≥ 1” under column “59” means that $59^r | n$ for some $r ≥ 1$. Finally, the last column lists the minimum possible size for $|G| = n$.

| Case | 3 | 7 | 13 | 19 | 29 | 43 | 59 | $|G|$ at least |
|------|---|---|----|----|----|----|----|----------------|
| 1    | 1 | 1 | 1  | 1  | 1  | 0  | ≥ 1 | $1.09 \times 10^{695735}$ |
| 2    | 1 | 1 | 1  |    |    |    | 0  | $4.43 \times 10^{873009}$  |
| 3    | 1 | 1 |    |    |    | ≥ 2| 0  | $3.36 \times 10^{1587242}$ |
| 4    | 1 | 1 |    |    |    |    | 0  | $1.51 \times 10^{414856}$  |
| 5    | 1 | 0 |    |    |    |    | ≥ 1| $2.63 \times 10^{210575}$  |
| 6    | 1 | 0 | 0  |    |    |    |    | $1.49 \times 10^{289074}$  |
| 7    | 1 |    |    |    | ≥ 2| 0  |    | $8.17 \times 10^{336}$      |
| 8    | 2 | ≥ 1| ≥ 1|    |    |    |    | $3.00 \times 10^{3586670}$ |
| 9    | 2 | ≥ 1| 0  |    |    |    |    | $2.55 \times 10^{1969368}$ |
| 10   | ≥ 2|    |    |    |    |    |    | $1.23 \times 10^{690177}$  |
| 11   | ≥ 3| 1  |    |    |    |    | ≥ 1| $6.62 \times 10^{536}$      |
| 12   | ≥ 3| 1  |    |    |    |    | 0  | $3.80 \times 10^{6904}$     |
| 13   | ≥ 3| 1  |    |    |    | 0  | 0  | $1.15 \times 10^{1762}$     |
| 14   | 3 | ≥ 2|    |    |    | ≥ 1|    | $2.60 \times 10^{1715286}$ |
| 15   | ≥ 4| ≥ 2|    |    |    | 0  | 0  | $6.10 \times 10^{1963}$     |
| 16   | ≥ 3| ≥ 2| 0  |    |    |    |    | $9.65 \times 10^{13878}$   |

Table 1: The 16 cases along with the minimum size of $|G|$ in each case.
\[ \mathcal{P}_1 = \{2, 3, 7, 17, 29, 59, 137, 257, 1097, 1889, 3779, 4013, 7559, 17477, 55697, 60449, 65537, 74597, 79187, 120899, 120929, 140417, 140837, 149057, 157217, 241667, 241793, \ldots \} \]

For the other cases, we have:
\[ \mathcal{P}_2 = \{2, 3, 7, 17, 29, 137, 233, 257, 467, 929, 1097, 1973, 3947, 7433, 7457, 13457, 14867, \ldots \} \]
\[ \mathcal{P}_3 = \{2, 3, 7, 17, 29, 59, 137, 233, 257, 467, 929, 1097, 1889, 1973, 3779, 3947, 4013, 7433, \ldots \} \]
\[ \mathcal{P}_4 = \{2, 3, 7, 17, 113, 137, 227, 239, 257, 449, 479, 953, 1097, 1583, 1907, 1913, 3167, 3347, \ldots \} \]
\[ \mathcal{P}_5 = \{2, 3, 13, 17, 53, 107, 137, 257, 443, 677, 857, 887, 1097, 1697, 3329, 5417, 6659, 6857, \ldots \} \]
\[ \mathcal{P}_6 = \{2, 3, 17, 97, 103, 137, 193, 257, 389, 409, 619, 769, 773, 823, 1097, 1237, 1543, 1553, \ldots \} \]
\[ \mathcal{P}_7 = \{2, 3, 7, 17, 29, 59, 113, 137, 197, 227, 233, 239, 257, 449, 467, 479, 827, 929, 953, 1097, \ldots \} \]
\[ \mathcal{P}_8 = \{2, 3, 7, 13, 17, 29, 53, 59, 107, 113, 137, 197, 227, 233, 239, 257, 443, 449, 467, 479, \ldots \} \]
\[ \mathcal{P}_9 = \{2, 3, 7, 17, 29, 43, 59, 97, 103, 113, 137, 173, 193, 197, 227, 232, 239, 257, 337, 347, \ldots \} \]
\[ \mathcal{P}_{10} = \{2, 3, 13, 17, 19, 37, 53, 73, 79, 97, 103, 107, 109, 137, 149, 157, 163, 193, 232, 229, 257, \ldots \} \]
\[ \mathcal{P}_{11} = \{2, 3, 7, 13, 17, 19, 29, 37, 53, 59, 73, 79, 97, 103, 107, 109, 137, 149, 157, 163, 193, 223, \ldots \} \]
\[ \mathcal{P}_{12} = \{2, 3, 7, 13, 17, 19, 37, 53, 73, 79, 97, 103, 107, 109, 113, 127, 137, 149, 157, 163, 173, \ldots \} \]
\[ \mathcal{P}_{13} = \{2, 3, 7, 13, 17, 19, 37, 43, 53, 59, 73, 79, 97, 103, 107, 109, 137, 149, 157, 163, 173, \ldots \} \]
\[ \mathcal{P}_{14} = \{2, 3, 7, 17, 19, 29, 39, 113, 137, 197, 227, 233, 239, 257, 449, 467, 479, 647, 827, 929, \ldots \} \]
\[ \mathcal{P}_{15} = \{2, 3, 7, 13, 17, 19, 29, 37, 43, 53, 59, 73, 79, 97, 103, 107, 109, 113, 127, 137, 149, \ldots \} \]
\[ \mathcal{P}_{16} = \{2, 3, 7, 13, 17, 29, 37, 43, 53, 59, 73, 79, 97, 103, 107, 109, 113, 127, 137, 149, 157, \ldots \} \]

