Distinguishing Labelling of Partial Actions

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Abstract

A generalization of $G$–sets, called partial $(G, \alpha)$–sets, are the sets that admit an action of partial maps on their subsets. The $(G, \alpha)$–sets are expressed, up to partial $G$–isomorphism, in terms of stabilizers of the elements of a partial $G$–transversal in [7]. The main aim of this paper is to give some applications of this theorem to partial $(G, \alpha)$–sets and to distinguishing labelling of partial actions.

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1. Introduction

First, we recall the definition of a partial action on a set.

Definition. Let $G$ be a group and $X$ a set. A partial action of $G$ on $X$ is a pair $\alpha = \{\{D_g\}_{g \in G}, \{\alpha_g\}_{g \in G}\}$, where for each $g \in G$, $D_g$ is a subset of $X$ and $\alpha_g : D_g^{-1} \rightarrow D_g$ is a bijective map, satisfying the following three properties for each $g, h \in G$:

(i) $D_1 = X$, and $\alpha_1 = Id_X$, the identity map on $X$,
(ii) $\alpha_g(D_g^{-1} \cap D_h) = D_g \cap D_{gh}$,
(iii) $\alpha_g(\alpha_h(x)) = \alpha_{gh}(x)$ for $x \in D_h^{-1} \cap D_{h^{-1}g^{-1}}$. 
If $\alpha$ is a partial action of $G$ on $X$, then we say that $X$ is a partial $(G, \alpha)$–set.

In case of a ring, $D_g$'s are taken to be ideals and maps to be ring homomorphisms. Partial actions on algebras were studied by M. Dokuchaev and R. Exel [2] M. Dokuchaev, M. Ferrero and A. Paques [3] and on semialgebras by Sharma et. al. [4,5,6,7]. In this paper, we introduce the concept of distinguishing numbers for partial actions and derive some results regarding these as applications of the structure theorem for partial $G$–sets.

2. Preliminaries

We recall some definitions and results from [7].

Let $X$ be a partial $(G, \alpha)$–set and $x \in X$. Then the set $G_x = \{ g \in G | x \in D_{g^{-1}} \}$ is a stabilizer of $x$ in $G$. The stabilizer of an element of $X$ becomes a subgroup of $G$ similar to that in the group actions.

The set $O_G(x) = \{ \alpha_g^{-1}(x) | x \in D_g \}$ is the orbit of $x$ in $G$. Two orbits in a partial $(G, \alpha)$–set are either identical or disjoint. Therefore, if $y_1, y_2 \in O_G(x)$, then $O_G(y_1) = O_G(y_2)$. Let $X'$ be a subset of $X$. Then $X'$ is said to be a partial $(G, \alpha)$–subset of $X$ if $x \in X' \cap D_g$ implies $\alpha_g^{-1}(x)$. The orbit $O_G(x)$ is a partial $(G, \alpha)$–subset of $X$.

**Definition 2.1.** Let $X$ be a partial $(G, \alpha)$–set and $X'$ a partial $(G, \alpha')$–set. Then a function $\beta : X \to X'$ is said to be a partial $G$–map if for $x \in D_{g^{-1}}$, $\beta(x) \in D_{g'^{-1}}$ and $\beta(\alpha_g(x)) = \alpha_g'(\beta(x))$. Further, $X$ is said to be partial $G$–isomorphic to $X'$ if $\beta$ is a bijective $G$–map with $\beta(D_g) \supseteq D_{g'}$.

**Remark.** Let $X$ be a partial $(G, \alpha)$–set, which is partial $G$–isomorphic to a $(G, \alpha')$–set $X'$. Then the image of a $G$–subset of $X$ is a $G$–subset of $X'$.

**Lemma 2.2.** Let $G^x = \{ g^{-1} \in G | x \in D_g \}$, $x \in X$. Then the set $G^x/G_x = \{ g^{-1} G_x | x \in D_g \}$ becomes a partial $G$–set. Further, $G^x/G_x$ is partial $G$–isomorphic to $O_G(x)$.

