On the Pentagon Relations of Valued Quivers

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Abstract

In this paper, we provide a combinatorial proof of the pentagon relations for valued quivers and their associated clusters for coefficient free cluster algebras.

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1 Introduction

Cluster algebras were introduced by S. Fomin and A. Zelevinsky in [5, 6, 1]. The original motivation was to find an algebraic combinatorial framework to study canonical basis and total positivity. Each cluster algebra $\mathcal{A}$ is uniquely determined by a class of combinatorial data called seeds. A seed in a field of rational functions $\mathbb{F}$ is a pair of an algebraically independent set of rational
functions that generates $\mathfrak{F}$ along with a valued quiver. Any two elements in the class of seeds of $\mathcal{A}$ can be obtained from each other by applying a sequence of combinatorial operations called mutations.

Recently, many new results have been established relating mutations of quivers (directed graphs) with a huge class of subjects where cluster algebras have subsequently turned out to be relevant, see for example [4, 8, 10].

The relation between mutations and valued quivers morphisms has not been given enough attention. The main goal of this paper is providing a combinatorial detailed proof for the pentagon relations, [3, Section 2], in the case of valued quivers. Which we use to provide an answer for the following question

**Question 1.1** ([11, Question 3.2]) For which valued quiver $\Gamma$ and a valued quiver automorphism $\phi$ are there a particular sequence of mutations $\mu$ such that $\phi(\Gamma) = \mu(\Gamma)$?

The paper consists of two sections. The introduction section which also contains some basic concepts and notations of valued quivers and cluster algebras. The second section is devoted for the main result of the article which is Lemma 2.2 where we have the proof of the pentagon relations for valued quivers. We conclude the paper by Theorem 2.6 which provides a way to associate a particular sequence of mutations with every quiver automorphism for simply-laced quivers.

Throughout the paper, $K$ is a field with zero characteristic and $\mathcal{F} = K(t_1, t_2, \ldots, t_n)$ is the field of rational functions in $n$ independent (commutative) variables over $K$. Let $Aut_K(\mathcal{F})$ denote the automorphism group of $\mathcal{F}$ over $K$ and Let $S_n$ be the symmetric group on $n$ letters. We always denote $(b_{ij})$ for the square matrix $B$ and $[1, n] = \{1, 2, \ldots, n\}$.

### 1.1 Basic definitions and notations

For more details about the material of this section refer to [2, 9, 5, 6, 1, 11].

**Definition 1.2 (Valued quivers)**

- A valued quiver of rank $n$ is a quadruple $\Gamma = (Q_0, Q_1, V, d)$, where
  - $Q_0$ is a set of $n$ vertices labeled by numbers from the set $[1, n]$;
  - $Q_1$ is called the set of arrows of $\Gamma$ and consists of ordered pairs of vertices, that is $Q_1 \subset Q_0 \times Q_0$;
  - $V = \{(v_{ij}, v_{ji}) \in \mathbb{N} \times \mathbb{N}|(i, j) \in Q_1\}$, $V$ is called the valuation of $\Gamma$;
  - $d = (d_1, \cdots, d_n)$, where $d_i$ is a positive integer for each $i$, such that $d_i v_{ij} = v_{ji}d_j$, for every $i, j \in [1, n]$. In the following we will omit $d$ as it does not play an essential role in the content of this paper.
In the case of \((i, j) \in Q_1\), then there is an arrow oriented from \(i\) to \(j\) and in notation we shall use the symbol \(\cdot_i^{(v_{ij}, v_{ji})} \cdot_j\). In such case the two vertices \(i\) and \(j\) are said to be adjacent. If \(v_{ij} = v_{ji} = 1\) we simply write \(\cdot_i \rightarrow \cdot_j\).

- If \(v_{ij} = v_{ji}\) for every \((v_{ij}, v_{ji}) \in V\) then \(\Gamma\) is called a **equally valued quiver**;

- An equally valued quiver \(\Gamma\) with \(v_{ij} = v_{ji} = 1\) for every \((v_{ij}, v_{ji}) \in V\) is called a **simply-laced quiver**.

In this paper, we moreover assume that \((i, i) \notin Q_1\) for every \(i \in Q_0\), and if \((i, j) \in Q_1\) then \((j, i) \notin Q_1\). Also, we shall assume that \(\Gamma = (Q_0, Q_1, V, d)\) is connected, in the sense that, for every \(v, v' \in \Gamma\), there is a sequence \(v = v_1, \ldots, v_l = v'\) of vertices such that \(b_{v_tv_{t+1}} \neq 0\), i.e., \(v_t\) and \(v_{t+1}\) are adjacent vertices for \(i = 1, \ldots, l - 1\).

**Remarks 1.3**

1. Every quiver \(Q\) without loops nor 2-cycles corresponds to an equally valued quiver which has an arrow \((i, j)\) if there is at least one arrow directed from \(i\) to \(j\) in \(Q\) and with the valuation \((v_{ij}, v_{ji}) = (t, t)\), where \(t\) is the number of arrows from \(i\) to \(j\).

2. Every valued quiver of rank \(n\) corresponds to a skew symmetrizable integer matrix \(B(\Gamma) = (b_{ij})_{i,j \in [1,n]}\) given by

\[
b_{ij} = \begin{cases} 
  v_{ij}, & \text{if } (i, j) \in Q_1, \\
  0, & \text{if neither } (i, j) \text{ nor } (j, i) \text{ is in } Q_1, \\
  -v_{ij}, & \text{if } (j, i) \in Q_1.
\end{cases}
\]  

Conversely, given a skew symmetrizable \(n \times n\) matrix \(B\), a valued quiver \(Q_B\) can be easily defined such that \(B(\Gamma_B) = B\). This gives rise to a bijection between the skew-symmetrizable \(n \times n\) integral matrices \(B\) and the valued quivers with set of vertices \([1, n]\), up to isomorphism fixing the vertices.

**Definition 1.4 (Valued quiver mutations)** Let \(\Gamma\) be a valued quiver. The mutation \(\mu_k(\Gamma)\) at a vertex \(k\) is defined through Fomin-Zelevinsky’s mutation of the associated skew-symmetrizable matrix. The mutation of a skew symmetrizable matrix \(B = (b_{ij})\) on the direction \(k \in [1, n]\) is given by \(\mu_k(B) = (b'_{ij})\), where

\[
b'_{ij} = \begin{cases} 
  -b_{ij}, & \text{if } k \in \{i, j\}, \\
  b_{ij} + \text{sign}(b_{ik}) \max(0, b_{ik}b_{kj}), & \text{otherwise}.
\end{cases}
\]
The following remarks provide a set of rules that are adequate to be used to calculate mutations of valued quivers without using their associated skew-symmetrizable matrix.

