On Gamma-Derivations in the Projective Product of Gamma Rings

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Abstract

This paper highlights many enlightening results on various Gamma-derivations in the projective product of Gamma-rings. If \((X, \Gamma)\) is the projective product of two Gamma-rings \((X_1, \Gamma_1)\) and \((X_2, \Gamma_2)\), a pair of derivations \(D_1\) and \(D_2\) on \((X_1, \Gamma_1)\) and \((X_2, \Gamma_2)\) respectively with the property of general Gamma-derivation/semi-derivation/generalized Gamma-derivation/Jordan derivation/generalized Jordan derivation/inner derivation/generalized inner derivation can be extended to a derivation \(D\) on \((X, \Gamma)\) having the same respective property. The converse problems are also studied fruitfully. The similar results can be investigated in case of the projective product of \(n\) number of Gamma-rings.

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1. Introduction

An extensive generalized concept of classical rings set forth the notion of a gamma ring theory which is an emerging field of research. The research work of classical ring theory to the gamma ring theory has been drawn interest of many algebraist and prominent mathematicians over the globe to determine many basic properties of gamma ring and to enrich the field of research in algebra. The different researchers on this field have been doing a significant contribution to this field from its inception. This field, which has evolved as an extension of general
ring theory does not only cover a small area with an independent life, but also serves as a unifying thread interlacing many other branches such as Banach spaces, C*-algebras, Dynamical systems, Quantum theory etc., and thus it suggests a very wide of scope of doing research, [1, 2, 4, 6, 8, 10]. The concept of a gamma ring was first introduced by Nobusawa [9] and also shown that gamma rings are more general than rings. Barnes [2] weakened slightly the conditions in the definition of gamma ring in the sense of Nobusawa. They obtained a large number of important basic properties of gamma rings in various ways and determined some prominent results of gamma rings. We next put forward some basic concepts which are absolutely necessary for our main results.

2. Basic Concepts

Definition 2.1: Let \( (X, \Gamma) \) be a gamma ring. Then an additive mapping \( D : X \to X \) is called a **Gamma-derivation** if \( D(x\alpha y) = D(x)\alpha y + x\alpha D(y) \ \forall \ x, y \in X \text{ and } \alpha \in \Gamma \).

Definition 2.2: Let \( (X, \Gamma) \) be a gamma ring. Then an additive mapping \( f : X \to X \) is called a **generalized Gamma-derivation** on \( (X, \Gamma) \) if \( f(x\alpha y) = f(x)\alpha y + x\alpha D(y) \ \forall \ x, y \in X \text{ and } \alpha \in \Gamma \) and \( D \) is a derivation on \( (X, \Gamma) \).

Definition 2.3: Let \( (X, \Gamma) \) be a gamma ring. Then an additive mapping \( T : X \to X \) s.t. \( T(x\alpha y\beta x) = x\alpha T(y)\beta x \ \forall \ x, y \in X \text{ and } \alpha, \beta \in \Gamma \) is called a **centralizer**.

Definition 2.4: An additive mapping \( J : X \to X \) is called a **Jordan Gamma-derivation** on \( (X, \Gamma) \) if \( J(x\alpha x) = J(x)\alpha x + x\alpha D(x) \ \forall \ x \in X \text{ and } \alpha \in \Gamma \).

Definition 2.5: An additive mapping \( J : X \to X \) is called a **Jordan generalized Gamma-derivation** on \( (X, \Gamma) \) if there exists a derivation \( D : X \to X \) such that \( J(x\alpha x) = J(x)\alpha x + x\alpha D(x) \ \forall \ x \in X \text{ and } \alpha \in \Gamma \).

Definition 2.6: Let \( (X, \Gamma) \) be a gamma ring, then an additive mapping \( d : X \to X \) is called a **Gamma-semiderivation** associated with a function \( g : X \to X \) if for all \( x, y \in X \text{ and } \alpha \in \Gamma \),
\[
d(x\alpha y) = d(x)\alpha g(y) + x\alpha d(y) = d(x)\alpha y + g(x)\alpha d(y)
\]
and \( d(g(x)) = g(d(x)) \).

