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On Points and Lines of the Left Hall Plane of Order 9

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Abstract

In this paper, the points and lines of the projective plane over a left nearfield of order 9 are determined.

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1 Introduction

Projective geometry provides a geometric way to study vector spaces. Indeed, a projective spaces over a skew field K is nothing else than the lattice of proper non-trivial subspaces of a vector spaces over K. There are a lot of examples of projective planes over field ([1,2,4]). Finite projective ring planes represent a well-studied, important and venerable branch of algebraic geometry. There are four known projective planes of order 9: the Desarguesian plane, a nearfield plane, the dual of the nearfield plane and the Hughes plane of order 9. One of the best known classes of non-Desarguesian planes is the class of Hall planes (see Hall [2]). Kirkpatrick [5] has considered a type of plane which is a generalization of the Hall planes and which he calls generalized Hall planes.

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It is well known that every projective plane has also an algebraic structure obtained by coordinatization. Conversely, certain algebraic structures can be used to construct projective planes.

In this paper, we first consider the projective plane of order 9 coordinated by elements of a left nearfield of order 9 that is Let $(F_3, +, .)$ be the field of integers modulo 3. Let

$$S = \{a + \lambda b : a, b \in F_3, \lambda \notin F_3\}$$

and consider the addition and multiplication on S given by

$$(a + \lambda b) \oplus (c + \lambda d) = (a + c) + \lambda(b + d) \tag{1}$$

$$(a + \lambda b) \odot (c + \lambda d) = \begin{cases} ac + \lambda(ad), & \text{if} \quad b = 0 \\ ac - b^{-1}df(a) + \lambda(bc - (a-1)d), & \text{if} \quad b \neq 0 \end{cases} \tag{2}$$

where, $a, b \in F_3$, $\lambda \notin F_3$ and $f(t) = t^2 + 1$ is a irreducible polynom on F_3.

For the sake of sorthness if we use ab instead of $a + \lambda b$ in equation (1) and (2) then addition and multiplication tables as follows:

$\oplus$	00	01	02	10	11	12	20	21	22
00	00	01	02	10	11	12	20	21	22
01	01	02	00	11	12	10	21	22	20
02	02	00	01	12	10	11	22	20	21
10	10	11	12	20	21	22	00	01	02
11	11	12	10	21	22	20	01	02	00
12	12	10	11	22	20	21	02	00	01
20	20	21	22	00	01	02	10	11	12
21	21	22	20	01	02	00	11	12	10
22	22	20	21	02	00	01	12	10	11

$\odot$	00	01	02	10	11	12	20	21	22
00	00	00	00	00	00	00	00	00	00
01	00	20	10	01	21	11	02	22	12
02	00	10	20	02	12	22	01	11	21
10	00	01	02	10	11	12	20	21	22
11	00	12	21	11	20	02	22	01	10
12	00	22	11	12	01	20	21	10	02
20	00	02	01	20	22	21	10	12	11
21	00	11	22	21	02	10	12	20	01
22	00	21	12	22	10	01	11	02	20

The system $(S, \oplus, \odot)$ is a left nearfield of order 9.

Now, we are considering the projective plane of order 9 coordinated by elements of the above left nearfield.

2 The Plane P_2S

This paper is deals with determination of the points and lines of the projective plane of order 9 coordinated by elements of the above left nearfield.

Definition 2.1 *While N and D are two distinct sets whose elements are called as the points and the lines, respectively and o is the incidence relation between N and D ; then the ordered triple (N, D, o) is called as geometrical structure. (N, D, o) satisfying the following three axioms is called a projective plane and denoted by P. If N is finite, projective plane P is called as finite projective plane.*

P1. Any distinct two points are incident with just one line.

P2. Any two lines are incident with at least one point.

P3. There exists four points of which no three are collinear.

The order of P is defined to be the number of points on any line of projective plane $P = (N, D, o)$ minus 1. If the order of a finite projective plane is q, total number of its points and lines is equal and $n^2 + n + 1$.