**Remarks 1.5**

1. Let $\Gamma = (Q_0, Q_1, V)$ be a valued quiver. The mutation $\mu_k(\Gamma)$ at the vertex $k$ can be described using the mutation of $B(\Gamma)$ as follows: Let $\mu_k(\Gamma) = (Q'_0, Q'_1, V')$ be the valued quiver obtained from $\Gamma$ by applying mutation at the vertex $k$. We obtain $Q'_1$ and $V'$, by altering $Q_1$ and $V$, based on the following rules

   (a) replace the pairs $(i, k)$ and $(k, j)$ with $(k, i)$ and $(j, k)$ respectively and switch the components of the ordered pairs of their valuations;

   (b) if $(i, k), (k, j) \in Q_1$, such that at least $i$ or $j$ is in $Q_0$ but $(j, i) \notin Q_1$ (respectively $(i, j) \in Q_1$) add the pair $(i, j)$ to $Q'_1$, and give it the valuation $(v_{ik}v_{kj}, v_{ki}v_{jk})$ (respectively change its valuation to $(v_{ij} + v_{ik}v_{kj}, v_{ji} + v_{ki}v_{jk})$);

   (c) if $(i, k), (k, j)$ and $(j, i)$ in $Q_1$, then we have three cases

      i. if $v_{ik}v_{kj} < v_{ij}$, then keep $(j, i)$ and change its valuation to $(v_{ji} - v_{jk}v_{ki}, -v_{ij} + v_{ik}v_{kj})$;

      ii. if $v_{ik}v_{kj} > v_{ij}$, then replace $(j, i)$ with $(i, j)$ and change its valuation to $(-v_{ij} + v_{ik}v_{kj}, |v_{ji} - v_{jk}v_{ki}|)$;

      iii. if $v_{ik}v_{kj} = v_{ij}$, then remove $(j, i)$ and its valuation.

2. The mutation of valued quiver is again a valued quiver.

3. One can see that; $\mu_k^2(\Gamma) = \Gamma$ and $\mu_k(B(Q)) = B(\mu_k(Q))$ at each vertex $k$. The involution property of mutations is used to prove that mutations define an equivalence relations on the set of valued quivers of a certain rank. The equivalence class of a valued quiver $\Gamma$ is called the mutation class of $\Gamma$.

Unless it is otherwise said, in the rest of the paper, let $\Gamma = (Q_0, Q_1, V, d)$ be a valued quiver such that every arrow $(i, j) \in Q_1$ has a valuation $(b_{ij}, b_{ji})$.

**Example 1.6** Applying the mutations rules from Part (1) of Remark 1.5 on the following valued quiver at the vertex 2, we obtain

![Figure 1. Mutation of a valued quiver of rank 3.](image-url)
1.2 Cluster algebras

[6, Definition 2.3] A labeled seed of rank \( n \) in \( \mathcal{F} \) is a pair \((X, \Gamma)\) where \( X = (x_1, \ldots, x_n) \) is an \( n \)-tuple elements of \( \mathcal{F} \) forming a free generating set and \( \Gamma \) is a valued quiver of rank \( n \). In this case, \( X \) is called a cluster and elements of \( X \) are called cluster variables.

We will refer to labeled seeds simply as seeds, when there is no risk of confusion.

The definition of clusters above is a bit different from the definition of clusters given in [1] and [5].

**Definition 1.7 (Seed mutations)** Let \( p = (X, \Gamma) \) be a seed in \( \mathcal{F} \), and \( k \in [1, n] \). A new seed \( \mu_k(X, \Gamma) = (\mu_k(X), \mu_k(\Gamma)) \) is obtained from \((X, \Gamma)\) by setting \( \mu_k(X) = (x_1, \ldots, x'_k, \ldots, x_n) \) such that \( x'_k \) is defined by the so-called exchange relation:

\[
x_k x'_k = \prod_{i, (i, k) \in Q_1} x_i^{v_{ik}} + \prod_{i, (k, i) \in Q_1} x_i^{v_{ki}},
\]

where \( Q_1 \) is the set of arrows of \( \Gamma \). And \( \mu_k(\Gamma) \) is the mutation of \( \Gamma \) at the vertex \( k \).

1.3 Valued quivers automorphisms

**Definition 1.8** A valued quiver morphism \( \phi \) from \( \Gamma = (Q_0, Q_1, V) \) to \( \Gamma' = (Q'_0, Q'_1, V') \) is a pair of maps \((\sigma_\phi, \sigma_1)\) where \( \sigma_\phi : Q_0 \to Q'_0 \) and \( \sigma_1 : Q_1 \to Q'_1 \) such that \( \sigma_1(i, j) = (\sigma_\phi(i), \sigma_\phi(j)) \) and \((v_{ij}, v_{ji}) = (v'_{\sigma_\phi(i)\sigma_\phi(j)}, v'_{\sigma_\phi(j)\sigma_\phi(i)})\) for each \((i, j) \in Q_1\). If \( \phi \) is invertible then it is called a valued quiver isomorphism. In particular \( \phi \) is called a valued quiver automorphism of \( \Gamma \) if it is a valued quiver isomorphism from \( \Gamma \) to itself.