If \( g = 1 \) i.e the identity mapping on \( X \), then all Gamma-semiderivations associated with \( g \) are merely ordinary Gamma-derivations.

If \( g \) is an endomorphism of \( X \), then other examples of semiderivations are of the form \( d(x) = x - g(x) \).
On Gamma-derivations in the projective product of gamma rings

**Definition 2.7:** A Gamma-derivation $D$ is said to be **inner** if $\exists \ a \in X$ s.t. $D(xax) = aax - xaa$. A mapping $xax \to aax + xab$, where $a,b$ are fixed elements in $X$ and for all $\alpha \in \Gamma$ is called a **generalized inner derivation**.

**Definition 2.8:** Let $S$ be a nonempty subset of $X$ and let $d$ be a Gamma-derivation on $X$.

If $d(xay) = d(x)ad(y)$ [or $(xay) = d(y)ad(x)$], for all $x,y \in S \& \gamma \in \Gamma$, then $d$ is said to be a $\Gamma$-**homomorphism** [or an anti $\Gamma$-**homomorphism**] on $S$.

**Definition 2.9:** Let $(X_1, \Gamma_1)$ & $(X_2, \Gamma_2)$ be two gamma rings. Let $X = X_1 \times X_2$ and $\Gamma = \Gamma_1 \times \Gamma_2$. Then we define addition and multiplication on $X$ and $\Gamma$ by,

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2), (\alpha_1, \alpha_2) + (\beta_1, \beta_2) = (\alpha_1 + \beta_1, \alpha_2 + \beta_2)$$

and

$$(x_1, x_2)(\alpha_1, \alpha_2)(y_1, y_2) = (x_1\alpha_1y_1, x_2\alpha_2y_2)$$

for every $(x_1, x_2), (y_1, y_2) \in X$ and $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in \Gamma$

With respect to this addition and multiplication $(X, \Gamma)$ is a gamma ring. We call this gamma ring as the **Projective product of gamma rings**.

Since $(X_1, \Gamma_1)$ and $(X_2, \Gamma_2)$ are additive abelian groups, so obviously $X = X_1 \times X_2$ and $\Gamma = \Gamma_1 \times \Gamma_2$ are additive abelian groups. To show $(X, \Gamma)$ is a gamma ring, we need to show the following properties:

Let $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in X$

and $\alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2), \gamma = (\gamma_1, \gamma_2) \in \Gamma$ be any elements.

**Property-1:**

$$(x + y)az = ((x_1, x_2) + (y_1, y_2))(\alpha_1, \alpha_2)(z_1, z_2)$$

$$= ((x_1 + y_1), (x_2 + y_2))(\alpha_1, \alpha_2)(z_1, z_2)$$

$$= (x_1 + y_1)\alpha_1z_1, (x_2 + y_2)\alpha_2z_2$$

$$= (x_1\alpha_1z_1 + y_1\alpha_1z_1, x_2\alpha_2z_2 + y_2\alpha_2z_2)$$

$$= (x_1, x_2)(\alpha_1, \alpha_2)(z_1, z_2) + (y_1, y_2)(\alpha_1, \alpha_2)(z_1, z_2) = xaz + yaz$$

Thus we get, $(x + y)az = xaz + yaz$.

Similarly, $x(\alpha + \beta)z = xaz + x\beta z$ and, $x(\alpha + z) = x\alpha y + xaz$

**Property-2:**

$$(xay)\beta z = ((x_1, x_2)(\alpha_1, \alpha_2)(y_1, y_2))(\beta_1, \beta_2)(z_1, z_2)$$

$$= (x_1\alpha_1y_1, x_2\alpha_2y_2)(\beta_1, \beta_2)(z_1, z_2) = ((x_1\alpha_1y_1)\beta_1z_1, (x_2\alpha_2y_2)\beta_2z_2)$$

$$= (x_1\alpha_1(y_1\beta_1z_1), x_2\alpha_2(y_2\beta_2z_2))$$

[Since $(X_1, \Gamma_1), (X_2, \Gamma_2)$ are gamma rings]
Thus we get, \((xy)\beta z = x(\alpha y\beta z)\). Similarly, \(x\alpha(y\beta z) = x(\alpha y\beta)z\)