**Theorem 2.3 (structure theorem).** Let $X$ be a partial $(G, \alpha)$–set and $S$ a partial $G$–transversal in $X$. Then $X$ is $G$–isomorphic to $\bigcup_{s \in S} G^s/G_s$.

3. Applications

Let $X$ be a partial $(G, \alpha)$–set. Then $X$ is a $G$–free set if $G_x = 1$ for all $x \in X$.

**Lemma 3.1.** A partial $(G, \alpha)$–set is $G$–free if and only if it is a disjoint
union of copies of \( G^s, s \in S \).

**Proof.** From the Theorem 2.3, we have \( \cup_{s \in S} G^s/G_s \cong X \). We know that \( X \) is \((G, \alpha)\)-free partial \((G, \alpha)\)-set if and only if \( G_x = \{1\} \) for all \( x \in X \), which implies that \( \cup_{s \in S} G^s \cong X \).

**Theorem 3.2.** If \( \beta \) is a partial \((G, \alpha)\)-map from a partial \((G, \alpha)\)-set \( X \) to a partial \((G, \alpha')\)-set \( Y \), then \( G_x \subseteq G_{\beta(x)} \). The equality follows if \( \beta \) is injective.

**Proof.** Let \( g \in G_x \), then \( \alpha_g(x) = x \) for \( x \in D_{g^{-1}} \). As \( \beta \) is a partial \( G \)-map,

\[
\alpha'_g(\beta(x)) = \beta(\alpha_g(x)) = \beta(x) \quad \text{(because } x \in G_x),
\]

which implies that \( g \in G_{\beta(x)} \), proving that \( G_x \subseteq G_{\beta(x)} \) for \( x \in D_{g^{-1}} \). Let \( g \in G_{\beta(x)} \) for \( \beta(x) \in D_{g^{-1}} \). Then \( \alpha'_g(\beta(x)) = \beta(x) \). But \( \beta \) is a partial \((G, \alpha)\)-map, so again

\[
\beta(\alpha_g(x)) = \alpha'_g(\beta_g(x)) = \beta(x).
\]

As \( \beta \) is injective, \( \alpha_g(x) = x \), implying \( g \in G_x \). Therefore, \( G_{\beta(x)} \subseteq G_x \) for \( x \in D_{g^{-1}} \) and \( \beta(x) \in D_{g^{-1}} \).

**Corollary 3.3.** The only partial \((G, \alpha)\)-sets which have partial \((G, \alpha)\)-map to free partial \((G, \alpha)\)-sets are the free partial \((G, \alpha)\)-sets.

**Proof.** Let \( \beta : X \to Y \) be a partial \((G, \alpha)\)-map, where \( X \) is a partial \((G, \alpha)\)-set and \( Y \) is a free partial \((G, \alpha)\)-set. By Theorem 3.2, for \( x \in X, G_x \subseteq G_{\beta(x)} \). But \( Y \) is a free partial \((G, \alpha)\)-set. Therefore, \( G_{\beta(x)} = 1 \) for all \( \beta(x) \in Y \), which implies that \( G_x = 1 \) for all \( x \in X \). Hence, \( G_x \) is a free partial \((G, \alpha)\)-set.

**Lemma 3.4.** The intersection of two partial \((G, \alpha)\)-subsets of \( X \) is a partial \((G, \alpha)\)-subset of \( X \).

**Proof.** Let \( A \) and \( B \) be two partial \((G, \alpha)\)-subsets of \( X \) and \( x \in (A \cap B) \cap D_g \).

Then \( x \in A \cap D_g \) and \( B \cap D_g \). So \( \alpha_{g^{-1}}(x) \in A \cap B \). Hence \( A \cap B \) is a partial subset of \( X \).

**Lemma 3.5.** Let \( X \) be a partial \((G, \alpha)\)-set. Then \( P(X) \) can be made a partial \((G, \alpha)\)-set using the partial action \( \alpha \) of \( G \) on \( X \).