Let \( \phi \) be a valued quiver automorphism of \( \Gamma = (Q_0, Q_1, V) \). Then \( \phi \) induces a permutation \( \sigma_\phi \in S_n \). We can obtain a new valued quiver \( \phi(\Gamma) = (Q'_0, Q'_1, V') \) from \( \Gamma \) as follows

- \( Q'_0 \) is obtained by permuting the vertices of \( Q_0 \) using \( \sigma_\phi \);
- \( Q'_1 = \{(\sigma_\phi(i), \sigma_\phi(j)) | (i, j) \in Q_1\} \);
- For every \((\sigma_\phi(i), \sigma_\phi(j)) \in Q'_1\) we give the valuation \((v_{\sigma_\phi(i)\sigma_\phi(j)}, v_{\sigma_\phi(j)\sigma_\phi(i)})\);

**Remarks 1.9** The action of a quiver automorphism \( \phi \) on a quiver \( \Gamma \) is equivalent to the action of \( \sigma_\phi \) on the associated skew-symmetrizable matrix \( B(\Gamma) \) by permuting the rows and columns based on \( \sigma_\phi \). More precisely, if \( B(\Gamma) = (b_{ij})_{i,j \in [1,n]} \), then \( \sigma_\phi(B(\Gamma)) = (b_{\sigma_\phi(i)\sigma_\phi(j)})_{i,j \in [1,n]} \). For the sake of simplicity we will use \( \phi \) and \( \sigma_\phi \) as the same operator.
Example 1.10 Consider the following valued quiver $\Gamma$ with the action of the quiver automorphism $\phi$ that has an underlying permutation $\sigma_\phi = (123)$:

$$
\begin{array}{ccc}
\begin{array}{c}
1 \\
(4,1)
\end{array} & \xrightarrow{(2,1)} & \begin{array}{c} \\
2 \\
(1,2)
\end{array} \\
\begin{array}{c} \\
3
\end{array} & \xleftarrow{(2,1)} & \begin{array}{c} (4,1)\\(1,2)\end{array}
\end{array}
\xrightarrow{\phi} 
\begin{array}{ccc}
\begin{array}{c} \\
2 \\
(4,1)
\end{array} & \xrightarrow{(2,1)} & \begin{array}{c} \\
3 \\
(1,2)
\end{array} \\
\begin{array}{c} (1,2)\\(4,1)\\1
\end{array} & \xleftarrow{(2,1)} & \begin{array}{c} \\
2 \\
(1,2)
\end{array}
\end{array}
\end{array}
$$

Figure 2. Quiver automorphisms of valued quiver of rank 3.

2 The pentagon relations of valued quivers

In this section we provide the main result of this paper, a proof for the pentagon relations, Lemma 2.2.

Before stating the next lemma, we need to develop some notations. For a seed $p = (Y, \Gamma)$, the neighborhood of a cluster variable $y_i$ is denoted by $N_{\Gamma}(i)$ and is defined to be the union of the two sets $N_{\Gamma,+}(i)$ and $N_{\Gamma,-}(i)$. Where

$$
N_{\Gamma,+}(i) = \{y_j \in Y; (i, j) \in Q_1\} \quad \text{and} \quad N_{\Gamma,-}(i) = \{y_j \in Y; (j, i) \in Q_1\}.
$$

For $x, y \in Q_0$, let $\mu_{xy}$ denote the sequence of mutations $\mu_y \mu_x$ and $\Gamma_{xy}$ be the valued quiver $\mu_y(\mu_x(\Gamma))$.

Remarks 2.1 From Remark 1.5, we have the following rules which govern the changes that occur on the neighborhood of a cluster variable $y_i$ after applying the mutation $\mu_i$ on the seed $p = (Y, \Gamma)$:

1. $N_{\Gamma,+}(i) = N_{\Gamma,-}(i)$ and $N_{\Gamma,-}(i) = N_{\Gamma,+}(i)$;

2. For any three vertices $i, j, k \in Q_0$, we have the following cases

   (a) if $j \in N_{\Gamma,+}(i)$ and $k \in N_{\Gamma,-}(i)$ and either $b_{kj} = 0$ or $j \in N_{\Gamma,+}(k)$, then $j \in N_{\Gamma,+}(k)$;

   (b) if $j \in N_{\Gamma,+}(i)$, $k \in N_{\Gamma,-}(i) \cap N_{\Gamma,+}(j)$, then we have two cases

      i. If $b_{ij}b_{ki} - b_{kj} > 0$, then $j \in N_{\Gamma,+}(k)$;

      ii. If $b_{ij}b_{ki} - b_{kj} < 0$, then $j \in N_{\Gamma,-}(k)$.

Let $\sigma_{ij}$ be the transposition in the permutation group $\mathfrak{S}_n$ that fixes every vertex except $i$ and $j$.

Lemma 2.2 (The pentagon relations) Let $p = (X, \Gamma)$ be a seed of rank $n$. Then for every $(i, j) \in Q_1$ such that $b_{ij} = b_{ji} = 1$, we have

$$
\mu_{ijij}(p) = \mu_{jiji}(p) = \sigma_{ij}(p). \quad (4)
$$
Proof. In the following we will prove $\mu_{ijji}(p) = \sigma_{ij}(p)$ for the two components of the seed $p$ starting with the valued graph $\Gamma$. Without loss of generality, we assume that $j \in N_{\Gamma-}(i)$. It would be adequate to prove that for every $k \in N_{\Gamma}(i)$ and $t \in N_{\Gamma}(j)$, we have

(A)
$$N_{\mu_{ijji,+}(\Gamma)}(k) \setminus \{i\} = N_{\Gamma,+}(k) \setminus \{i\} \quad \text{and} \quad N_{\mu_{ijji,-}(\Gamma)}(k) \setminus \{i\} = N_{\Gamma,-}(k) \setminus \{i\};$$

(B)
$$N_{\mu_{ijji,+}(\Gamma)}(t) \setminus \{j\} = (N_{\Gamma,+}(t) \setminus \{j\} \quad \text{and} \quad N_{\mu_{ijji,-}(\Gamma)}(t) \setminus \{j\} = N_{\Gamma,-}(t) \setminus \{j\}. $$

Such that

1. the following two sets equations are satisfied
$$N_{\mu_{ijji}(\Gamma)}(k) = (N_{\Gamma}(k)) \cup \{j\} \setminus \{i\} \quad \text{and} \quad N_{\mu_{ijji}(\Gamma)}(t) = (N_{\Gamma}(t)) \cup \{i\} \setminus \{j\};$$

2. for any vertex $x \in Q_0 \setminus \{i, j\}$, the arrows $(x, k)$ and $(k, x)$ in $\Gamma$ will keep their valuation in $\mu_{ijji}(\Gamma)$;

3. the valuations of the arrows $(i, k)$ and $(k, i)$ (respectively $(j, k)$ and $(k, j)$) in $\mu_{ijji}(\Gamma)$ are the same valuation of $(j, k)$ and $(k, j)$ in $\Gamma$ (respectively $(i, k)$ and $(k, i)$);

4. the statements (2) and (3) after replacing $k$ with $t$ and switching $i$ and $j$ are still satisfied.