2.1 Points and Lines of P_2S

Definition 2.2 *Let $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ be . The 91 points of P_2S are the elements of the set*

$$\{(x, y) \ : \ x, y \in S\} \cup \{(m) : m \in S\} \cup \{(\infty)\}.$$

The points of the form (x, y) are called proper points, and the unique point (∞) and the points of the form (m) are called ideal points. The 91 lines of P_2S are defined to be set of points satisfying one of the three conditions:

$$\begin{aligned} &[m, k] = \{(x, y) \in S^2 : y = m \odot x \oplus k\} \cup \{(m)\} \\ &[\lambda] = \{(x, y) \in S^2 : \ x = \lambda\} \cup \{(\infty)\} \\ &[\infty] = \{(m) \in S\} \cup \{(\infty)\} \end{aligned}$$

The 81 lines having form $y = m \odot x \oplus k$ and 9 lines having equation of the form $x = \lambda$ are called the proper lines and the unique line $[\infty]$ is called the ideal line.

The system of points, lines and incidence relation given above defines a projective plane of order 9, which is the left nearfiled plane.

Since the 81 lines having form $y = m \odot x \oplus k$, 9 lines having equation of the form $x = \lambda$ and the unique line $[\infty]$, we obtain the full list of the points and lines as fallows:

$[\infty]$	$(00), (01), (02), (20), (21), (10), (11), (12), (22), (\infty)$

$[00]$	$(00, 00), (00, 01), (00, 02), (00, 20), (00, 21), (00, 10), (00, 11), (00, 12), (00, 22), (\infty)$

