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Directional Tubular Surfaces

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Abstract

In this paper, we introduce a new version of tubular surfaces. We first define a new adapted frame along a space curve, and denote this the q-frame. We then reveal the relationship between the Frenet frame and the q-frame. We give a parametric representation of a directional tubular surface using the q-frame. Finally, some comparative examples are shown to confirm the effectiveness of the proposed method.

Mathematics Subject Classification: 53A04, 53A05

Keywords: Frenet frame, pipe surface, tube, adapted frame

1 Introduction

It is well known that the tubular (pipe) surface is defined as the envelope of the set of spheres with radius r which are centered at a spine curve $\alpha(t)$

[9]. The importance of the tubular surface lies in the fact that it is used in many practical applications in computer aided geometric design. The tubular surface can be parameterized using the Frenet frame, however, this frame is undefined wherever the curvature vanishes, such as at points of inflection or along straight sections of the curve [13]. Thus, various alternative methods have been proposed for computing the tubular surfaces [11]. Klok [2] defined the sweep surfaces using rotation-minimizing frames. A robust computation of the rotation minimizing frame for sweep surfaces was introduced by Wang [3].

There are a number of different adapted frames along a space curve, such as the parallel transport frame [1,12] and the Frenet frame [5]. Although the Frenet frame is the most well-known frame along a space curve, its rotation about the tangent of a general spine curve often leads to undesirable twists in the tubular surface modelling. Owing to its minimal twist, the Bishop frame is widely used in computer graphics, however, it is not easy to compute [3].

Let $\alpha(t)$ be a space curve with a non-vanishing second derivative. The Frenet frame is defined as follows,

$$\mathbf{t} = \frac{\alpha'}{\|\alpha'\|}, \mathbf{b} = \frac{\alpha' \wedge \alpha''}{\|\alpha' \wedge \alpha''\|}, \mathbf{n} = \mathbf{b} \wedge \mathbf{t}. \tag{1}$$

The curvature κ and the torsion τ are given by

$$\kappa = \frac{\|\alpha' \wedge \alpha''\|}{\|\alpha'\|^3}, \tau = \frac{\det(\alpha', \alpha'', \alpha''')}{\|\alpha' \wedge \alpha''\|^2}. \tag{2}$$

The well-known Frenet formulas are given by

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{bmatrix} = v \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}, \tag{3}$$

where

$$v = \|\alpha'(t)\|. \tag{4}$$

In order to construct the 3D curve offset, Coquillart [6] introduced the quasi-normal vector of a space curve. The quasi-normal vector is defined for each point of the curve, and lies in the plane perpendicular to the tangent of the curve at this point [7].

2 q-frame Along a Space Curve

In this section, as an alternative to the Frenet frame we define a new adapted frame along a space curve, the q-frame. Given a space curve $\alpha(t)$ the q-frame consists of three orthonormal vectors, these being the unit tangent vector $\mathbf{t}$, the quasi-normal $\mathbf{n}_q$ and the quasi-binormal vector $\mathbf{b}_q$. The q-frame $\{\mathbf{t}, \mathbf{n}_q, \mathbf{b}_q, \mathbf{k}\}$ is given by

$$\mathbf{t} = \frac{\alpha'}{\|\alpha'\|}, \mathbf{n}_q = \frac{\mathbf{t} \wedge \mathbf{k}}{\|\mathbf{t} \wedge \mathbf{k}\|}, \mathbf{b}_q = \mathbf{t} \wedge \mathbf{n}_q \tag{5}$$

where $\mathbf{k}$ is the projection vector.

For simplicity, we have chosen the projection vector $\mathbf{k} = (0, 0, 1)$ in this paper. However, the q-frame is singular in all cases where $\mathbf{t}$ and $\mathbf{k}$ are parallel. Thus, in those cases where $\mathbf{t}$ and $\mathbf{k}$ are parallel the projection vector $\mathbf{k}$ can be chosen as $\mathbf{k} = (0, 1, 0)$ or $\mathbf{k} = (1, 0, 0)$.