In the following we will show equations (5) and (6) above only for $k$ and the proof for $t$ is quite similar. Without loss of generality, we assume that $k \in N_{\Gamma,+}(i)$. Throughout the proof we will frequently use the fact that the mutation of skew symmetrizable matrix is again skew symmetrizable. One can see that $N_{\Gamma}(k)$ is partitioned into the following five subsets of $Q_0$

1. $N_{1,\Gamma} = N_{\Gamma,+}(k) \cap N_{\Gamma,+}(i)$;
2. $N_{2,\Gamma} = N_{\Gamma,-}(k) \cap N_{\Gamma,+}(i)$;
3. $N_{3,\Gamma} = N_{\Gamma,+}(k) \cap N_{\Gamma,-}(i)$;
4. $N_{4,\Gamma} = N_{\Gamma,-}(k) \cap N_{\Gamma,-}(i)$;
5. $N_{5,\Gamma} = N_{\Gamma}(k) \setminus N_{\Gamma}(i)$. In addition to these five sets we will consider the following two sets of vertices
The strategy of the proof is applying the sequence of mutations $\mu_{ijji}$ on $\Gamma$ and observe the changes in the vertices of the above seven sets along with the valuations of all the arrows of the form $(x_{k_{ij}}, x_{k_{ji}}) \rightarrow_y$ where $x$ is a vertex in $N_{t, \Gamma}$ for some $t = 1, \ldots, 7$ and $y$ is one of the vertices $\{i, j, k\}$. Notice that, since the valuation of $(i, j)$ is $(1, 1)$ then the valuation of $(i, j)$ or $(j, i)$ will stay the same during applying the steps of the sequence $\mu_{ijji}$. The vertices in each set of the above sets will be altered similarly, so we will pick a representative from each set. Thus, in addition to the vertices $i, j$ and $k$, we will focus on the changes on seven vertices labeled by $\{x; x \in N_{x, \Gamma}, x = 1, \ldots, 7\}$. Furthermore, we will assume that $x \in N_{\Gamma, -}(j)$ for every $x = 1, \ldots, 7$. All the calculations of all other cases are quite similar.

1. Apply $\mu_i$ on $\Gamma$. Then $j$ will be added to $N_{\Gamma, -}(k)$ with valuation $(b_{ik}, b_{ki})$ and the arrows $(j, i), (i, k)$ and $(i, x), \ x = 1, \ldots, 7$ will be reversed with switching the components of each pair of valuation. Some other specific changes which will occur to the sets $N_{x, \Gamma}, x = 1, \ldots, 7$ are as follows:

   (a) $N_{1, \Gamma} \Rightarrow N_{1, \Gamma_i} = N_{\Gamma, +}(k) \cap N_{\Gamma, -}(i)$. There will be two cases for altering the arrow $(1, j)$:
   
   i. if $b_{i1} - b_{j1} > 0$ and $b_{1j} - b_{1i} < 0$ then the arrow $(1, j)$ in $\Gamma$ will be replaced with $(j, 1)$ in $\Gamma_i$ with valuation $(b_{i1} - b_{j1}, b_{1i} - b_{1j})$;
   
   ii. if $b_{i1} - b_{j1} < 0$ and $b_{1j} - b_{1i} > 0$ then the arrow $(1, j)$ in $\Gamma$ stays the same in $\Gamma_i$ with valuation $(b_{1j} - b_{1i}, b_{j1} - b_{i1})$.

   (b) $N_{2, \Gamma} \Rightarrow N_{2, \Gamma_i} = N_{\Gamma, -}(k) \cap N_{\Gamma, -(i)}$. There will be two cases for altering the arrow $(2, j)$:

   i. if $b_{22} - b_{2j} > 0$ and $b_{2j} - b_{2i} < 0$ then the arrow $(2, j)$ in $\Gamma$ will be replaced with $(j, 2)$ in $\Gamma_i$ with valuation $(b_{22} - b_{2j}, b_{2i} - b_{2j})$;
   
   ii. if $b_{22} - b_{2j} < 0$ and $b_{2j} - b_{2i} > 0$ then the arrow $(2, j)$ in $\Gamma$ stays the same in $\Gamma_i$ with valuation $(b_{2j} - b_{2i}, b_{j2} - b_{2j})$.

   (c) $N_{3, \Gamma} \Rightarrow N_{3, \Gamma_i} = N_{\Gamma, -(i)} \cap N_{\Gamma, -(j)}$. There will be two cases for altering the arrow $(k, 3)$:

   i. if $b'_{3k} = b_{3i}b_{ik} - b_{3k} > 0$ and $b'_{k3} = b_{k3} - b_{ki}b_{3i} < 0$ then the arrow $(k, 3)$ in $\Gamma$ will be replace with $(3, k)$ in $\Gamma_i$ with valuation $(b_{3i}b_{ik} - b_{3k}, b_{k3}b_{3i} - b_{k3})$;
   
   ii. if $b'_{3k} < 0$ and $b'_{k3} > 0$ then the arrow $(k, 3)$ in $\Gamma$ stays the same in $\Gamma_i$ with valuation $(b_{k3} - b_{ki}b_{3i}, b_{3k} - b_{3i}b_{ik})$.
(d) $N_{4,\Gamma} \Rightarrow N_{4,\Gamma_i} = N_{\Gamma,-}(k) \cap N_{\Gamma,+}(i) \cap N_{\Gamma,-}(j)$ and the arrow $(4, k)$ will stay the same in $\Gamma_i$ with the new valuation $(b_{4k} + b_{4i}b_{ik}, b_{4k} + b_{4i}b_{ik})$;

(e) $N_{5,\Gamma} \Rightarrow N_{5,\Gamma_i} = N_{\Gamma}(k) \setminus N_{\Gamma}(i)$;

(f) $N_{6,\Gamma} \Rightarrow N_{6,\Gamma_i} = N_{\Gamma,+}(i) \cap N_{\Gamma,-}(k)$;

(g) $N_{7,\Gamma} \Rightarrow N_{7,\Gamma_i} = N_{\Gamma,-}(i) \setminus N_{\Gamma}(k)$.

