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Characterizations of Strongly Associative Group Algebras

Jonas Gonçalves Lopes

Department of Mathematics
Federal University of Rio Grande do Norte
Natal, RN CEP 59078-900, Brazil

Abstract

Given a partial action α of a group G on the group algebra FH, where H is a finite group and F is an arbitrary field whose characteristic p divides the order of H, we investigate the associativity question of the partial crossed product $FH *_\alpha G$. If $FH *_\alpha G$ is associative for any G and any α, then FH is called strongly associative. We characterize the strongly associative modular group algebras FH.

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1 Introduction

The Following three definitions are due to Dokuchaev and Exel.

Definition 1.1. *Let G be a group with identity element 1 and $\mathcal{A}$ an associative non-unital (i.e., non-necessarily unital) algebra. A partial action α of G on $\mathcal{A}$ is a pair $\alpha = (\{\mathcal{A}_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$, formed by a collection of (two-sided) ideals $\mathcal{A}_g$ of $\mathcal{A}$, and a collection of isomorphisms of algebras*

$$\alpha_g : \mathcal{A}_{g^{-1}} \to \mathcal{A}_g$$

which satisfy the following conditions for every $g, h \in G$:

(i) $\mathcal{A}_1 = \mathcal{A}$ *and* α_1 *is the identity automorphism of* $\mathcal{A}$*;*

(ii) $\mathcal{A}_{(gh)^{-1}} \supseteq \alpha_h^{-1}(\mathcal{A}_h \cap \mathcal{A}_{g^{-1}})$*;*

(iii) $\alpha_g \circ \alpha_h(x) = \alpha_{gh}(x)$ *for each* $x \in \alpha_h^{-1}(\mathcal{A}_h \cap \mathcal{A}_{g^{-1}})$*.*

Definition 1.2. *Given a partial action α of a group G on a F-algebra $\mathcal{A}$, the partial skew group ring of $\mathcal{A}$ and G by α, written $\mathcal{A} *_\alpha G$, is the set of all finite formal sums $\{\sum_{g\in G} a_g\delta_g : a_g \in \mathcal{A}_g\}$, where δ_g are symbols. Addition is defined in the obvious way, and multiplication is determined by*

$$(a_g\delta_g) \cdot (b_h\delta_h) = \alpha_g(\alpha_{g^{-1}}(a_g)b_h)\delta_{gh}.$$

Definition 1.3. *We say that an algreba $\mathcal{A}$ is strongly associative if for any group G and an arbitrary partial action α of G on $\mathcal{A}$ the partial skew group ring $\mathcal{A} *_\alpha G$ is associative.*

Dokuchaev and Exel [see 4, Theorem 3.1] proved that if $\alpha = (\{\mathcal{A}_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is an arbitrary partial action of a group G on $\mathcal{A}$, then $\mathcal{A}$ is strongly associative if and only if the equality

$$(\alpha_g \circ R_c \circ \alpha_{g^{-1}}) \circ L_a = L_a \circ (\alpha_g \circ R_c \circ \alpha_{g^{-1}}) \qquad (*)$$

is valid on $\mathcal{A}_g$ for every $g \in G$ and all $a, c \in \mathcal{A}$.

If H is a finite group and F is a field whose characteristic $p > 0$ does not divide the order of H, we know that the group algebra FH is semiprime. By [4, Corollary 3.4], we conclude that FH is strongly associative.

Now, if p divides $|H|$, then FH is not semiprime. In [4], Dokuchaev and Exel suggested researching the problem of characterization of strongly associative modular group algebras. In [9], [10] and [11] we characterize the strongly associative modular group algebras FH for several classes of groups H over algebraically closed fields of characteristic $p > 0$.

The main purpose of this article is to characterize the strongly associative modular group algebras FH with H being an arbitrary finite group and F an arbitrary field of characteristic $p > 0$. We demonstrate (see Theorem 2.7) that if H is a finite group and F is a field whose characteristic p divides $|H|$, then the group algebra FH is strongly associative if and only if H is one of the following two types:

(1) H is cyclic of order 2 or 3;

(2) H is a Frobenius group with complement P and kernel H', where P is a Sylow p-subgroup of H, $|P| = p = 2$ or $|P| = p = 3$.

Now we recall some definitions.

A finite group H is called p-solvable (p is a prime) if H has a subnormal series

$$H_0 = H \supseteq H_1 \supseteq H_2 \supseteq \ldots \supseteq H_r = \{1\}$$

such that H_{i-1}/H_i is a p-group or a p'-group for each i.