**Proof.** Define \( \bar{D}_g = \{ A \cap D_g \mid A \text{ is a partial } (G, \alpha)\text{-subset of } X \} \) and \( \bar{\alpha}_{g^{-1}} : \bar{D}_g \to \bar{D}_{g^{-1}} \) by \( \bar{\alpha}_{g^{-1}}(A \cap D_g) = \{ \alpha_{g^{-1}}(x) \mid x \in A \cap D_g \} \). It suffices to verify three properties of the Definition of partial \( G \)-set.

(i) To verify the first property of the definition, we define a map from \( \bar{\alpha}_1 : \bar{D}_1 \to \bar{D}_1 \) by \( \bar{\alpha}_1(T) = \{ \alpha_1(x) \mid x \in T \} = T \) where \( \bar{D}_1 = P(X) \) and \( T \in \bar{D}_1 \). Clearly \( \bar{\alpha}_1 \) is the identity map on \( \bar{D}_1 \).

(ii) To verify the second property of the definition, that is, \( \bar{\alpha}_{g^{-1}}(\bar{D}_g \cap \bar{D}_h) = \bar{D}_{g^{-1}} \cap \bar{D}_{h^{-1}} \).
Let $X' \in \bar{D}_g \cap \bar{D}_h$. That is, $X' \in \bar{D}_g$ and $X' \in \bar{D}_h$. Then $X' = A \cap D_g = B \cap D_h$, where $A$ and $B$ are the partial $(G, \alpha)$–subsets of $X$. Therefore, for $x \in X'$, $\alpha_{g^{-1}}(x) \in \alpha_{g^{-1}}(X')$ implies that $x \in (A \cap B) \cap (D_g \cap D_h)$. So\
\[ \alpha_{g^{-1}}(x) \in \alpha_{g^{-1}}(D_g \cap D_h) = D_{g^{-1}} \cap D_{g^{-1}h}. \]  (1)
Also $x \in A \cap B$ and $A \cap B$ being a partial $(G, \alpha)$–subset of $X$, implies $\alpha_{g^{-1}}(x) \in A \cap B$.

From (1) and (2), we get $\alpha_{g^{-1}}(x) \in ((A \cap B) \cap (D_{g^{-1}} \cap D_{g^{-1}h})), \text{ which implies,}$\
\[ \alpha_{g^{-1}}(X') \subseteq \bar{D}_{g^{-1}} \cap \bar{D}_{g^{-1}h}, \]  (3)

Conversely, let $X' \in \bar{D}_{g^{-1}} \cap \bar{D}_{g^{-1}h}$. Then $X' \in (A \cap B) \cap (D_{g^{-1}} \cap D_{g^{-1}h})$. Let $x \in X'$. Thus $x \in (A \cap B)$ and $(D_{g^{-1}} \cap D_{g^{-1}h})$, so that $y = \alpha_g(x) \in A \cap B$ (by Lemma 3.4), therefore\
y \in \alpha_g(D_{g^{-1}} \cap D_{g^{-1}h}) = D_g \cap D_h.

Let\
\[ Y = \{ y | y = \alpha_g(x), x \in X' \} \subseteq (A \cap D_g) \cap (B \cap D_h) \in \bar{D}_g \cap \bar{D}_h. \]

By definition of $\alpha_{g^{-1}}$, we have\
\[ \alpha_{g^{-1}}(Y) = \{ \alpha_{g^{-1}}(y), y \in Y \} = \{ x | x \in X' \} = X'. \] (iii) To verify (iii), we have to show\
\[ (\alpha_{g^{-1}}(\alpha_{h^{-1}}(X'))) = \alpha_{g^{-1}}(X') \text{ for } X' \in \bar{D}_h \cap \bar{D}_h. \]

As $X' \in \bar{D}_h \cap \bar{D}_h$. Without loss of generality, there exists a partial $(G, \alpha)$-subset $A$ of $X$ such that $X' = A \cap D_h \cap D_{hg}$. Let $x \in X' = A \cap D_h \cap D_{hg}$, then\
\[ \alpha_{g^{-1}}(X') = \{ \alpha_{g^{-1}}(x) | x \in A \cap D_h \cap D_{hg} \}. \]

