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Sequence Generator of Two Second Homotopy Modules

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Abstract

This article discusses about sequence generators of two second homotopy modules. We consider group presentation $\langle y, z\,; y^n, z^n\rangle$ and $\langle b, c; b^n, (bc)^n\rangle$. It's shown that $\langle y, z\,; y^n, z^n\rangle$ and $\langle b, c; b^n, (bc)^n\rangle$ are isomorphic, and there are sequence of generators from $\pi_2(\langle y, z\,; y^n, z^n\rangle)$ to $\pi_2(\langle b, c; b^n, (bc)^n\rangle)$. We use operations on picture to show that.

Mathematics Subject Classification: 14F35, 14H30, 20F05, 20M05, 57M05

Keywords: Second homotopy module, Tietze transformation, Generator

1 Introduction

A picture over P is called a set of generator $\pi_2(\text{P})$ if $\{[\text{P}]:\text{P} \in \boldsymbol{P}\}$ generate $\mathbb{Z}G-$module $\pi_2(\text{P})$ [1]. Therefore, set generator $\boldsymbol{P}$ is generator iff each spherical picture over P can be transformation to empty picture by using operation on picture [2].

The computation of generator $\pi_2(\text{P})$ was done by [2] for generator of one second homotopy module. Besides, the computation generator of two second homotopy modules discussed by [7]. Therefore, in this paper is given properties that can be used to compute generators of second homotopy module to another second homotopy module which is the group presentation is isomorphic.

We use Tietze transformation to prove that the presentation group are isomorphic. Definition of Tietze transformation can generally be seen in some textbooks, as in [3] and [4]. Definition of Tietze transformation can generally be seen as follows:

Definition 1.1 (Definition of Tietze Transformation)
Let $\mathrm{P}_1 = \langle \boldsymbol{x}\,; \boldsymbol{r} \rangle$ *and* $\mathrm{P}_2 = \langle \boldsymbol{y}\,; \boldsymbol{s} \rangle$ *be two presentations of the group* G.

(T1) *If the word S is derivable from* $\boldsymbol{r}$, *then add S to the list of relators*; $\langle \boldsymbol{x}\,; \boldsymbol{r} \rangle \to \langle \boldsymbol{x}\,; S, \boldsymbol{r} \rangle$

(T2) *If the word S is derivable from* $\boldsymbol{r}$, *remove S from the list relators*; $\langle \boldsymbol{x}\,; S, \boldsymbol{r} \rangle \to \langle \boldsymbol{x}\,; \boldsymbol{r} \rangle$

(T3) *If R is word in the* $\boldsymbol{x}$, *and y is some symbol not in the generating set, add y to the generating set and add word* $y^{-1}R \in \boldsymbol{r},\ y \in \boldsymbol{x}$ *to the relator set.*

(T4) *If there is a relator of the form* $y^{-1}R \in \boldsymbol{r},\ y \in \boldsymbol{x}$ *with y not appearing in R, delete this relator and delete y from the generating set, replacing all order occurences of y in the relator words with* R.

Tietze transformation does not change the group defined by a presentation, as mentioned below:

Theorem 1.2 [5] *Suppose that the groups presented by the two presentations* $\langle \boldsymbol{x}; \boldsymbol{r} \rangle$ *and* $\langle \boldsymbol{y}; \boldsymbol{s} \rangle$ *are isomorphic. Then there is a sequence of Tietze transformations leading from one of these to the other. If these presentations are both finite the sequence can be taken to be a finite number of single step.*

We use operations on picture to prove that there are sequence generator of two second homotopy module. Picture and operations on picture can be seen on [6].

2 Sequence Generator From $\pi_2(\langle y, z\,; y^n, z^n \rangle)$ to $\pi_2(\langle b, c; b^n, (bc)^n \rangle)$.

Lemma 2.1
Let $\mathrm{P}_1 = \langle y, z; y^n, z^n \rangle$ *and* $\mathrm{P}_2 = \langle b, c; b^n, (bc)^n \rangle$, *for* $n \in Z$ *and* $n \geq 1$ *be two presentations group. Then* P_1 *isomorphic to* P_2 *and there are sequence of generators from second homotopy module of* P_1 *to second homotopy module of* P_2.