[01]	(01, 00), (01, 01), (01, 02), (01, 10), (01, 11), (01, 12), (01, 20), (01, 21), (01, 22), (∞)
[02]	(02, 00), (02, 01), (02, 02), (02, 10), (02, 11), (02, 12), (02, 20), (02, 21), (02, 22), (∞)
[10]	(10, 00), (10, 01), (10, 02), (10, 10), (10, 11), (10, 12), (10, 20), (10, 21), (10, 22), (∞)
[11]	(11, 00), (11, 01), (11, 02), (11, 10), (11, 11), (11, 12), (11, 20), (11, 21), (11, 22), (∞)
[12]	(12, 00), (12, 01), (12, 02), (12, 10), (12, 11), (12, 12), (12, 20), (12, 21), (12, 22), (∞)
[20]	(20, 00), (20, 01), (20, 02), (20, 10), (20, 11), (20, 12), (20, 20), (20, 21), (20, 22), (∞)
[21]	(21, 00), (21, 01), (21, 02), (21, 10), (21, 11), (21, 12), (21, 20), (21, 21), (21, 22), (∞)
[22]	(22, 00), (22, 01), (22, 02), (22, 10), (22, 11), (22, 12), (22, 20), (22, 21), (22, 22), (∞)
[00, 00]	(00, 00), (01, 00), (02, 00), (10, 00), (11, 00), (12, 00), (20, 00), (21, 00), (22, 00), (00)
[00, 01]	(00, 01), (01, 01), (02, 01), (10, 01), (11, 01), (12, 01), (20, 01), (21, 01), (22, 01), (00)
[00, 02]	(00, 02), (01, 02), (02, 02), (10, 02), (11, 02), (12, 02), (20, 02), (21, 02), (22, 02), (00)
[00, 10]	(00, 10), (01, 10), (02, 10), (10, 10), (11, 10), (12, 10), (20, 10), (21, 10), (22, 10), (00)
[00, 11]	(00, 11), (01, 11), (02, 11), (10, 11), (11, 11), (12, 11), (20, 11), (21, 11), (22, 11), (00)
[00, 12]	(00, 12), (01, 12), (02, 12), (10, 12), (11, 12), (12, 12), (20, 12), (21, 12), (22, 12), (00)
[00, 20]	(00, 20), (01, 20), (02, 20), (10, 20), (11, 20), (12, 20), (20, 20), (21, 20), (22, 20), (00)
[00, 21]	(00, 21), (01, 21), (02, 21), (10, 21), (11, 21), (12, 21), (20, 21), (21, 21), (22, 21), (00)
[00, 22]	(00, 22), (01, 22), (02, 22), (10, 22), (11, 22), (12, 22), (20, 22), (21, 22), (22, 22), (00)
[01, 00]	(00, 00), (01, 20), (02, 10), (10, 01), (11, 21), (12, 11), (20, 02), (21, 22), (22, 12), (01)
[01, 01]	(00, 01), (01, 21), (02, 11), (10, 02), (11, 22), (12, 12), (20, 00), (21, 20), (22, 10), (01)
[01, 02]	(00, 02), (01, 22), (02, 12), (10, 00), (11, 20), (12, 10), (20, 01), (21, 21), (22, 11), (01)
[01, 10]	(00, 10), (01, 00), (02, 20), (10, 11), (11, 01), (12, 21), (20, 12), (21, 02), (22, 22), (01)
[01, 11]	(00, 11), (01, 01), (02, 21), (10, 12), (11, 02), (12, 22), (20, 10), (21, 00), (22, 20), (01)
[01, 12]	(00, 12), (01, 02), (02, 22), (10, 10), (11, 00), (12, 20), (20, 11), (21, 01), (22, 21), (01)
[01, 20]	(00, 20), (01, 10), (02, 00), (10, 21), (11, 11), (12, 01), (20, 22), (21, 12), (22, 02), (01)
[01, 21]	(00, 21), (01, 11), (02, 01), (10, 22), (11, 12), (12, 02), (20, 20), (21, 10), (22, 00), (01)
[01, 22]	(00, 22), (01, 12), (02, 02), (10, 20), (11, 10), (12, 00), (20, 21), (21, 11), (22, 01), (01)
[02, 00]	(00, 00), (01, 10), (02, 20), (10, 02), (11, 12), (12, 22), (20, 01), (21, 11), (22, 21), (02)
[02, 01]	(00, 01), (01, 11), (02, 21), (10, 00), (11, 10), (12, 20), (20, 02), (21, 12), (22, 22), (02)
[02, 02]	(00, 02), (01, 12), (02, 22), (10, 01), (11, 11), (12, 21), (20, 00), (21, 10), (22, 20), (02)
[02, 10]	(00, 10), (01, 20), (02, 00), (10, 12), (11, 22), (12, 02), (20, 11), (21, 21), (22, 01), (02)
[02, 11]	(00, 11), (01, 21), (02, 01), (10, 10), (11, 20), (12, 00), (20, 12), (21, 22), (22, 02), (02)
[02, 12]	(00, 12), (01, 22), (02, 02), (10, 11), (11, 21), (12, 01), (20, 10), (21, 20), (22, 00), (02)
[02, 20]	(00, 20), (01, 00), (02, 10), (10, 22), (11, 02), (12, 12), (20, 21), (21, 01), (22, 11), (02)
[02, 21]	(00, 21), (01, 01), (02, 11), (10, 20), (11, 00), (12, 10), (20, 22), (21, 02), (22, 12), (02)
[02, 22]	(00, 22), (01, 02), (02, 12), (10, 21), (11, 01), (12, 11), (20, 20), (21, 00), (22, 10), (02)
[10, 00]	(00, 00), (01, 01), (02, 02), (10, 10), (11, 11), (12, 12), (20, 20), (21, 21), (22, 22), (10)
[10, 01]	(00, 01), (01, 02), (02, 00), (10, 11), (11, 12), (12, 10), (20, 21), (21, 22), (22, 20), (10)
[10, 02]	(00, 02), (01, 00), (02, 01), (10, 12), (11, 10), (12, 11), (20, 22), (21, 20), (22, 21), (10)
[10, 10]	(00, 10), (01, 11), (02, 12), (10, 20), (11, 21), (12, 22), (20, 00), (21, 01), (22, 02), (10)
[10, 11]	(00, 11), (01, 12), (02, 10), (10, 21), (11, 22), (12, 20), (20, 01), (21, 02), (22, 00), (10)
[10, 12]	(00, 12), (01, 10), (02, 11), (10, 22), (11, 20), (12, 21), (20, 02), (21, 00), (22, 01), (10)
[10, 20]	(00, 20), (01, 21), (02, 22), (10, 00), (11, 01), (12, 02), (20, 10), (21, 11), (22, 12), (10)
[10, 21]	(00, 21), (01, 22), (02, 20), (10, 01), (11, 02), (12, 00), (20, 11), (21, 12), (22, 10), (10)
[10, 22]	(00, 22), (01, 20), (02, 21), (10, 02), (11, 00), (12, 01), (20, 12), (21, 10), (22, 11), (10)
[11, 00]	(00, 00), (01, 12), (02, 21), (10, 11), (11, 20), (12, 02), (20, 22), (21, 01), (22, 10), (11)