We can define the Euclidean angle θ between the principal normal $\mathbf{n}$ and quasi-normal $\mathbf{n}_q$ vectors. Then, as one can see immediately, the relation matrix may be expressed as

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}. \tag{6}$$

Thus,

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix}. \tag{7}$$

Let $\alpha(s)$ be a curve that is parameterized by arc length s. Differentiating (6) with respect to s, then substituting (3) and (7) into the results gives the variation equations of the q-frame in the following form

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}_q' \\ \mathbf{b}_q' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n}_q \\ \mathbf{b}_q \end{bmatrix}, \tag{8}$$

where the q-curvatures are

$$k_1 = \kappa\cos\theta, k_2 = -\kappa\sin\theta, k_3 = d\theta + \tau. \tag{9}$$

Fig. 1a and 1b show the q-frame and the rotation minimizing frame(RMF) of the curve $r(t) = (2t, t^2, t^3/3)$. Note that the behavior of the q-frame is similar to that of the RMF. It is well known that computing the rotation minimizing frame along the curve is difficult, although both of the frames have similar accuracy.

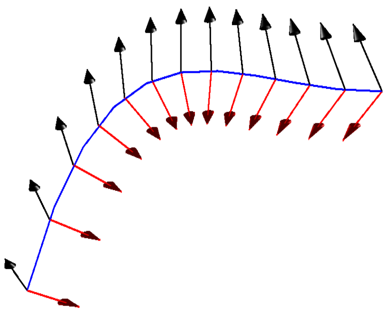

Figure 1a: The rotation minimizing frame along the curve. The normal(red) and the binormal(black) vectors are shown.

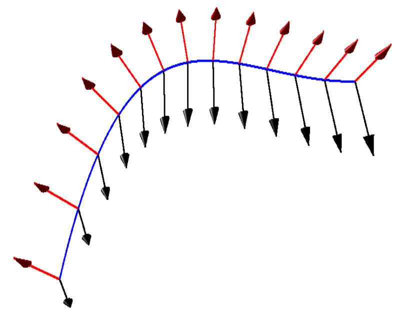

Figure 1b: The q-frame along the curve. The normal(red) and the binormal(black) vectors are shown.

3 D-Tubular Surfaces

In this section, we introduce a new form of tubular surface, and call this surface a directional tubular surface, or D-tubular surface for short. The D-tubular surface, at a distance r from the spine curve $\alpha(s)$, may be represented as

$$\psi^r(s,v) = \alpha(s) + r(\cos v\mathbf{n}_q + \sin v\mathbf{b}_q). \tag{10}$$

It is easy to see that

$$\|\psi^r - \alpha(s)\| = r, \tag{11}$$

where $\|\|$ denotes the Euclidean norm.

The partial derivatives of $\psi^r(s,v)$, with respect to s and v, are determined by

$$\psi^r_s = (1 - r(k_1\cos v + k_2\sin v))\mathbf{t} - rk_3\sin v\mathbf{n}_q + rk_3\cos v\mathbf{b}_q \tag{12}$$

and

$$\psi^r_v = r(\cos v\mathbf{b}_q - \sin v\mathbf{n}_q). \tag{13}$$

By taking the cross product of (12) and (13) we get

$$\psi^r_s \wedge \psi^r_v = -r(1 - r(k_1\cos v + k_2\sin v))(\cos v\mathbf{n}_q + \sin v\mathbf{b}_q). \tag{14}$$

Thus,

$$\|\psi^r_s \wedge \psi^r_v\| = \pm r(1 - r(k_1\cos v + k_2\sin v)). \tag{15}$$

From (14) and (15), we obtain the unit normal vector of the D-tubular surface

$$U = \mp(\cos v\mathbf{n}_q + \sin v\mathbf{b}_q). \tag{16}$$

Corollary 3.1. It is well know that the points where $\psi^r_s \wedge \psi^r_v = 0$ are singular. It follows that the singular points of the D-tubular surface can be obtained by

$$r = \frac{1}{k_1\cos v + k_2\sin v}. \tag{17}$$

We can then state the following theorem:

Theorem 3.1. The Gaussian and mean curvatures of the D-tubular surface are given by

$$K=\frac{-(k_1\cos v+k_2\sin v)}{r(1-r(k_1\cos v+k_2\sin v))} \tag{18}$$

and

$$2H=\mp\frac{2r(k_1\cos v+k_2\sin v)-1}{r(1-r(k_1\cos v+k_2\sin v))}. \tag{19}$$

Proof: From (12) and (13), the components $E=\langle\psi_s^r,\psi_s^r\rangle$, $F=\langle\psi_s^r,\psi_v^r\rangle$ and $G=\langle\psi_v^r,\psi_v^r\rangle$ of the first fundamental form are obtained by

$$E=(1-r(k_1\cos v+k_2\sin v))^2+r^2k_3^2 \tag{20}$$

and

$$F=r^2k_3, G=r^2. \tag{21}$$

Similarly we can derive the components $L=\langle\psi_{ss}^r,U\rangle$, $M=\langle\psi_{sv}^r,U\rangle$ and $N=\langle\psi_{vv}^r,U\rangle$ of the second fundamental form as

$$L=\mp\left[(k_1\cos v+k_2\sin v)(1-r(k_1\cos v+k_2\sin v))-rk_3^2\right] \tag{22}$$

and

$$M=\pm rk_3, N=\pm r. \tag{23}$$

It is well known that the Gaussian and mean curvatures of a surface are given by

$$K=\frac{LN-M^2}{EG-F^2}, 2H=\frac{LG-2MF+NE}{EG-F^2}. \tag{24}$$

By substituting (20)-(23) into (24), the Gaussian and mean curvatures of the D-tubular surface are obtained by

$$K=\frac{-(k_1\cos v+k_2\sin v)}{r(1-r(k_1\cos v+k_2\sin v))} \tag{25}$$

and

$$2H=\mp\frac{1-2r(\cos xk_1+\sin xk_2)}{r(1-r(k_1\cos x+k_2\sin x))}, \tag{26}$$

respectively.

4 Examples

In this section, we present several examples to highlight the advantages of this new approach.

Example 4.1. It is well known that the Frenet frame ($\mathbf{n}$ and $\mathbf{b}$) is not defined at points where the curvature of the curve is zero. Hence, the analytical expression of the tubular surface around a straight line is not obtainable.

In this example, we have obtained the analytical expression of the D-tubular surface generated by a line with a q-frame, as shown in Fig. 2. Now, let us consider a spine curve(line) parameterized by

$$\alpha(t) = (t, t, 0). \tag{27}$$

From (5), It is easy to see that,

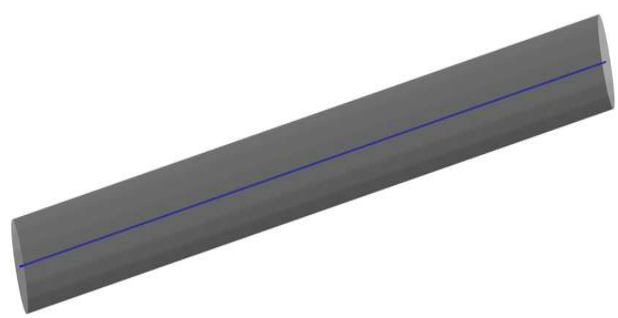

Figure 2: The D-tubular surface generated by the line with the q-frame.

$$\mathbf{t} = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0), \mathbf{n}_q = (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0) \tag{28}$$

and

$$\mathbf{b}_q = (0, 0, -1). \tag{29}$$

For $r = 3$, the D-tubular surface is parameterized by

$$\psi^r(t, v) = (t + 3\frac{\sqrt{2}}{2}\cos v, t - 3\frac{\sqrt{2}}{2}\cos v, -3\sin v). \tag{30}$$

Note that, compared to tubular surfaces generated by the RMF, the analytic expressions of D-tubular surfaces can be easily obtained.