Proof.
The proof is done in two steps. First, proves that P_1 is isomorphic to P_2. Furthermore, prove that there are sequence of generator of $\pi_2(\mathrm{P}_1)$ to generator of $\pi_2(\mathrm{P}_2)$

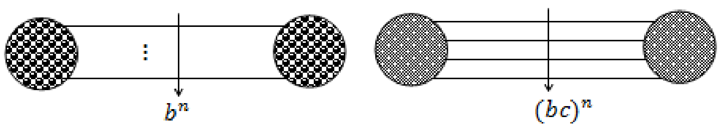
Tietze transformation of $\langle y, z; y^n, z^n \rangle \xrightarrow{\text{Tietze}} \langle b, c; b^n, (bc)^n \rangle$

$[T3]$ $\langle b, y, z\,; y^n, z^n, b = y\rangle$
Add the generator b to the set of generator with relation $b = y$.
$[T3]$ $\langle b, c, y, z\,; y^n, z^n, b = y, c = b^{-1}z\rangle$
Add the generator c to the set of generator with relation $c = b^{-1}z$.
$[T1]$ $\langle b, c, y, z\,; y^n, z^n, b = y, c = b^{-1}z, b^n\rangle$
Add b^n to the set of relator since it's derivable from $b = y$, that is $b = y \Leftrightarrow b^n = y^n = 1$.
$[T1]$ $\langle b, c, y, z\,; y^n, z^n, b = y, c = b^{-1}z, b^n, (bc)^n\rangle$
Add $(bc)^n$ to the set of relator since it's derivable from $c = b^{-1}z$, that is, $c = b^{-1}z \Leftrightarrow bc = z \Leftrightarrow (bc)^n = z^n = 1$.
$[T2]$ $\langle b, c, y, z\,; z^n, b = y, c = b^{-1}z, b^n, (bc)^n\rangle$
Delete relator y^n since it's derivable from $b = y$.
$[T2]$ $\langle b, c, y, z\,; b = y, c = b^{-1}z, b^n, (bc)^n\rangle$
Delete relator z^n since it's derivable from $c = b^{-1}z$
$[T4]$ $\langle b, c, y\,;\ c = b^{-1}z, b^n, (bc)^n\rangle$
Delete generator y since $y = b$.
$[T4]$ $\langle b, c\,;\ b^n, (bc)^n\rangle$
Delete generator z since $c = b^{-1}z \Leftrightarrow z = bc$.

Thus there are sequence of Tietze transformations from P_1 to P_2, so both these presentations group are isomorphic (see Theorem 1.2).

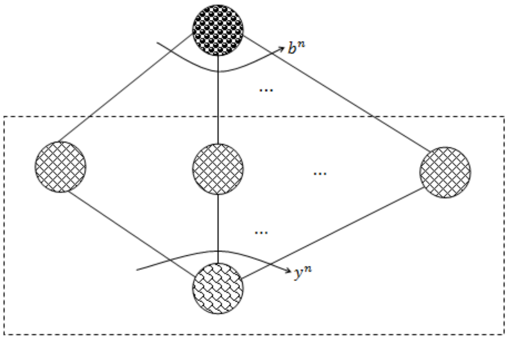
Next, we show there are sequence of generator from $\pi_2(\mathrm{P}_1)$ *to* $\pi_2(\mathrm{P}_2)$. According to [2] generator of second homotopy module $\pi_2(\langle y, z\,; y^n, z^n\rangle)$ is a generator that contains disk y^n and z^n, namely:

Meanwhile generator of $\pi_2(\langle b, c\,;\ b^n, (bc)^n\rangle)$ are a generator that contains disk b^n and disk$(bc)^n$, namely:

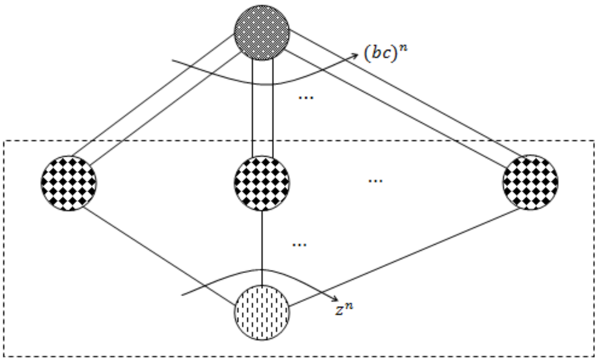


We have the same generators for $\pi_2(\langle b, c, y, z\,; y^n, z^n, b = y, c = b^{-1}z\rangle)$ and $\pi_2(\langle y, z\,; y^n, z^n\rangle)$ based on Corollary 1 [7].

Based on Theorem 1 [7], for $\pi_2(\langle b, c, y, z\,; y^n, z^n, b = y, c = b^{-1}z, b^n\rangle)$ there is a disk containing the new generator b^n. The generators are P_1, P_2 and P_3,, where P_3 is:

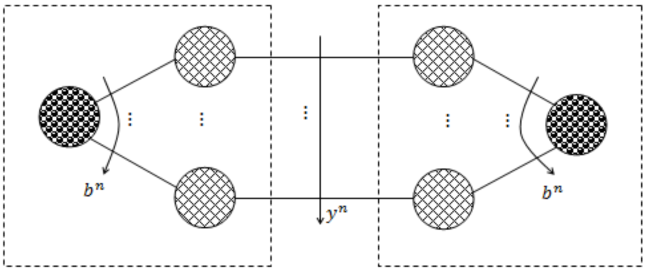


Furthermore, for $\pi_2(\langle b,c,y,z\,;y^n,z^n,b=y,c=b^{-1}z,b^n,(bc)^n\rangle$, there is a new generator that contains the disk $(bc)^n$ (Theorem 1 [7]). That generators are P_1,P_2,P_3 and P_4, where P_4 is:

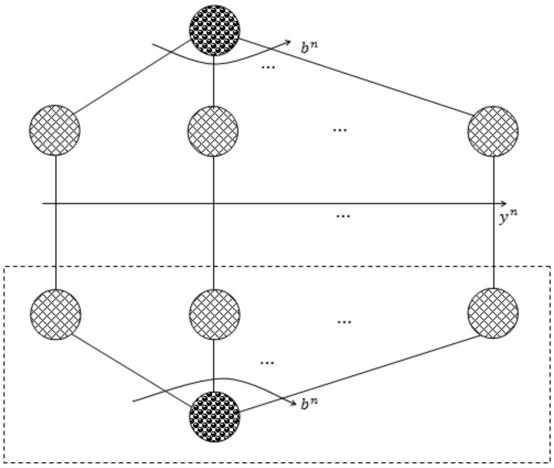


For $\pi_2(\langle b,c,y,z\,;z^n,b=y,c=b^{-1}z,b^n,(bc)^n\rangle)$ there are the elimination of relator y^n and replace the disk into the disk y^n to disk b^n. Generators which contains y^n are P_1 dan P_3. These new generators namely respectively $P_1{}'$ and $P_3{}'$.