[11, 01]	(00, 01) , (01, 10) , (02, 22) , (10, 12) , (11, 21) , (12, 00) , (20, 20) , (21, 02) , (22, 11) , (11)
[11, 02]	(00, 02) , (01, 11) , (02, 20) , (10, 10) , (11, 22) , (12, 01) , (20, 21) , (21, 00) , (22, 12) , (11)
[11, 10]	(00, 10) , (01, 22) , (02, 01) , (10, 21) , (11, 00) , (12, 12) , (20, 02) , (21, 11) , (22, 20) , (11)
[11, 11]	(00, 11) , (01, 20) , (02, 02) , (10, 22) , (11, 01) , (12, 10) , (20, 00) , (21, 12) , (22, 21) , (11)
[11, 12]	(00, 12) , (01, 21) , (02, 00) , (10, 20) , (11, 02) , (12, 11) , (20, 01) , (21, 10) , (22, 22) , (11)
[11, 20]	(00, 20) , (01, 02) , (02, 11) , (10, 01) , (11, 10) , (12, 22) , (20, 12) , (21, 21) , (22, 00) , (11)
[11, 21]	(00, 21) , (01, 00) , (02, 12) , (10, 02) , (11, 11) , (12, 20) , (20, 10) , (21, 22) , (22, 01) , (11)
[11, 22]	(00, 22) , (01, 01) , (02, 10) , (10, 00) , (11, 12) , (12, 21) , (20, 11) , (21, 20) , (22, 02) , (11)
[12, 00]	(00, 00) , (01, 22) , (02, 11) , (10, 12) , (11, 01) , (12, 20) , (20, 21) , (21, 10) , (22, 02) , (12)
[12, 01]	(00, 01) , (01, 20) , (02, 12) , (10, 10) , (11, 02) , (12, 21) , (20, 22) , (21, 11) , (22, 00) , (12)
[12, 02]	(00, 02) , (01, 21) , (02, 10) , (10, 11) , (11, 00) , (12, 22) , (20, 20) , (21, 12) , (22, 01) , (12)
[12, 10]	(00, 10) , (01, 02) , (02, 21) , (10, 22) , (11, 11) , (12, 00) , (20, 01) , (21, 20) , (22, 12) , (12)
[12, 11]	(00, 11) , (01, 00) , (02, 22) , (10, 20) , (11, 12) , (12, 01) , (20, 02) , (21, 21) , (22, 10) , (12)
[12, 12]	(00, 12) , (01, 01) , (02, 20) , (10, 21) , (11, 10) , (12, 02) , (20, 00) , (21, 22) , (22, 11) , (12)
[12, 20]	(00, 20) , (01, 12) , (02, 01) , (10, 02) , (11, 21) , (12, 10) , (20, 11) , (21, 00) , (22, 22) , (12)
[12, 21]	(00, 21) , (01, 10) , (02, 02) , (10, 00) , (11, 22) , (12, 11) , (20, 12) , (21, 01) , (22, 20) , (12)
[12, 22]	(00, 22) , (01, 11) , (02, 00) , (10, 01) , (11, 20) , (12, 12) , (20, 10) , (21, 02) , (22, 21) , (12)
[20, 00]	(00, 00) , (01, 02) , (02, 01) , (10, 20) , (11, 22) , (12, 21) , (20, 10) , (21, 12) , (22, 11) , (20)
[20, 01]	(00, 01) , (01, 00) , (02, 02) , (10, 21) , (11, 20) , (12, 22) , (20, 11) , (21, 10) , (22, 12) , (20)
[20, 02]	(00, 02) , (01, 01) , (02, 00) , (10, 22) , (11, 21) , (12, 20) , (20, 12) , (21, 11) , (22, 10) , (20)
[20, 10]	(00, 10) , (01, 12) , (02, 11) , (10, 00) , (11, 02) , (12, 01) , (20, 20) , (21, 22) , (22, 21) , (20)
[20, 11]	(00, 11) , (01, 10) , (02, 12) , (10, 01) , (11, 00) , (12, 02) , (20, 21) , (21, 20) , (22, 22) , (20)
