Some Results About Very Good Homomorphisms between Hypergroups\(^1\)

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Abstract

Let \( f \) be a surjective very good homomorphism from a hypergroup \( H \) to a hypergroup \( H' \). In this paper we prove first that if \( H' \) has a scalar identity \( \varepsilon' \) and \( A = f^{-1}(\varepsilon') \), then \( H/A \cong H' \). Next, a series of results about hypergroups corresponding to that of groups are obtained by foregoing result.

Mathematics Subject Classification: 20N20

Keywords: Hypergroups, Normal sub-hypergroups, Invertible sub-hypergroups, Very good homomorphism

1 Introduction

The concept of a hypergroup which is based on the notion of hyperoperation was introduced by Marty in [4] and studied extensively by many mathematicians. Hypergroup theory extends some well-known results in group theory and introduces new topics leading to a wide variety of applications, as well as to broadening of the fields of investigation. Surveys of the theory can be

\(^1\)This work is supported by Scientific Research Foundation of Sichuan Provincial Education Department(No:14ZA0314) and Scientific Research Foundation of Sichuan Provincial Education Department(No:15ZA0345).
found in the books of Corsini [1], Davvaz and Leoreanu-Fotea [3], Corsini and
Leoreanu [2].

A nonempty subset $A$ of a hypergroup $H$ is called a sub-hypergroup if it is a hypergroup. An element $\varepsilon$ of $H$ is called an identity element if, for all $x \in H$, $x \in xe \cap ex$. We denote with $x/y$ the set of \{z|x \in yz\} and denote with $y \setminus x$ the set of \{z|x \in yz\}. A function $f : H \to H'$ is called a homomorphism if $f(ab) \subseteq f(a) \circ f(b)$ for all $a$ and $b$ in $H$. We say that $f$ is a good homomorphism if for all $a$ and $b$ in $H$, $f(ab) = f(a) \circ f(b)$. We say that $f$ is a very good homomorphism if $f$ is a surjective good homomorphism and for all $a$ and $b$ in $H$, $f(a/b) = f(a) / f(b)$ and $f(a \setminus b) = f(a) \setminus f(b)$.

In this paper, we extends Homomorphism Theorem, Corresponding Theorem of subgroups in group theory and get a series of analogous results in hypergroup theory.

2 Main results

In the first place, we introduce some terms needed in this paper which are extended from group theory.

**Definition 2.1** ([1]62 Definition) Let $A$ be a sub-hypergroup of a hypergroup $H$, if for every $x \in H$, $xA = Ax$, then $A$ is normal sub-hypergroup in $H$.

**Definition 2.2** ([1]35 Definition)Let $A$ be a sub-hypergroup of a hypergroup $H$, if for every $(x, y) \in H \times H$, $y \in xA$ implies $x \in yA$ and $y \in Ax$ implies $x \in Ay$, then $A$ is invertible sub-hypergroup in $H$.

**Definition 2.3** ([1]21 Definition)Let $H$ and $H'$ be hypergroups, $f$ be a surjective function from $H$ to $H'$. We call $f$ a very good homomorphism if for every $(x, y) \in H \times H$, $f(xy) = f(x)f(y)$, $f(x/y) = f(x)/f(y)$ and $f(x \setminus y) = f(x) \setminus f(y)$ are valid.

**Definition 2.4** Let $H$ and $H'$ be hypergroups. If $f$ is a bijective from $H$ to $H'$ such that $f(xy) = f(x)f(y)$ for every $(x, y) \in H \times H$, then $H$ is said to be isomorphic to $H'$, in which case we may write $H \cong H'$.

Now we present some new results about above concepts.

**Lemma 2.1** Let $H$ and $H'$ be hypergroups, $f$ be a very good homomorphism from $H$ to $H'$. Then

1. $A'$ is a normal sub-hypergroup of $H'$ if and only if $f^{-1}(A')$ is a normal sub-hypergroup of $H$.

2. $A'$ is an invertible sub-hypergroup of $H'$ if and only if $f^{-1}(A')$ is an invertible sub-hypergroup of $H$.
Results about very good homomorphisms between hypergroups

Proof: Let $A'$ be a sub-hypergroup of $H'$ and $A = f^{-1}(A')$.

Since $f(A \circ A) = f(A) \circ f(A) = A'$, so $A \circ A \subseteq A$. Let $(x, y) \in A \times A$. Then there is a $z' \in A'$ such that $f(y) \in z'f(x)$, so $z' \in f(y)/f(x) = f(y/x)$. Hence there is a $z \in y/x$ such that $f(z) = z'$, it implies that $z \in A$, so $y \in Ax$. Similarly, we can get $y \in xA$ too. Thus $A$ is a sub-hypergroup of $H$.