**Property-3:**

Let \(x\alpha y = 0 \quad \forall x, y \in X \Rightarrow (x_1, x_2)(\alpha_1, \alpha_2)(y_1, y_2) = 0 \quad \forall x_1, x_2, y_1, y_2 \in X_2\)

\[\Rightarrow (x_1\alpha_1 y_1, x_2\alpha_2 y_2) = 0 = (0,0)\]

\[\Rightarrow x_1\alpha_1 y_1 = 0, x_2\alpha_2 y_2 = 0 \quad \forall x_1, y_1 \in X_1 \& x_2, y_2 \in X_2\]

\[\Rightarrow \alpha_1 = 0, \alpha_2 = 0 \quad \text{[(}X_1, \Gamma_1\text{),}(X_2, \Gamma_2\text{) are \textit{\Gamma-} rings]}\]

\[\Rightarrow (\alpha_1, \alpha_2) = (0,0) \Rightarrow \alpha = 0\]

Thus we get, \(x\alpha y = 0 \quad \forall x, y \in X\) implies \(\alpha = 0\)

Hence \((X, \Gamma)\) is a gamma ring which is known as the **Projective Product of Gamma-rings**.

Many authors have discussed very interesting results on various derivations on Gamma-rings, and on the basis of these results, we have extended to the Projective Product of Gamma-rings., \([2, 4, 7, 11]\).

### 3. Main Results

**Theorem 3.1:** Let \((X_1, \Gamma_1)\) and \((X_2, \Gamma_2)\) be two gamma rings and \((X, \Gamma)\) be their Projective product. Then we get the following results:

**1.** Every pair of Gamma- derivations \(D_1\) and \(D_2\) on \((X_1, \Gamma_1)\) and \((X_2, \Gamma_2)\) respectively give rise to a Gamma-derivation \(D\) on \((X, \Gamma)\).

**Proof:** We define a mapping \(D : X \rightarrow X\) by \(D(x) = D((x_1, x_2)) = (D_1(x_1), D_2(x_2))\). Clearly, \(D\) is an well defined mapping. We show that \(D\) is a derivation on \((X, \Gamma)\).

Let \(x = (x_1, x_2), y = (y_1, y_2) \in X\) and \(\alpha = (\alpha_1, \alpha_2) \in \Gamma\) be any elements. Then

\[D(x + y) = D((x_1, x_2) + (y_1, y_2)) = D((x_1 + y_1), (x_2 + y_2))\]

\[= (D_1(x_1 + y_1), D_2(x_2 + y_2))\]

\[= (D_1(x_1) + D_1(y_1), D_2(x_2) + D_2(y_2)) \quad \text{[Since \(D_1\) and \(D_2\) are additive mappings]}\]

\[= (D_1(x_1), D_2(x_2)) + (D_1(y_1), D_2(y_2))\]

\[= D((x_1, x_2)) + D((y_1, y_2)) = D(x) + D(y)\]
Thus, \( D(x + y) = D(x) + D(y) \forall x, y \in X \) which implies that \( D \) is additive.

Again, \( D(xay) = D((x_1, x_2)(\alpha_1, \alpha_2)(y_1, y_2)) = D((x_1 \alpha_1 y_1, x_2 \alpha_2 y_2)) \)

\[ = (D_1(x_1 \alpha_1 y_1), D_2(x_2 \alpha_2 y_2)) \]

\[ = (D_1(x_1)\alpha_1 y_1 + x_1\alpha_1 D_1(y_1), D_2(x_2)\alpha_2 y_2 + x_2\alpha_2 D_2(y_2)) \]  [Since \( D_1 \) and \( D_2 \) are Gamma-derivations on \((X_1, \Gamma_1)\) and \((X_2, \Gamma_2)\) respectively]

\[ = (D_1(x_1)\alpha_1 y_1, D_2(x_2)\alpha_2 y_2) + (x_1\alpha_1 D_1(y_1), x_2\alpha_2 D_2(y_2)) \]

\[ = (D_1(x_1), D_2(x_2))(\alpha_1, \alpha_2)(y_1, y_2) + (x_1, x_2)(\alpha_1, \alpha_2)(D_1(y_1), D_2(y_2)) \]

\[ = D((x_1, x_2))(\alpha_1, \alpha_2)(y_1, y_2) + (x_1, x_2)(\alpha_1, \alpha_2)D((y_1, y_2)) \]

\[ = D(x)\alpha y + x\alpha D(y) \]