For $x \in A \cap D_h \cap D_{hg}$, we have $\alpha_{g^{-1}}(x) = \alpha_{g^{-1}}(\alpha_{h^{-1}}(x))$. So\
\[ \alpha_{g^{-1}}(\alpha_{h^{-1}}(X')) = \alpha_{g^{-1}}(\{ \alpha_{h^{-1}}(x), x \in A \cap D_h \cap D_{hg} \} = \alpha_{g^{-1}}(Y), Y = \{ y | y = \alpha_{h^{-1}}(x) \text{ and } x \in X' \} = \{ \alpha_{g^{-1}}(y) | y \in Y \} = \{ y | y = \alpha_{h^{-1}}(x) \text{ and } x \in X' \} = \{ \alpha_{g^{-1}}(\alpha_{h^{-1}}(x)) | x \in X' \} = \{ \alpha_{g^{-1}}(x') | x \in X' \} = \alpha_{g^{-1}}(X'). \]

Hence, $P(X)$ is a partial $(G, \alpha)$–set.
4. Distinguishing labelling of the partial actions

Following [1], we define

**Definition 4.1.** Let $G$ be a group, $X$ a $(G, \alpha)$-set and $\alpha$ be a proper partial action of $G$ on $X$. For a +ve integer $r$, an $r$-labelling $\phi : X \to \{1, 2, 3, 4, \ldots, r\}$ is said to be $r$-distinguishing with respect to the partial action $\alpha$ of $G$ on $X$ if for each $g \in G$, $\alpha_g \neq \alpha_1|_{D_g}$, there is an element $x \in D_g$ such that $\phi(x) \neq \phi(\alpha_g^{-1}(x))$.

**Theorem 4.2.** Let $X$ be a proper partial $(G, \alpha)$-set, $\bigcap_{g \in G} D_{g^{-1}} \neq \phi$ and $O_G(x) \subseteq \bigcap_{g \in G} D_{g^{-1}}$ for some $x \in X$. If $G_x$ is normal, then $O_G(x)$ can be distinguished with two labels.

**Proof.** Define

$$\phi(y) = \begin{cases} 
1, & \text{if } y = x \\
2, & \text{for all } y \neq x \in O_G(x)
\end{cases}$$

Suppose that for $g \in G$, $\alpha_g$ does not distinguish $O_G(x)$, so it must fix $x$, that is, $\alpha_g(x) = x$, for $x \in D_{g^{-1}}$, so $g \in G_x$. Let $x \neq y \in O_G(x)$. Then there must exist $x \neq z \in O_G(x)$ such that $\alpha_g(y) = z$. Thus for $y, z \in O_G(x)$, both not equal to $x$, there are $g_1, g_2 \in G$ both not equal to $g$ such that $\alpha_{g_1}(y) = x, \alpha_{g_2}(z) = x$. Also $z = \alpha_{g_1}(y)$, which implies that $z = \alpha_g(\alpha_{g_1}^{-1}(x))$, that is, $\alpha_g(\alpha_{g_1}^{-1}(x)) = \alpha_{g_2}^{-1}(x)$. Since $x \in \bigcap_{g \in G} D_{g^{-1}}$, we have $\alpha_{g_2}^{-1}(x) = \alpha_{g_1}^{-1}(x)$. Further $G_x$ is normal, and $g \in G_x$, we have $g_1 g_1^{-1} g_2^{-1} \in G_x$, that is, there exist $h \in G_x$ such that $g_1 g_2^{-1} = h$, implying $g_1^{-1} = g_2^{-1} h$ which together with $\alpha_{g_2}^{-1}(x) = \alpha_{g_1}^{-1}(x)$ implies that $\alpha_{g_1}^{-1}(x) = \alpha_{g_1}^{-1}(x) = \alpha_{g_1}^{-1}(x)$. So $\alpha_{g_2}^{-1}(x) = \alpha_g((\alpha_{g_1}^{-1}(x)) = \alpha_g(y) = z$ and $\alpha_{g_1}^{-1}(x) = \alpha_{g_1}^{-1}(\alpha_h(x)) = \alpha_{g_1}^{-1}(x) = y$. This proves that $y = z$ and hence $\alpha_g = \alpha_1$ on $O_G(x)$.