Let $A'$ be a normal sub-hypergroup of $H'$, $(x, y) \in H \times H$ and $y \in Ax$. Then $f(y) \in f(Ax) = f(A)f(x) = f(x)f(A) = f(x)A'$, so $f(x)f(y) \cap A' \neq \phi$. Since $f(x)f(y) = f(x/y)$, there exists a $(z, z') \in x/y \times A'$ such that $f(z) = z'$. Hence $z \in A$. Thus $y \in xz \subseteq xA$, and then we have $Ax \subseteq xA$. Similarly, we can get $xA \subseteq Ax$. That is to say $A$ is a normal sub-hypergroup of $H$.

Now, set $A'$ be an invertible sub-hypergroup of $H'$ and $(x, y) \in H \times H$. If $y \in Ax$, then $f(y) \in f(Ax) = f(A)f(x)$, Since $A'$ is invertible, we have that $f(x) \in A'f(y)$. Hence $f(x)/f(y) \cap A' \neq \phi$. Since $f(x)/f(y) = f(x/y)$, there exists a $(z, z') \in x/y \times A'$ such that $f(z) = z'$. Hence $x \in zy \subseteq Ay$. Similarly, we can prove that if $y \in xA$, then $x \in yA$. That is, $A$ is invertible in $H$.

On the other hand, let $f$ be surjective good homomorphism. It is easy to prove that $B$ is a sub-hypergroup (normal sub-hypergroup) of $H$ implies that $f(B)$ is a sub-hypergroup (normal sub-hypergroup) of $H'$. If $f$ is a surjective very good homomorphism and $B$ is an invertible sub-hypergroup of $H$, then we can easily prove that $f(B)$ is an invertible sub-hypergroup of $H'$ too. Thus if $f^{-1}(A')$ is a sub-hypergroup (normal sub-hypergroup or invertible sub-hypergroup) of $H$, we have that $A'$ is a sub-hypergroup (normal sub-hypergroup or invertible sub-hypergroup) of $H'$. The proof of the Lemma is now complete.

Lemma 2.2 Let $H$ be a hypergroup, and let $A$ be a normal and invertible sub-hypergroup of $H$. Set $H/A = \{Ax | x \in H\}$ equipped with the operation: $Ax \circ Ay = \{Az | z \in xy\}$. Then $H/A$ is a hypergroup with a scalar identity $A$, and $f(x) = Ax$ is a very good homomorphism from $H$ to $H/A$.

Proof: Obviously, $H/A$ is a hypergroup with a scalar identity $A$ and $f$ is a surjective good homomorphism. So we need only to prove that $f$ is a very good homomorphism.

If $Az \in f(x/y)$, then there is a $z_1 \in x/y$ such that $Az = Az_1$. Since $z_1 \in x/y$, we have $x \in z_1y$, $Ax \subseteq Az_1y$. Hence $Ax \in Az_1 \circ Ay = Az \circ Ay$, and $Az \in Ax/Ay = f(x)/f(y)$. Therefore $f(x/y) \subseteq f(x)/f(y)$.

If $Az \in Ax/Ay$, then $Ax \in Az \circ Ay$. Hence there is a $x_1 \in zy$, such that $Ax = Ax_1$. As $x_1 \in zy$, we have $Ax_1 \subseteq Az_1y$ and $Ax \subseteq Az_1y$. Since $A$ is invertible, so $x \in Az_1y$. Hence there is a $z_1 \in Az$ such that $x \in z_1y$. Thus we get $z_1 \in x/y$ and $Az_1 \in f(x/y)$. On the other hand, since $z_1 \in Az$, we have $Az = Az_1$, that is $Az \in f(x/y)$ and hence $f(x)/f(y) \subseteq f(x/y)$. This implies that $f(x/y) = f(x)/f(y)$. 
By the same argument we have that \( f(x \setminus y) = f(x) \setminus f(y) \). Therefore \( f \) is a very good homomorphism from \( H \) to \( H/A \).

We now can extend the Homomorphism Theorem of subgroups in group theory to hypergroup theory.

**Theorem 2.3** Let \( H \) and \( H' \) be hypergroups, and let \( f \) be a very good homomorphism from \( H \) to \( H' \). If \( H' \) has a scalar identity \( \varepsilon' \), and \( A = f^{-1}(\varepsilon') \), then \( H/A \cong H' \).

**Proof:** It is easy to see that \( \varepsilon' \) is a normal and invertible sub-hypergroup of \( H' \). By Lemma 2.1, \( A \) is a normal and invertible sub-hypergroup of \( H \). By Lemma 2.2, \( H/A \) is a hypergroup. Define a mapping \( g : H/A \rightarrow H' \) by \( g(Ax) = f(x) \). Obviously, this is a well-defined mapping and \( g(Ax \circ Ay) = g(Ax)g(Ay) \). It is remained to prove that \( g \) is an injection.