Thus, \( D(xay) = D(x)\alpha y + x\alpha D(y) \forall x, y \in X \) and \( \alpha \in \Gamma \)

So \( D \) is a Gamma-derivation on \((X, \Gamma)\) and hence the result.

(II) Two Gamma semi-derivations \( d_1 \) and \( d_2 \) on \((X_1, \Gamma_1)\) and \((X_2, \Gamma_2)\) respectively give rise to a Gamma semi-derivation \( d \) on \((X, \Gamma)\).

**Proof**: Let \( d_1 \) be a Gamma-semi-derivation on \((X_1, \Gamma_1)\) associated with the function \( g_1: X_1 \to X_1 \) and \( d_2 \) be a Gamma-semi-derivation on \((X_2, \Gamma_2)\) associated with the function \( g_2: X_2 \to X_2 \).

We define the functions \( d: X \to X \) and \( g: X \to X \) by

\[ d(x) = d((x_1, x_2)) = (d_1(x_1), d_2(x_2)) \quad \text{and} \]

\[ g(x) = g((x_1, x_2)) = (g_1(x_1), g_2(x_2)) \quad \text{for all} \quad x = (x_1, x_2) \in X \]

Then clearly \( d \) and \( g \) are well defined as well as \( d \) is additive.

Let \( x = (x_1, x_2), y = (y_1, y_2) \in X \) and \( \alpha = (\alpha_1, \alpha_2) \in \Gamma \) be any elements. Then,

\[ d(xay) = d((x_1, x_2)(\alpha_1, \alpha_2)(y_1, y_2)) = d((x_1 \alpha_1 y_1, x_2 \alpha_2 y_2)) \]

\[ = (d_1(x_1 \alpha_1 y_1), d_2(x_2 \alpha_2 y_2)) \]

\[ = (d_1(x_1)\alpha_1 g_1(y_1) + x_1\alpha_1 d_1(y_1), d_2(x_2)\alpha_2 g_2(y_2) + x_2\alpha_2 d_2(y_2)) \]  [Since \( d_1 \) and \( d_2 \) are Gamma semi-derivations on \((X_1, \Gamma_1)\) and \((X_2, \Gamma_2)\) respectively]

\[ = (d_1(x_1)\alpha_1 g_1(y_1), d_2(x_2)\alpha_2 g_2(y_2)) + (x_1\alpha_1 d_1(y_1), x_2\alpha_2 d_2(y_2)) \]
We shall show that

Similarly we can show that, \( d(xay) = d(x)\alpha g(y) + x\alpha d(y) \forall x, y \in X \) and \( \alpha \in \Gamma \)

Again, \( d(g(x)) = d\left(g\left( (x_1, x_2) \right) \right) = d\left((g_1(x_1), g_2(x_2)) \right) = (d_1(g_1(x_1)), d_2(g_2(x_2))) \) [Since \( d_1 \) and \( d_2 \) are semiderivations on \( (X_1, \Gamma_1) \) and \( (X_2, \Gamma_2) \) respectively]

\( g\left( (d_1(x_1), d_2(x_2)) \right) = g\left( (x_1, x_2) \right) = g\left( d(x) \right) \)

Thus, we get, \( d(g(x)) = g\left( d(x) \right) \forall x \in X \)

Hence \( d \) is a Gamma semi-derivation on \( (X, \Gamma) \) associated with the function \( g \) and hence the required result.