2. Apply $\mu_j$ on $\Gamma_i$. The arrows $(i, j)$, $(j, k)$ and $(j, x)$ or $(x, j)$, $x = 1, \cdots, 7$ will be reversed with switching the components of their valuations. The arrow $(i, k)$ will be omitted as it will have $(0, 0)$ valuation. The seven sets $N_{k,\Gamma_i}$, $k = 1, \cdots, 7$ will be altered as follows

(a) $N_{1,\Gamma} \Rightarrow N_{1,\Gamma_i} = N_{\Gamma,+}(k) \cap N_{\Gamma,-}(i)$. We have two cases for $(1, i)$:

i. if $(j, 1)$ is in $\Gamma_i$, then $(1, i)$ will have the valuation $(b_{1j}, b_{j1})$;

ii. if $(1, j)$ is in $\Gamma_i$, then $(1, i)$ will keep its valuation $(b_{1i}, b_{ji})$.

We will have two cases for the arrow $(k, 1)$:

i. if $(1, j)$ is in $\Gamma_i$, then $(k, 1)$ will keep its valuation $(b_{k1}, b_{1k})$;

ii. if $(j, 1)$ is in $\Gamma_i$, then we have two cases for the arrow $(k, 1)$:

A. if $b_{k1} - b_{ki}b_{i1} + b_{ki}b_{j1} \geq 0$, then $(k, 1)$ will stay in $\mu_{ji}(\Gamma)$ and will have the valuation $(b_{k1} - b_{ki}b_{i1} + b_{ki}b_{j1}, b_{1k} - b_{j1}b_{ik} + b_{ji}b_{ik})$;

B. if $b_{k1} - b_{ki}b_{i1} + b_{ki}b_{j1} < 0$, then $(k, 1)$ will be reversed in $\mu_{ji}(\Gamma)$ and will have the valuation $(b_{1j}b_{ik} - b_{1k} - b_{1i}b_{ik}, +b_{ki}b_{i1} - b_{ki}b_{j1} - b_{k1})$.

(b) $N_{2,\Gamma} \Rightarrow N_{2,\Gamma_i} = N_{\Gamma,-}(k) \cap N_{\Gamma,-}(j)$. We have two cases for $(2, i)$:

i. if $(j, 2)$ is in $\Gamma_i$, then $(2, i)$ will have the valuation $(b_{2j}, b_{j2})$;

ii. if $(2, j)$ is in $\Gamma_i$, then $(2, i)$ will keep its valuation $(b_{2i}, b_{ji})$.

We will have two cases for the arrow $(2, k)$:

i. if $(j, 2)$ is in $\Gamma_i$, then $(2, k)$ will keep its valuation $(b_{2k}, b_{k2})$;

ii. if $(2, j)$ is in $\Gamma_i$, then the arrow $(2, k)$ will stay in $\mu_{ji}(\Gamma)$ and will have the valuation $(b_{2k} - b_{2i}b_{ik} + b_{2j}b_{ik}, b_{k2} + b_{ki}b_{j2} - b_{ki}b_{j2})$, this is because $b_{2k} - b_{2i}b_{ik} + b_{2j}b_{ik} \geq 0$ due to $b_{2j} - b_{2i} > 0$.

(c) $N_{3,\Gamma} \Rightarrow N_{3,\Gamma_i} = N_{\Gamma,+}(i) \cap N_{\Gamma,+}(j)$. The arrow $(3, j)$ in $\Gamma_i$ will be reversed in $\Gamma_{ij}$ with switching the two components of its valuation.

We will have two cases for the arrow $(k, 3)$:

i. if $b_{3k}'' = b_{3j}b_{ik} - b_{3k} + b_{j3}b_{ik} > 0$ then the arrow $(3, k)$ stays in $\mu_{ji}(\Gamma)$ with valuation $(b_{3i}b_{ik} - b_{3k} + b_{j3}b_{ik}, b_{k3} + b_{ki}b_{j3} - b_{k3})$;

ii. if $b_{3k}'' < 0$ then the arrow will be reversed and get the valuation $(b_{k3} - b_{ki}b_{3i} - b_{ki}b_{j3}, b_{3k} - b_{3i}b_{ik} - b_{j3}b_{ik})$. 


4. Apply $\mu_i$ to $\mu_{ij}(Q)$. The seven sets $N_{k,\Gamma_{ij}}$, $k = 1, \cdots, 7$ will be altered as follows

(a) $N_{1,\Gamma_{ij}} \Rightarrow N_{1,\Gamma_{ij}} = N_{\Gamma_{ij}} = N_{\Gamma_{ij}}(i) \cap N_{\Gamma_{ij}}(j)$. The arrow $(1, 2)$ will be reversed with switching the components of their valuations. There are two cases for $(1, 2)$ and $(1, 3)$ and with no change in $(1, 3)$ or $(1, 4)$ and $(1, 6)$ will be reversed with switching the components of their valuations. The seven sets $N_{k,\Gamma_{ij}}$, $k = 1, \cdots, 7$ will be altered as follows

(b) $N_{2,\Gamma_{ij}} \Rightarrow N_{2,\Gamma_{ij}} = N_{\Gamma_{ij}} = N_{\Gamma_{ij}}(i) \cap N_{\Gamma_{ij}}(j)$ and with no change in $(1, 3)$ or $(1, 4)$ and with no change in $(1, 3)$ or $(1, 4)$ and with no change in $(1, 4)$ and (i, 6) will not be reversed and will be given the valuation $(b_{ij} + b_{i3}, b_{ij} + b_{i4})$;

(c) $N_{3,\Gamma_{ij}} \Rightarrow N_{3,\Gamma_{ij}} = N_{\Gamma_{ij}} = N_{\Gamma_{ij}}(i) \cap N_{\Gamma_{ij}}(j)$ and with no change in $(1, 3)$ or $(1, 4)$ and with no change in $(1, 3)$ or $(1, 4)$ and with no change in $(1, 4)$ and (i, 6) will not be reversed and will be given the valuation $(b_{ij} + b_{i3}, b_{ij} + b_{i4})$;

(d) $N_{4,\Gamma_{ij}} \Rightarrow N_{4,\Gamma_{ij}} = N_{\Gamma_{ij}} = N_{\Gamma_{ij}}(i) \cap N_{\Gamma_{ij}}(j)$ and with no change in $(1, 3)$ or $(1, 4)$ and with no change in $(1, 3)$ or $(1, 4)$ and with no change in $(1, 4)$ and (i, 6) will not be reversed and will be given the valuation $(b_{ij} + b_{i3}, b_{ij} + b_{i4})$;

(e) $N_{5,\Gamma_{ij}} \Rightarrow N_{5,\Gamma_{ij}} = N_{\Gamma_{ij}} = N_{\Gamma_{ij}}(i)$;

(f) $N_{6,\Gamma_{ij}} \Rightarrow N_{6,\Gamma_{ij}} = N_{\Gamma_{ij}} = N_{\Gamma_{ij}}(i) \cap N_{\Gamma_{ij}}(j)$;

(g) $N_{7,\Gamma_{ij}} \Rightarrow N_{7,\Gamma_{ij}} = N_{\Gamma_{ij}}(j)$. 