Let \( g(Ax) = g(Ay) \). Then \( f(x) = f(y) = \varepsilon'f(y) \), and hence \( \varepsilon' \in f(x)/f(y) = f(x/y) \). Therefore there is a \( z \in x/y \) such that \( f(z) = \varepsilon' \). This implies that \( x \in zy \subseteq Ay \). Now \( A \) is invertible, we have \( Ax = Ay \). That is, \( g \) is an injection.

**Lemma 2.4** Let \( H \) and \( H' \) and \( H'' \) be hypergroups, if \( f_1 \) is a very good homomorphism from \( H \) to \( H' \), \( f_2 \) is very good homomorphism from \( H' \) to \( H'' \), then \( f_2f_1 \) is a very good homomorphism from \( H \) to \( H'' \).

**Proof.** For any \( (x, y) \in H \times H \), we have

\[
f_2f_1(xy) = f_2(f_1(xy)) = f_2(f_1(x)f_1(y)) = f_2(f_1(x))f_2(f_1(y)) = f_2f_1(x)f_2f_1(y).
\]

Similarly,

\[
f_2f_1(x/y) = f_2(f_1(x)/f_1(y)) = f_2f_1(x)/f_2f_1(y),
\]

and

\[
f_2f_1(x \setminus y) = f_2f_1(x) \setminus f_2f_1(y).
\]

Hence \( f_2f_1 \) is a very good homomorphism from \( H \) to \( H'' \).

By Lemma 2.2, Theorem 2.3 and Lemma 2.4, we can get immediately the following two Corollaries:

**Corollary 2.1** Let \( H \) and \( H' \) be hypergroups, \( f \) be a very good homomorphism from \( H \) to \( H' \). If \( A' \) is a normal and invertible sub-hypergroup of \( H' \), \( A = f^{-1}(A') \), then \( H/A \cong H'/A' \).

**Corollary 2.2** Let \( H \) be a hypergroup. If \( A \) and \( B \) are normal and invertible sub-hypergroups of \( H \) and \( A \subseteq B \), then \( (H/A)/(B/A) \cong H/B \).
Now we extend the Corresponding Theorem of subgroups in group theory to hypergroup theory. For this reason, we need the following Lemma:

**Lemma 2.5** Let $H$ and $H'$ be hypergroups, and let $f$ be a very good homomorphism from $H$ to $H'$. Suppose that $H'$ has a scalar identity $\varepsilon'$, $A = f^{-1}(\varepsilon')$, $A_1$ and $A_2$ are sub-hypergroups of $H$. If $A \subseteq A_1$, $A \subseteq A_2$ and $f(A_1) = f(A_2)$, then $A_1 = A_2$.

Proof. If $x \in A_1$, then there is a $y \in A_2$ such that $f(x) = f(y) = f(x/y)$, so there is a $z \in x/y$ satisfying $f(z) = \varepsilon$. Therefore we get $x \in zy \subseteq Ay \subseteq A_2$, which implies that $A_1 \subseteq A_2$. By the same argument, we have $A_2 \subseteq A_1$. Thus $A_1 = A_2$.

By Lemma 2.1, and Lemma 2.5, we can get the following Theorem:

**Theorem 2.6** Let $H$ and $H'$ be hypergroups, and let $f$ be a very good homomorphism from $H$ to $H'$. Suppose that $H'$ has a scalar identity $\varepsilon'$, $A = f^{-1}(\varepsilon')$. Write $G = \{B | A \subseteq B, B$ is a sub-hypergroup(normal sub-hypergroup or invertible sub-hypergroup) of $H\}$, and $G' = \{B' | \varepsilon' \in B', B'$ is a sub-hypergroup(normal sub-hypergroup or invertible sub-hypergroup) of $H'\}$. Then $f$ is a bijective from $G$ to $G'$.

By Lemma 2.2, Lemma 2.4 and Theorem 2.6, we can get the following Corollary:

**Corollary 2.3** Let $H$ and $H'$ be hypergroups, and let $f$ be a very good homomorphism from $H$ to $H'$. Suppose that $A'$ is a normal and invertible sub-hypergroup of $H'$, $A = f^{-1}(A')$. Write $G = \{B | A \subseteq B, B$ is a sub-hypergroup(normal sub-hypergroup or invertible sub-hypergroup) of $H\}$, and $G' = \{B' | A' \subseteq B', B'$ is a sub-hypergroup(normal sub-hypergroup or invertible sub-hypergroup) of $H'\}$. Then $f$ is a bijective from $G$ to $G'$.

**References**


Received: January 2, 2015; Published: February 1, 2015