In what follows all groups H will be finite and we write $O_{p'}(H)$ for the maximal normal p'-subgroup of H, while $O_p(H)$ stands for the maximal normal p-subgroup of H. The inverse image of $O_p(H/O_{p'}(H))$ in H is denoted by $O_{p',p}(H)$. A group H is called p-constrained if the inverse image of $C_{H/O_{p'}(H)}(O_{p',p}(H)/O_{p'}(H))$ in H is contained in $O_{p',p}(H)$. It is well known (see [14]) that any p-solvable group is p-constrained. H' denotes the commutator subgroup of H and $J(R)$ is the Jacobson radical of a ring R.

A transitive permutation group H in which only the identity fixes more than one letter, but the subgroup fixing a letter is nontrivial, is called a *Frobenius group*.

Let $\mathcal{A}$ be a finite-dimensional algebra over a field F. A central idempotent $e \in \mathcal{A}$ is called *centrally primitive* if $e \neq 0$ and e can not to be written as a sum of two nonzero ortogonal central idempotents in $\mathcal{A}$.

There exists a direct decomposition

$$\mathcal{A} = B_1 \oplus \cdots \oplus B_n$$

of $\mathcal{A}$ into indecomposable two-sided ideals $B_i \neq 0$ with $B_iB_j = 0$ for $i \neq j$.

Write $1 = e_1 + \cdots + e_n$ with $e_i \in B_i$. Then the e_i's are mutually orthogonal centrally primitive idempotents and $B_i = \mathcal{A}e_i = e_i\mathcal{A}$.

The ideal B_i is called a *block of* $\mathcal{A}$ and e_i is called a *block idempotent of* $\mathcal{A}$.

Let H be a finite group and F an arbitrary field of characteristic $p > 0$. The principal block of FH is the block that contains the trivial FH-module 1_H or 1_{FH} (the module for the 1-representation of H or the trivial representation of H over F of degree 1, $h \mapsto (1), \forall\, h \in H$).

Let V be an FH-module. By the kernel of V, written Ker$V$, we understand the kernel of the (modular) representation of H afforded by V. Following Brauer (see [2]), we define the kernel of B, written Ker$B$, by

$$\mathrm{Ker}B = \bigcap_{V \in X} \mathrm{Ker}V,$$

where X is the set of all irreducible FH-modules in B.

Now a group H is called p-nilpotent if H has a normal p'-subgroup M such that H/M is a p-group. Expressed otherwise, H is p-nilpotent if $H = O_{p',p}(H)$.

2 THE PROBLEM OF STRONG ASSOCIATIVITY OF FH

The following two examples are due to Lopes.

Example 2.1. *Let $H = \langle y : y^2 = 1\rangle$ be a cyclic group of order 2 and F a field of characteristic 2. Then the group algebra FH is strongly associative. Indeed, the unique nontrivial ideal of FH is $J(FH) = FH(y-1)$, by [8, Lemma 17.13 (ii)]. Let G be any group and $\alpha = (\{\mathcal{A}_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ an arbitrary partial action of G on FH. Then, for every $g \in G$, we have $\mathcal{A}_g = FH$ or $\mathcal{A}_g = J(FH)$. Since H is a 2-group, it folllows that $J(FH) = \Delta(H)$ (the augmentation ideal of FH). By [10, Lemma 2.1] we obtain that the condition $(*)$ is satisfied. Therefore FH is strongly associative.*

Example 2.2. *Let $H = \langle y : y^3 = 1\rangle$ be the ciclic group of order 3 and F a field of characteristic 3. Then the group algebra FH is strongly associative. Indeed, the unique nontrivial ideals of FH are $J(FH) = FH(y-1)$ and $J(FH)^2 = FH(y-1)^2$. Let G be any group and $\alpha = (\{\mathcal{A}_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ an arbitrary partial action of G on FH. In this case, $\forall\ g \in G$, we have $\mathcal{A}_g = FH$ or $\mathcal{A}_g = J(FH)$, or $\mathcal{A}_g = J(FH)^2$. Since H is a 3-group it follows that $J(FH) = \Delta(H)$. Thus by [10, Lemma 2.1], the equality $(*)$ is valid on $\mathcal{A}_g = FH$ and on $\mathcal{A}_g = J(FH)$. Now, let $\mathcal{A}_g = J(FH)^2$. Since $dim_F J(FH)^2 = 1$, we easily see that $\alpha_g \circ R_c \circ \alpha_{g^{-1}}$ and L_a commute on $\mathcal{A}_g$, $\forall$ $a, c \in FH$. Therefore FH is strongly associative.*

We will use the following results of Lopes (see [10]).