3. Apply $\mu_i$ on $\mu_{ij}(Q)$. The arrow $(j, i)$, $(1, i)$, $(7, i)$, $(i, 3)$, $(i, 4)$ and $(i, 6)$ will be reversed with switching the components of their valuations. The seven sets $N_{k,\Gamma_{ij}}$, $k = 1, \cdots, 7$ will be altered as follows

(a) $N_{1,\Gamma_{ij}} \Rightarrow N_{1,\Gamma_{ij}} = N_{\Gamma_{ij}}(k) \cap N_{\Gamma_{ij}}(i)$. The arrow $(4, 5)$ will get a new valuation $(b_{ik} + b_{ik}, b_{ik} + b_{ik}, b_{ik} + b_{ik})$ in $\mu_{ij}(\Gamma)$;

(b) $N_{2,\Gamma_{ij}} \Rightarrow N_{2,\Gamma_{ij}} = N_{\Gamma_{ij}}(k) \cap N_{\Gamma_{ij}}(i)$;

(c) $N_{3,\Gamma_{ij}} \Rightarrow N_{3,\Gamma_{ij}} = N_{\Gamma_{ij}}(k) \cap N_{\Gamma_{ij}}(i)$ and will be given the valuation $(b_{ij}, b_{ij}, b_{ij}, b_{ij})$;

(d) $N_{4,\Gamma_{ij}} \Rightarrow N_{4,\Gamma_{ij}} = N_{\Gamma_{ij}}(k) \cap N_{\Gamma_{ij}}(j)$. The arrow $(4, 5)$ will get a new valuation $(b_{ik} + b_{ik}, b_{ik} + b_{ik}, b_{ik} + b_{ik})$ in $\mu_{ij}(\Gamma)$;

(e) $N_{5,\Gamma_{ij}} \Rightarrow N_{5,\Gamma_{ij}} = N_{\Gamma_{ij}}(k) \cap N_{\Gamma_{ij}}(i)$;

(f) $N_{6,\Gamma_{ij}} \Rightarrow N_{6,\Gamma_{ij}} = N_{\Gamma_{ij}}(k) \cap N_{\Gamma_{ij}}(i)$ and will be given the valuation $(b_{ij}, b_{ij}, b_{ij}, b_{ij})$;

(g) $N_{7,\Gamma_{ij}} \Rightarrow N_{7,\Gamma_{ij}} = N_{\Gamma_{ij}}(j)$.

4. Apply $\mu_j$ on $\mu_{ij}(Q)$. The seven sets $N_{k,\Gamma_{ij}}$, $k = 1, \cdots, 7$ will be altered as follows

(a) $N_{1,\Gamma_{ij}} \Rightarrow N_{1,\Gamma_{ij}} = N_{\Gamma_{ij}}(k) \cap N_{\Gamma_{ij}}(i)$. The arrow $(1, 2)$ (respectively to $(1, j)$), $(i, j)$ will be reversed with switching the components of their valuations. There are two cases for $(1, i)$:

i. if $(1, j)$ is in $\Gamma_{ij}$ then there will be no change will occur to $(1, i)$ and it will keep its valuation $(b_{1i}, b_{1i})$;

ii. if $(j, 1)$ is in $\Gamma_{ij}$ then $(1, i)$ will stay in $\Gamma_{ij}$ with the new valuation $(b_{1i}, b_{1i})$.

We will have two cases for $(k, 1)$:

i. if $(j, 1)$ is in $\Gamma_{ij}$ then there will be no change will occur to $(k, 1)$ and it will keep its valuation $(b_{k1}, b_{k1})$;

ii. if $(1, j)$ is in $\Gamma_{ij}$ then $(k, 1)$ will stay in $\Gamma_{ij}$ with the same valuation $(b_{k1}, b_{k1})$.
(b) \( N_{2, \Gamma_{ijij}} \Rightarrow N_{2, \Gamma_{ijij}} = N_{\Gamma, -(k)} \cap N_{\Gamma, +}(i) \). The arrow \((2, j)\) (respectively to \((j, 2)\), \((i, j)\) will be reversed with switching the components of their valuations. There are two cases for \((i, 2)\):

i. if \((2, j)\) is in \(\Gamma_{ijij}\) then there will be no change will occur to \((i, 2)\) and it will keep its valuation \((b_{j2}, b_{2j})\);

ii. if \((j, 2)\) is in \(\Gamma_{ijij}\) then \((i, 2)\) will stay in \(\Gamma_{ijij}\) with the same valuation \((b_{j2}, b_{2j})\).

We will have two cases for \((2, k)\):

i. if \((2, j)\) is in \(\Gamma_{ijij}\) then there will be no change will occur to \((2, k)\) and it will keep its valuation \((b_{kj}, b_{kj})\);

ii. if \((j, 2)\) is in \(\Gamma_{ijij}\) then \((2, k)\) will stay in \(\Gamma_{ijij}\) with the same valuation \((b_{kj}, b_{kj})\).

(c) \( N_{3, \Gamma_{ijij}} \Rightarrow N_{3, \Gamma_{ijij}} = N_{\Gamma, +}(k) \cap N_{\Gamma, -(j)} \cap N_{\Gamma, +}(i) \). The arrow \((j, 3), (i, j)\) and \((j, k)\) will be reversed with switching the components of their valuations. The arrow \((3, i)\) will be reversed and get the new valuation \((b_{j3}, b_{3j})\). The arrow \((3, k)\) will be reserved and get the valuation \((b_{k3}, b_{3k})\).