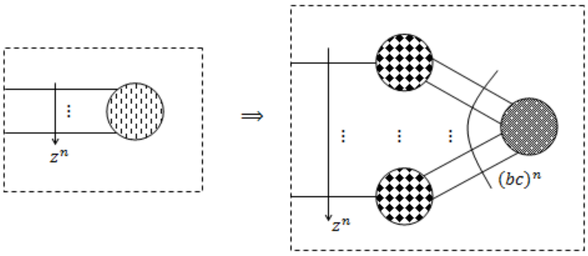
Generator $P_1{}'$:



Generator $P_3{}'$:

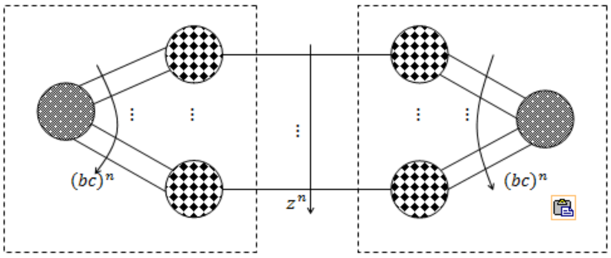


For $\pi_2(\langle b, c, y, z\,; b = y, c = b^{-1}z, b^n, (bc)^n\rangle)$, deletion of z^n and replace the disk z^n into the disk $(bc)^n$ (Corollary 1[7])

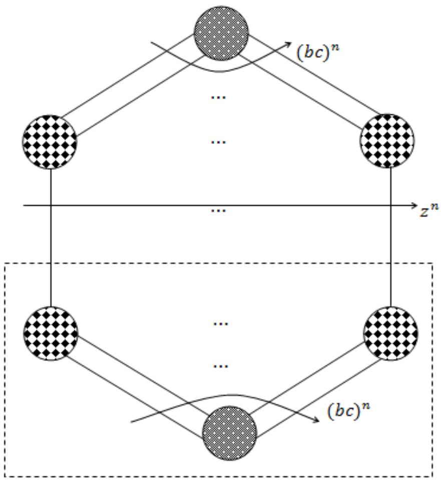


Generators containing z^n n are generator P_2 and P_4. These new generators namely respectively $P_2{}'$ and $P_4{}'$.

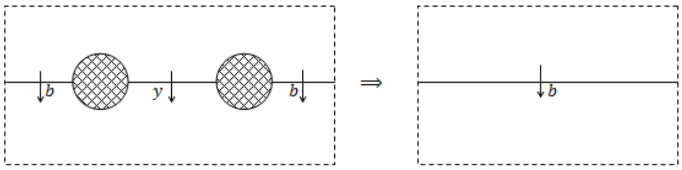
Generator $P_2{}'$:



Generator P_4':

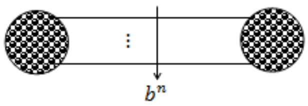


For $\pi_2(\langle b, c, y\,;\ c = b^{-1}z, b^n, (bc)^n\rangle)$, deletion of generator y and replace y to b (Corollary 2 [7]).

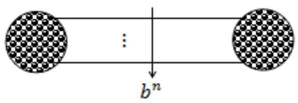


Generators containing y are P_1' and P_3'. These new generators namely respectively P_1'' and P_3''.

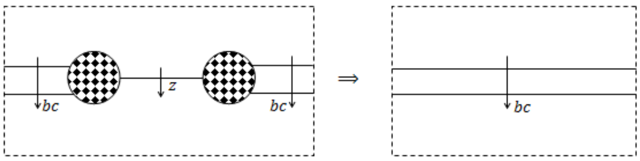
Generator P_1'':



Generator P_3'':

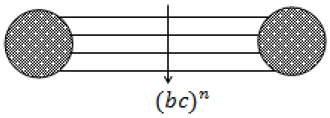


Next, for $\pi_2(\langle b, c\,;\ b^n, (bc)^n\rangle)$, deletion of the generator z and replace z into bc (Corollary 2 [7]).

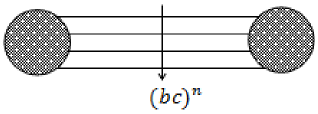


Generators containing z are generator P_2' and P_4'. These new generators namely respectively P_2'' dan P_4''.

Generator P_2'':



Generator P_4'':



So the generators are obtained that contains disk b^n and disk $(bc)^n$.■

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