(III) For every generalized Gamma-derivations \( f_1 \) and \( f_2 \) on \( (X_1, \Gamma_1) \) and \( (X_2, \Gamma_2) \) respectively give rise to a generalized Gamma-derivation \( f \) on \( (X, \Gamma) \).

**Proof:** Let \( f_1 \) be a generalised derivation on \( (X_1, \Gamma_1) \) with respect to the Gamma-derivation \( d_1: X_1 \rightarrow X_1 \) and \( f_2 \) be a generalized Gamma-derivation on \( (X_2, \Gamma_2) \) with respect to the Gamma-derivation \( d_2: X_2 \rightarrow X_2 \).

We define the mappings \( f: X \rightarrow X \) and \( d: X \rightarrow X \) by

\[
\begin{align*}
  f(x) &= f\left( (x_1, x_2) \right) = (f_1(x_1), f_2(x_2)) \\
  d(x) &= d\left( (x_1, x_2) \right) = (d_1(x_1), d_2(x_2)) \text{ for all } x = (x_1, x_2) \in X
\end{align*}
\]

Then obviously \( f \) is an additive mapping and \( d \) is a Gamma-derivation on \( X \).

We shall show that \( f \) is a generalized derivation on \( X \) with respect to the derivation \( d \) on \( X \).

Let \( x = (x_1, x_2), y = (y_1, y_2) \in X \) and \( \alpha = (\alpha_1, \alpha_2) \in \Gamma \) be any elements. Then

\[
\begin{align*}
  f(xay) &= f\left( (x_1, x_2)(\alpha_1, \alpha_2)(y_1, y_2) \right) = f\left( (x_1\alpha_1y_1, x_2\alpha_2y_2) \right) \\
  &= \left( f_1(x_1\alpha_1y_1), f_2(x_2\alpha_2y_2) \right) \\
  &= \left( f_1(x_1)\alpha_1y_1 + x_1\alpha_1d_1(y_1), f_2(x_2)\alpha_2y_2 + x_2\alpha_2d_2(y_2) \right) \text{ [Since } f_1 \text{ and } f_2 \text{ are generalized derivations on } (X_1, \Gamma_1) \text{ and } (X_2, \Gamma_2) \text{ respectively]} \\
  &= \left( f_1(x_1)\alpha_1y_1, f_2(x_2)\alpha_2y_2 \right) + (x_1\alpha_1d_1(y_1), x_2\alpha_2d_2(y_2)) \\
  &= \left( f_1(x_1), f_2(x_2) \right)(\alpha_1, \alpha_2)(y_1, y_2) + (x_1, x_2)(\alpha_1, \alpha_2)(d_1(y_1), d_2(y_2))
\end{align*}
\]
\[ f((x_1, x_2)) = f(x_1, x_2)(\alpha_1, \alpha_2) + f(x_1, x_2)(\alpha_1, \alpha_2) d((y_1, y_2)) \]

\[ f(x)ay + xad(y) \]

Thus, \( f(xy) = f(x)ay + xad(y) \) \( \forall x, y \in X \) and \( \alpha \in \Gamma \)

Hence \( f \) is a generalized Gamma-derivation on \( X \) with respect to the Gamma-derivation \( d \) on \( X \).

(IV) Two inner Gamma-derivations \( d_1 \) and \( d_2 \) on \( (X_1, \Gamma_1) \) and \( (X_2, \Gamma_2) \) respectively give rise to an inner Gamma-derivation \( d \) on \( (X, \Gamma) \).

Proof: Let \( d_1 \) be an inner Gamma-derivation on \( (X_1, \Gamma_1) \) with respect to the element \( \alpha \in X_1 \) and \( d_2 \) be an inner Gamma-derivation on \( (X_2, \Gamma_2) \) with respect to the element \( \beta \in X_2 \). We define a mapping \( d: X \to X \) by \( d(x) = d((x_1, x_2)) = (d_1(x_1), d_2(x_2)) \) \( \forall x = (x_1, x_2) \in X \). Then, \( d \) is well defined as well as additive.