(d) \( N_{4, \Gamma_{ijij}} \Rightarrow N_{4, \Gamma_{ijij}} = N_{\Gamma, -(k)} \cap N_{\Gamma, -(j)} \). The arrow \((j, 4), (i, j)\) and \((k, j)\) will be reversed with switching the components of their valuations. The arrow \((4, k)\) will stay in \(\Gamma_{ijij}\) with the new valuation \((b_{k4}, b_{4k})\). The arrow \((4, i)\) will be reserved and get the valuation \((b_{4j}, b_{4j})\).

(e) \( N_{5, \Gamma_{ijij}} \Rightarrow N_{5, \Gamma_{ijij}} = N_{\Gamma}(k) \setminus N_{\Gamma}(i) \);

(f) \( N_{6, \Gamma_{ijij}} \Rightarrow N_{6, \Gamma_{ijij}} = N_{\Gamma, -(j)} \setminus N_{\Gamma}(k) \);

(g) \( N_{7, \Gamma_{ijij}} \Rightarrow N_{7, \Gamma_{ijij}} = N_{\Gamma, +}(j) \setminus N_{\Gamma}(k) \).

5. Apply \(\mu\) on \(\mu_{ijij}(Q)\). The arrow \((j, i), (i, 1), (i, 2), (i, 3), (i, 4), (i, 6)\) and \((7, i)\) will be reversed with switching the components of their valuations. The seven sets \(N_{k, \Gamma_{ijij}}, k = 1, \ldots, 7\) will be altered as follows

(a) \( N_{1, \Gamma_{ijij}} \Rightarrow N_{\Gamma, +}(k) \cap N_{\Gamma, +}(j) \cap N_{\Gamma, -(i)} \). Whether \((j, 1)\) or \((1, j)\) is in \(\Gamma_{ijij}\) they will be replaced by \((j, 1)\) in \(\Gamma_{ijij}\) with valuation \((b_{1j}, b_{j1})\).

(b) \( N_{2, \Gamma_{ijij}} \Rightarrow N_{2, \Gamma} = N_{\Gamma, -(k)} \cap N_{\Gamma, +}(j) \cap N_{\Gamma, -(i)} \). Whether \((j, 2)\) or \((2, j)\) is in \(\Gamma_{ijij}\) they will be replaced by \((j, 2)\) in \(\Gamma_{ijij}\) with valuation \((b_{2j}, b_{2j})\).

(c) \( N_{3, \Gamma_{ijij}} \Rightarrow N_{3, \Gamma} = N_{\Gamma, +}(k) \cap N_{\Gamma, -(j)} \cap N_{\Gamma, -(i)} \). The arrow \((j, 3)\) will be replaced with \((3, j)\) in \(\Gamma_{ijij}\) with valuation \((b_{3j}, b_{3j})\).

(d) \( N_{4, \Gamma_{ijij}} \Rightarrow N_{4, \Gamma_{ijij}} = N_{\Gamma, -(k)} \cap N_{\Gamma, -(j)} \). The arrow \((4, j)\) will be stay the same in \(\Gamma_{ijij}\) with valuation \((b_{4j}, b_{4j})\).
(e) \( N_{5, r_{ij}} \Rightarrow N_{5, r_{ij}} = N_r(k) \setminus N_r(i) \);

(f) \( N_{6, r_{ij}} \Rightarrow N_{6, r_{ij}} = N_{r_{-}}(j) \setminus N_r(k) \);

(g) \( N_{7, r_{ij}} \Rightarrow N_{7, r_{ij}} = N_{r_{+}}(j) \setminus N_r(k) \).

From step (5) one can see that by comparing the valued quivers \( \Gamma \) and \( \Gamma_{ij} \), the vertex \( k \) can move from the neighborhood of \( i \) to the neighborhood of \( j \) with all its own neighborhood with no changes in the valuations. Since \( k \) was chosen randomly from \( N_r(i) \) then every vertex \( N_r(i) \) will be moved to \( N_r(j) \) in the same way as \( k \) similarly with the vertices of \( N_r(j) \). Therefore \( \sigma_{ij}(\Gamma) = \Gamma_{ij} \).

Secondly, we will prove identity (4) for the elements of \( X \). Consider the following monomials

\[
M_{r_{+}}(i) = \prod_{k, (i, k) \in Q_1 \setminus \{j\}} x_k^{v_{ik}}, \quad M_{r_{-}}(i) = \prod_{k, (k, i) \in Q_1 \setminus \{j\}} x_k^{v_{ki}},
\]

\[
M_{r_{+}}(j) = \prod_{k, (j, k) \in Q_1 \setminus \{i\}} x_k^{v_{jk}} \quad \text{and} \quad M_{r_{-}}(j) = \prod_{k, (k, j) \in Q_1 \setminus \{i\}} x_k^{v_{kj}}.
\]

One can see that applying the mutation sequence \( \mu_{ij} \) on the cluster \( X \) with initial cluster graph \( \Gamma \) will affect the cluster variables \( x_i \) and \( x_j \) only. In the following we will present the changes on the initial cluster \( (\ldots, x_i, \ldots, x_j, \ldots) \) after every step.

\[
\mu_i \Rightarrow (\ldots, \frac{M_{r_{+}}(i) + x_j M_{r_{-}}(i)}{x_i}, \ldots)
\]

\[
\mu_j \Rightarrow (\ldots, \mu_i(x_i), \ldots, \frac{M_{r_{+}}(j) + \mu_i(x_i) M_{r_{-}}(j)}{x_j}, \ldots)
\]

\[
= (\ldots, \mu_i(x_i), \ldots, \frac{M_{r_{+}}(i) M_{r_{-}}(j) + M_{r_{-}}(i) M_{r_{-}}(j) x_j + M_{r_{+}}(i) M_{r_{+}}(j) x_i}{x_i x_j}, \ldots)
\]

\[
\mu_j \Rightarrow (\ldots, \mu_j(x_j), \ldots, \frac{M_{r_{-}}(j) M_{r_{+}}(i)}{\mu_i(x_j)}, \ldots, \mu_j(x_j), \ldots)
\]

\[
= (\ldots, \mu_j(x_j), \ldots, \frac{M_{r_{-}}(j) + x_i M_{r_{+}}(j)}{x_j}, \ldots, \mu_j(x_j), \ldots)
\]

\[
\mu_i \Rightarrow (\ldots, \mu_{ij}(x_i), \ldots, \frac{M_{r_{-}}(j) M_{r_{-}}(i) + \mu_{ij}(x_i) M_{r_{+}}(j)}{\mu_{ij}(x_j)}, \ldots)
\]

\[
= (\ldots, \mu_{ij}(x_i), \ldots, x_i, \ldots)
\]

\[
\mu_j \Rightarrow (\ldots, \frac{M_{r_{-}}(j) + x_i M_{r_{+}}(j)}{\mu_{ij}(x_i)}, \ldots, x_i, \ldots)
\]

\[
= (\ldots, x_j, \ldots, x_i, \ldots)
\]
Which finishes the proof of $\mu_{ijij}(p) = \sigma_{ij}(p)$. The proof of $\mu_{ijij}(p) = \sigma_{ij}(p)$ is quite similar.