[20, 12]	(00, 12) , (01, 11) , (02, 10) , (10, 02) , (11, 01) , (12, 00) , (20, 22) , (21, 21) , (22, 20) , (20)
[20, 20]	(00, 20) , (01, 22) , (02, 21) , (10, 10) , (11, 12) , (12, 11) , (20, 00) , (21, 02) , (02, 21) , (20)
[20, 21]	(00, 21) , (01, 20) , (02, 22) , (10, 11) , (11, 10) , (12, 12) , (20, 01) , (21, 00) , (02, 22) , (20)
[20, 22]	(00, 22) , (01, 21) , (02, 20) , (10, 12) , (11, 11) , (12, 10) , (20, 02) , (21, 01) , (02, 20) , (20)
[21, 00]	(00, 00) , (01, 11) , (02, 22) , (10, 21) , (11, 02) , (12, 10) , (20, 12) , (21, 20) , (02, 01) , (21)
[21, 01]	(00, 01) , (01, 12) , (02, 20) , (10, 22) , (11, 00) , (12, 11) , (20, 10) , (21, 21) , (02, 02) , (21)
[21, 02]	(00, 02) , (01, 10) , (02, 21) , (10, 20) , (11, 01) , (12, 12) , (20, 11) , (21, 22) , (02, 00) , (21)
[21, 10]	(00, 10) , (01, 21) , (02, 02) , (10, 01) , (11, 12) , (12, 20) , (20, 22) , (21, 00) , (02, 11) , (21)
[21, 11]	(00, 11) , (01, 22) , (02, 00) , (10, 02) , (11, 10) , (12, 21) , (20, 20) , (21, 01) , (02, 12) , (21)
[21, 12]	(00, 12) , (01, 20) , (02, 01) , (10, 00) , (11, 11) , (12, 22) , (20, 21) , (21, 02) , (02, 10) , (21)
[21, 20]	(00, 20) , (01, 01) , (02, 12) , (10, 11) , (11, 22) , (12, 00) , (20, 02) , (21, 10) , (02, 21) , (21)
[21, 21]	(00, 21) , (01, 02) , (02, 10) , (10, 12) , (11, 20) , (12, 01) , (20, 00) , (21, 11) , (02, 22) , (21)
[21, 22]	(00, 22) , (01, 00) , (02, 11) , (10, 10) , (11, 21) , (12, 02) , (20, 01) , (21, 12) , (02, 20) , (21)
[22, 00]	(00, 00) , (01, 21) , (02, 12) , (10, 22) , (11, 10) , (12, 01) , (20, 11) , (21, 02) , (02, 20) , (22)
[22, 01]	(00, 01) , (01, 22) , (02, 10) , (10, 20) , (11, 11) , (12, 02) , (20, 12) , (21, 00) , (02, 21) , (22)
[22, 02]	(00, 02) , (01, 20) , (02, 11) , (10, 21) , (11, 12) , (12, 00) , (20, 10) , (21, 01) , (02, 22) , (22)
[22, 10]	(00, 10) , (01, 01) , (02, 22) , (10, 02) , (11, 20) , (12, 11) , (20, 21) , (21, 12) , (02, 00) , (22)
[22, 11]	(00, 11) , (01, 02) , (02, 20) , (10, 00) , (11, 21) , (12, 12) , (20, 22) , (21, 10) , (02, 01) , (22)
[22, 12]	(00, 12) , (01, 00) , (02, 21) , (10, 01) , (11, 22) , (12, 10) , (20, 20) , (21, 11) , (02, 02) , (22)
[22, 20]	(00, 20) , (01, 11) , (02, 02) , (10, 12) , (11, 00) , (12, 21) , (20, 01) , (21, 22) , (02, 10) , (22)
[22, 21]	(00, 21) , (01, 12) , (02, 00) , (10, 10) , (11, 01) , (12, 22) , (20, 02) , (21, 20) , (02, 11) , (22)
[22, 22]	(00, 22) , (01, 10) , (02, 01) , (10, 11) , (11, 02) , (12, 20) , (20, 00) , (21, 21) , (02, 12) , (22)

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