Let \( p_{j,k} \) denote the \( k \)th element of \( \mathcal{P}_j \). For example, \( p_{1,9} = 1097 \). We focus now specifically on Case 11. We show that in this case \(|G| = n \) is at least \( 6.62 \times 10^{336} \) and that at least 110 primes divide \( n \). A computer calculation gives

\[
\frac{2^{14}}{2^{14} - 1} \cdot \frac{3^3}{3^3 - 1} \cdot \prod_{k=3}^{110} \frac{p_{11,k}}{p_{11,k} - 1} = \frac{16384}{16383} \cdot \frac{27}{26} \cdot \frac{7}{6} \cdot \frac{13}{12} \cdots \frac{2687}{2686} \\
\approx 1.999813 \\
< 2.
\]

[In the above calculation, if we had taken the index in the product to terminate at 111 instead of 110 (i.e. if we had multiplied a factor of \( p_{11,111}/(p_{11,111} - 1) = 2749/2748 \)), then the product would be greater than 2.] Thus, using Lemma 4 along with the fact that if \( a, b, n, t \in \mathbb{N} \) with \( a < b \) and \( n < t \) then \( a/(a-1) > b/(b-1) \) and \( a^n/(a^n - 1) > a/(a^2 - 1) \), we have that at least 110 primes from \( \mathcal{P}_{11} \) must divide \( n \) (including at least 109 odd primes). This means, again by
Lemma 4, that $2^{109}|n$. We conclude that

$$n > 2^{109} \cdot 3^3 \cdot \prod_{k=3}^{110} p_{11,k}$$

$$\approx 6.62 \times 10^{336}.$$

Cases 12, 13, 15, and 16 are similar to Case 11. For the remaining cases (i.e. Cases 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, and 14), we will focus on Case 1, which provides a typical example. A computer calculation gives

$$\frac{2^{14}}{2^{14} - 1} \cdot \prod_{k=2}^{72} \frac{p_{1,k}}{p_{1,k} - 1} = \frac{16384}{16383} \cdot \frac{3}{2} \cdot \frac{7}{6} \cdot \ldots \cdot \frac{13144259}{13144258}$$

$$\approx 1.9895795864$$

$$< 2.$$

Unlike Cases 11, 12, 13, 15, and 16, it appears that the product will not become greater than 2 even if we allow the index to approach infinity. However, to prove this would require sieve techniques. Therefore, we simply make use of the following calculation:

$$\frac{2^{14}}{2^{14} - 1} \cdot \prod_{k=2}^{93756} \frac{p_{1,k}}{p_{1,k} - 1} < \frac{2^{14}}{2^{14} - 1} \cdot \prod_{k=2}^{72} \frac{p_{1,k}}{p_{1,k} - 1} \cdot \left( \frac{p_{1,72}}{p_{1,72} - 1} \right)^{93756 - 72}$$

$$= \frac{2^{14}}{2^{14} - 1} \cdot \prod_{k=2}^{72} \frac{p_{1,k}}{p_{1,k} - 1} \cdot \left( \frac{13144259}{13144258} \right)^{93684}$$

$$\approx 1.999999925$$

$$< 2$$

By Lemma 4 we then have that at least 93756 primes from $\mathcal{P}_1$ must divide $n$ (including at least 97355 odd primes). Thus $2^{93755}|n$. We conclude that

$$n > 2^{93755} \cdot 13144259^{93684} \cdot \prod_{k=2}^{72} p_{1,k}$$

$$\approx 1.09 \times 10^{695135}.$$

\[\square\]

References


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