In the following we present two examples, first one shows that the pentagon relations are not necessarily satisfied if $b_{ij} \neq 1$. The second example provides a case of a valued quiver $\Gamma$ and a quiver automorphism $\phi$ where the action of $\phi$ on $\Gamma$ does not equal the action of any sequence of mutations, i.e., $\Gamma$ and $\phi(\Gamma)$ are not in the same mutations class.

**Example 2.3** Let $m$ be a natural number.

\[
\Gamma = \cdot_1 \xrightarrow{(m,m)} \cdot_2 \xrightarrow{(m,m)} \cdot_3 \xleftarrow{\mu_2} \cdot_1 \xrightarrow{(m,m)} \cdot_2 \xrightarrow{(m,m)} \cdot_3 \\
\mu_2 \cdot_1 \xrightarrow{(m,m)} \cdot_2 \xrightarrow{(m,m)} \cdot_3 \\
\mu_1 \cdot_1 \xleftarrow{(m,m)} \cdot_2 \xleftarrow{(m,m)} \cdot_3 \\
\mu_2 \cdot_1 \xleftarrow{(m^2,m^2)} \cdot_3 \\
\cdot_2 \xrightarrow{(m,m)} \cdot_3 \\
\cdot_2 \xrightarrow{(m^2,m^2)} \cdot_3 \\
\cdot_2 \xrightarrow{(m^2-m,m^2-m)} \cdot_3 \\
\]

One can see that the valued quiver $\Gamma$ does not satisfy identity (4) unless $m = 1$. However, if $m \neq 1$ the underlying simply-laced subgraph of $\Gamma$ that contains only the vertices 1, 2 and their neighborhoods (without valuations) satisfies identity (4).

**Example 2.4** [11, Example 3.3] Consider the valued quiver $\Gamma = \cdot_1 \xrightarrow{(2,2)} \cdot_2 \xrightarrow{(2,2)} \cdot_3$, and the quiver automorphism with underlying permutation $\sigma_{12}$. In [11], it was shown that there is no sequence of mutations $\mu$, such that $\mu(\Gamma) = \cdot_2 \xrightarrow{(2,2)} \cdot_1 \xrightarrow{(2,2)} \cdot_3$.

The following theorem provides some answer for Question 1.1, which is repeated in the following for the convenience of the reader.

**Question 2.5** [11, Question 3.2] For which valued quivers $Q$ and a quiver automorphism $\phi$ are there a sequence of mutations $\mu$ such that $\phi(Q) = \mu(Q)$?

In the following theorem we will use the notation $\mu_{[ij]} = \mu_{ijij}$ and Mut($Q$) for the mutations class of $Q$. 

---

---
Theorem 2.6 Let $Q$ be a simply laced quiver. Then, for every valued graph automorphism $\phi$ of $Q$ there is a sequence of mutations $\mu$ such that

$$\mu(Q) = \phi(Q).$$

(8)

In particular, The orbits of the permutations group action on $\text{Mut}(Q)$ are all embedded in $\text{Mut}(Q)$.

**Proof.** Let $Q = (Q_0, Q_1)$ be a simply laced quiver of rank $n$ and $\sigma_{v v'}$ be a transposition in $\mathfrak{S}_n$. Since $Q$ is a connected quiver then for any two vertices $v$ and $v'$ in $Q_0$, there is a sequence of vertices $v = v_1, \ldots, v_q = v'$ such that $v_t$ and $v_{t+1}$ are adjacent vertices for every $t \in [1, q - 1]$. Since $Q$ is simply laced quiver then for any $(i, j) \in Q_1$, $v_{ij} = v_{ji} = 1$. Hence, from Lemma 2.2, for $t \in [1, q - 1]$ the sequence of mutations $\mu_{v_tv_{t+1}}$ acts on $Q$ by switching the vertices $v_t$ and $v_{t+1}$. Let

$$\mu_{\sigma_{v v'}}(Q) = \mu_{\sigma_{v_1 v_2}}(Q) \cdots (\mu_{\sigma_{v_q v_{q-1}}})(\mu_{\sigma_{v_1 v_2}}) \cdots (\mu_{\sigma_{v_1 v_2}}).$$

(9)

Hence the following identity is satisfied

$$\sigma_{v v'}(Q) = \mu_{\sigma_{v v'}}(Q).$$

(10)

Then every transposition is associated to a particular sequence of mutations.

Now, let $\phi$ be a valued quiver automorphism of $Q$. Then it induces a permutation $\sigma_{\phi}$. Since the group $\mathfrak{S}_n$ is generated by transpositions, hence there are transpositions $\sigma_{i_1 i_2}, \sigma_{i_3 i_4}, \ldots, \sigma_{i_m i_{m-1}}$ such that $\sigma_{\phi}(Q) = \sigma_{i_1 i_2} \sigma_{i_3 i_4} \cdots \sigma_{i_m i_{m-1}}(Q)$. Therefore (10) implies that

$$\sigma_{\phi}(Q) = \sigma_{i_1 i_2} \sigma_{i_3 i_4} \cdots \sigma_{i_m i_{m-1}}(Q) = \mu_{\sigma_{i_1 i_2}} \cdots \mu_{\sigma_{i_{m-1} i_m}}(Q).$$

Which proves the first part of the statement by associating $\mu = \mu_{\sigma_{i_1 i_2}} \cdots \mu_{\sigma_{i_{m-1} i_m}}$ to $\phi$. The second part of the statement is immediate. □

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On the pentagon relations of valued quivers


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