**Theorem 4.3.** Let $X$ be a proper partial $(G, \alpha)$-set. If $X$ has a partial $(G, \alpha)$-orbit $O = \{x_1, x_2, x_3, \ldots, x_s\}$, $x_1, x_2, x_3, \ldots, x_s \subseteq \bigcap_{g \in G} D_{g^{-1}}$ that can be distinguished with $k$ labels and $\cap_{i=1}^s G_{x_i} = \{1\}$, then $X$ can be distinguished with $k$ labels.

**Proof.** If we label $X$ in such a way that $O_G(x)$ is $k$-distinguishing, then for $g \in G, \alpha_g \neq \alpha_1$ will act non-trivially on the partial $(G, \alpha)$-orbit, since $\cap_{i=1}^s G_{x_i} = \{1\}$. So the only $\alpha_g$ that stabilizes all $x_i$ in $O$ is $\alpha_1$. This implies that $X$ can be distinguished with $k$-labels.

**Theorem 4.4.** Let $X$ be a proper partial $(G, \alpha)$-set and $x_1, x_2, x_3, \ldots, x_t \in D_{g^{-1}}$ for all $g \in G$, are from $t$-different partial $(G, \alpha)$-orbits with respective
partial stabilizer subgroups $G_{x_1}, G_{x_2}, G_{x_3}, \ldots, G_{x_t}$. If $\cap_{i=1}^{t} G_{x_i} = \{1\}$, then $D_G(X) = 2$.

**Proof.** Define the labeling $\phi$ as follows:

$$\phi(x) = \begin{cases} 
1, & \text{if } x \in \{x_1, x_2, x_3, \ldots, x_t\} \\
2, & \text{otherwise}
\end{cases} \quad (1)$$

We want to show that for $g \in G, \alpha_g \neq \alpha_1$, there is at least one $x_i \in X \cap D_{g^{-1}}$ such that $\phi(\alpha_g(x)) \neq \phi(x)$. Since the intersection of the partial stabilizer subgroups of $x_1, x_2, x_3, \ldots, x_t$ is the identity, $\alpha_g$ must map at least one $x_i, 1 \leq i \leq t$, to another vertex in $O_{x_i}$, which by definition of $\phi$ is labelled 2. Thus $X$ has been distinguished with two labels.

**Theorem 4.5.** Let a group $G = \{g_1 = e, g_2, \ldots, g_p\}$ of order $p$ (prime) act partially on a set $X$. If the partial action of $G$ on $X$ is proper and non trivial for each $g \in G, g \neq 1, \alpha_g \neq \alpha_1|D_{g^{-1}}$. Then it is 2–distinguishable if and only if there exist $x_1, x_2, x_3, \ldots, x_p \in X$ with $x_i \in D_{g_i^{-1}}$ such that $O_G(x_i) \approx G_{x_i}$.

**Proof.** Define a labelling $\phi : X \mapsto \{1, 2\}$ by

$$\phi(x_i) = \begin{cases} 
1, & i = 1, 2, 3, \ldots, p \\
2, & \text{otherwise}
\end{cases} \quad (2)$$

Let $g_i \in G$ such that $\alpha_g \neq \alpha_1$. Then $O_G(x_i) \approx G_{x_i}$, using Structure theorem, give that $\alpha_g(x_i) \neq x_i$. So $x_i \neq \alpha_g(x_i) \in O_G(x_i)$, is labelled by 2. Therefore, the partial action of $G$ on $X$ is 2–distinguishable. Conversely, suppose that partial action of $G$ on $X$ is 2–distinguishable. Then for each $g \in G$, there exists $x_i \in D_{g^{-1}}$ such that $\alpha_g(x_i) \neq x_i$. Therefore $G_{x_i} \neq G$. Since order of $G$ is prime and $G_{x_i}$ is a subgroup of $G$, therefore $G_{x_i} = \{e\}$ and by using Structure Theorem, we get $O_G(x_i) \approx G_{x_i}$.

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