Let \( x = (x_1, x_2) \in X \) and \( \alpha = (\alpha_1, \alpha_2) \in \Gamma \) be any two elements. Then

\[ d(x \alpha x) = d((x_1, x_2)(\alpha_1, \alpha_2)(x_1, x_2)) = d((x_1 \alpha_1 x_1, x_2 \alpha_2 x_2)) \]

\[ = (d_1(x_1 \alpha_1 x_1), d_2(x_2 \alpha_2 x_2)) \]

\[ = (a \alpha_1 x_1 - x_1 \alpha_1 a, b \alpha_2 x_2 - x_2 \alpha_2 b) \] [Since \( d_1 \) and \( d_2 \) are inner derivations on \( (X_1, \Gamma_1) \) and \( (X_2, \Gamma_2) \) w.r.t. \( a \) and \( b \) respectively]

\[ = (a \alpha_1 x_1, b \alpha_2 x_2) - (x_1 \alpha_1 a, x_2 \alpha_2 b) \]

\[ = (a_1, a_2)(\alpha_1, \alpha_2)(x_1, x_2) - (x_1, x_2)(\alpha_1, \alpha_2)(a_1, a_2) \]

\[ = \text{max} - xam \text{ where } m = (a_1, a_2) \in X \]

Thus \( d \) is an inner derivation on \( (X, \Gamma) \) with respect to the element \( m \in X \).

(V) Every two Jordan derivations \( J_1 \) and \( J_2 \) on \( (X_1, \Gamma_1) \) and \( (X_2, \Gamma_2) \) respectively give rise to a Jordan derivation \( J \) on \( (X, \Gamma) \) defined by \( J_1 \) and \( J_2 \).

Proof: We define a map \( J: X \to X \) by \( J(x) = J((x_1, x_2)) = (J_1(x_1), J_2(x_2)) \) \( \forall x = (x_1, x_2) \in X \). Then \( J \) is an well defined as well as additive mapping.

Let \( x = (x_1, x_2) \in X \) and \( \alpha = (\alpha_1, \alpha_2) \in \Gamma \) be any two elements. Then

\[ J(x \alpha x) = J((x_1, x_2)(\alpha_1, \alpha_2)(x_1, x_2)) = J((x_1 \alpha_1 x_1, x_2 \alpha_2 x_2)) \]

\[ = (J_1(x_1 \alpha_1 x_1), J_2(x_2 \alpha_2 x_2)) \]
= (J_1(x_1) \alpha x_1 + x_1 \alpha J_1(x_1), J_2(x_2) \alpha x_2 + x_2 \alpha J_2(x_2)) \quad \text{[Since } J_1 \text{ and } J_2 \text{ are Jordan derivations on } (X_1, \Gamma_1) \text{ and } (X_2, \Gamma_2) \text{ respectively]}

= (J_1(x_1) \alpha x_1, J_2(x_2) \alpha x_2) + (x_1 \alpha J_1(x_1), x_2 \alpha J_2(x_2))

= (J_1(x_1), J_2(x_2))(\alpha_1, \alpha_2)(x_1, x_2) + (x_1, x_2)(\alpha_1, \alpha_2)(J_1(x_1), J_2(x_2))

= J((x_1, x_2))(\alpha_1, \alpha_2)(x_1, x_2) + (x_1, x_2)(\alpha_1, \alpha_2)J((x_1, x_2)) = J(x)\alpha x + x\alpha J(x)

Thus, J(x\alpha y) = J(x)\alpha x + x\alpha J(x) \forall x \in X \text{ and } \alpha \in \Gamma

So J is a Jordan derivation on (X, \Gamma) defined by J_1 and J_2 ; and hence the result.

Similarly we can show some other enlightening results highlighted below:

(VI) Every two generalized Jordan derivations J_1 and J_2 on (X_1, \Gamma_1) and (X_2, \Gamma_2) respectively give rise to a generalized Jordan derivation J on (X, \Gamma) constructed with the help of J_1 and J_2.

(VII) Every two generalized inner derivations on (X_1, \Gamma_1) and (X_2, \Gamma_2) respectively give rise to a generalized inner derivation on (X, \Gamma).