Example 4.2. Assume that the curve is given by

$$\alpha(t) = (t, t, t^9) \tag{31}$$

It is easy to see that the Frenet curvature κ and torsion τ of this curve is obtained by

$$\kappa = \frac{72\sqrt{2}t^7}{(2 + 81t^{16})^{\frac{3}{2}}}, \tau = 0 \tag{32}$$

respectively. Hence $\tau = 0$, the angle between the rotation minimizing frame(Bishop frame) and the Frenet frame is constant, therefore the Bishop frame is also not suitable for this example.

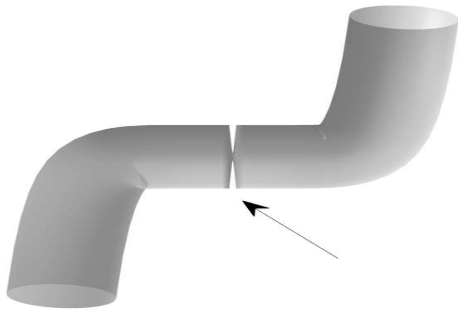

Figure 3a: The tubular surface.

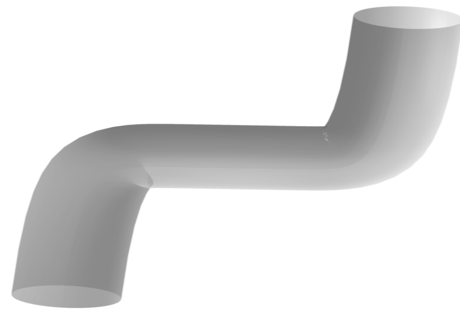

Figure 3b: The D-tubular surface.

The q frame can be calculated by

$$\begin{array}{rcl} \mathbf{t} & = & \frac{1}{\sqrt{2+81t^{16}}}(1,1,9t^8), \\ \mathbf{n}_q & = & \frac{1}{2}(\sqrt{2},-\sqrt{2},0), \\ \mathbf{b}_q & = & \frac{1}{2\sqrt{2+81t^{16}}}(9\sqrt{2}t^8, 9\sqrt{2}t^8, -2\sqrt{2}) \end{array} \tag{33}$$

For $r = 1$, the tubular and the D-tubular surfaces are shown in Fig. 3a and 3b, respectively.

Example 4.3. In this example, let us consider a more complex spine curve $\alpha(t)$, parameterized by

$$\alpha(t) = (\cos(\frac{3t}{10})(2+\cos(t)), \sin(\frac{3t}{10})(2+\cos(t)), -\sin(\frac{2t}{15})). \tag{34}$$

For $r = 0.2$, the tubular and the D-tubular surfaces are illustrated in Fig. 4a and 4b, respectively.

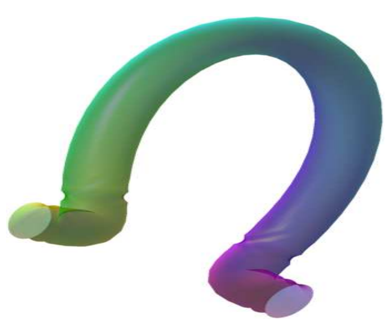

Figure 4a: The tubular surface.

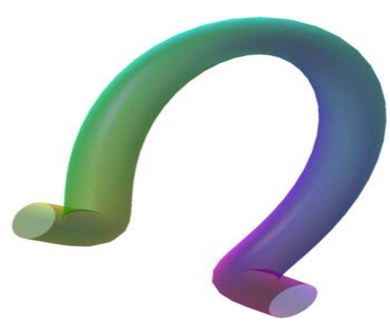

Figure 4b: The D-tubular surface.

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