(VIII) If \phi_1 and \phi_2 be two homomorphisms on (X_1, \Gamma_1) and (X_2, \Gamma_2) respectively, then there exist a homomorphism on (X, \Gamma) constructed with the help of \phi_1 and \phi_2.

Till now we have discussed the existence of various types of derivations on the Projective product of gamma rings based on the derivations on the component gamma rings. Now we shall discuss the converses of the above results.

**Theorem 3.2:** For every derivation D on (X, \Gamma), there exist derivations D_1 and D_2 on (X_1, \Gamma_1) and (X_2, \Gamma_2) respectively, where (X, \Gamma) is the Projective product of (X_1, \Gamma_1) and (X_2, \Gamma_2).

Furthermore, if D is semi-derivation/ generalized Gamma-derivation / inner Gamma- derivation / Jordan derivation / generalized Jordan derivation, then D_1 and D_2 are also so.

Proof: Let D be a derivation on (X, \Gamma). Let x_1 be any element of X_1 and let

\[ D((x_1, 0)) = (u_1, u_2) \]

We define a map D_1: X_1 \rightarrow X_1 by D_1(x_1) = u_1, i.e. by D_1(x) = fD((x, 0)) \text{ [i.e. the first component of } D((x, 0))] \text{. We shall show that } D_1 \text{ is a derivation on } (X_1, \Gamma_1).

Let x_1, x_2 \in X_1 be any two elements and \alpha_1 \in \Gamma_1, then
$D_1(x_1 + x_2) = fD((x_1 + x_2, 0)) = fD((x_1 + x_2, 0 + 0))$

$= fD((x_1, 0) + (x_2, 0)) = f[D((x_1, 0)) + D((x_2, 0))]$  [Since $D$ is additive]

$= fD((x_1, 0)) + fD((x_2, 0)) = D_1(x_1) + D_1(x_2)$

Thus we get, $D_1(x_1 + x_2) = D_1(x_1) + D_1(x_2) \forall x_1, x_2 \in X_1$ i.e $D_1$ is an additive.

Now, $D_1(x_1 \alpha_1 x_2) = fD((x_1 \alpha_1 x_2, 0)) = fD((x_1 \alpha_1 x_2, 0 \alpha_2 0))$, $\alpha_2 \in \Gamma_2$

$= f((x_1, 0)(\alpha_1, \alpha_2)(x_2, 0))$

$= fD(xy), where x = (x_1, 0), y = (x_2, 0) and \alpha = (\alpha_1, \alpha_2)$

$= f[D(x)ay + xD(y)]$  [since $D$ is a derivation on $(X, \Gamma)$]

$= f[D(xy) + xD(y)] = f[D((x_1, 0))((\alpha_1, \alpha_2)(x_2, 0))] + f[(x_1, 0)(\alpha_1, \alpha_2)D((x_2, 0))]$

$= f[D(xy) + xD(y)]$

$= fD(xy) + xD(y) = D_1(x_1)\alpha_1 x_2 + x_1 \alpha_1 D_1(x_2)$

Thus we get, $D_1(x_1 \alpha_1 x_2) = D_1(x_1)\alpha_1 x_2 + x_1 \alpha_1 D_1(x_2) \forall x_1, x_2 \in X_1 \& \alpha_1 \in \Gamma_1.$

So $D_1$ is a derivation on $(X_1, \Gamma_1)$ defined by the derivation $D$ on $(X, \Gamma)$.

Similarly defining a mapping, $D_2: X_2 \rightarrow X_2$ by $D_2(x) = sD((0, x))$, where $s$ represents the second component of $D((0, x))$, we can show that $D_2$ is a derivation on $(X_2, \Gamma_2)$.

Thus for every derivation $D$ on $(X, \Gamma)$ there exist derivations $D_1$ on $(X_1, \Gamma_1)$ and $D_2$ is on $(X_2, \Gamma_2)$ and hence the desired result.

The other results will follow by applying analogous mathematical techniques.

**Remark:** The above results can be extended to the projective product of n number of Gamma-rings.

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**References**


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