Lemma 2.1. *Let H be a finite group and $M \neq H$ a normal subgroup of H. Let F be a field of characteristic p. If p divides $|M|$, then the group algebra FH is not strongly associative.*

Proof. See [10, Lemma 2.2]. □

Theorem 2.1. *Let H be a nilpotent group and F a field whose characteristic p divides the order of H. Then the group algebra FH is strongly associative if and only if $|H| = 2$ or $|H| = 3$.*

Proof. See [10, Theorem 2.1]. □

In the next result we make a small correction of the Theorem 2.2 of [10].

Theorem 2.2. *Let H be a Frobenius group with complement P, where P is a Sylow p-subgroup of H, and let F be a field of characteristic $p > 0$. Then the group algebra FH is strongly associative if and only if the Frobenius kernel of H is H' and $|P| = p = 2$ or $|P| = p = 3$.*

Proof. Since H is a Frobenius group with complement P, there exists a normal subgroup M of H such that

$$H = MP, \quad M \cap P = \{1\}.$$

Now as $|H| = |M||P|$, we have that M is a p'-group. It obviously follows that $M = O_{p'}(H)$ and that H is p-constrained.

Let $e = |M|^{-1} \sum_{x \in M} x$. It is well known [see 1, Theorem 12.9 (a)] that

$$FH = FHe \oplus FH(1-e),$$

where $FHe \simeq F(H/M) \simeq FP$ and $FH(1-e) = \Delta(H, M)$.

Suppose that FH is strongly associative. We have by [10, Lemma 2.3] that FHe is strongly associative. Since $FHe \simeq FP$ it follows that FP is strongly associative. By Theorem 2.1 we conclude that $|P| = 2$ or $|P| = 3$.

Hence H/M is abelian and $H' \subseteq M$; consequently p does not divide $|H'|$. Thus $H' \cap P = \{1\}$. If $H' \neq M$, then $H'P \neq H$ is a normal subgroup of H and p divides $|H'P| = |H'||P|$. It follows by Lemma 2.1 that FH is not strongly associative, a contradiction. Therefore, $M = O_{p'}(H) = H'$, that is, the Frobenius kernel of H is H'. This completes the proof of the "only if" part. For proof of the "if" part, see [10, Theorem 2.2]. □

Now, we obtain the following result.

Theorem 2.3. *Let H be a finite p-nilpotent group and let F be a field of characteristic $p > 0$. Then the group algebra FH is strongly associative if and only if H is one of the following two types:*

(1) H is cyclic of order 2 or 3;

(2) H is a Frobenius group with complement P and kernel H', where P is a Sylow p-subgroup of H, $|P| = p = 2$ or $|P| = p = 3$.

Proof. Suppose that FH is strongly associative. Since H is p-nilpotent, there is a Sylow p-subgroup P of H such that $H = O_{p'}(H)P$.

If $O_{p'}(H) = \{1\}$, then $H = P$ and by Theorem 2.1, $|H| = 2$ or $|H| = 3$, so that H is of type (1).

Now, if $O_{p'}(H) \neq \{1\}$, note that p divides $|P|$ and $P \neq H$, hence P is not normal in H and consequently P is not nilpotent. Let

$$e_1 = |O_{p'}(H)|^{-1} \sum_{x \in O_{p'}(H)} x.$$

By [1, Theorem 12.9(a)] we obtain

$$FH = FHe_1 \oplus FH(1 - e_1), \qquad (**)$$

where $FHe_1 \simeq F(H/O_{p'}(H)) \simeq FP$ and $FH(1 - e_1) = \Delta(H, O_{p'}(H))$.

Since FH is strongly associative, we have by [10, Lemma 2.3] that FHe_1 is strongly associative. Now, $FHe_1 \simeq F(H/O_{p'}(H)) \simeq FP$ implies that FP is strongly associative. It follows by Theorem 2.1 that $|P| = 2$ or $|P| = 3$. By definition we have that the blocks of FH have p-defect $d = 1$ or $d = 0$.

Since $H/O_{p'}(H)$ is abelian, it follows as in the proof of the "only if" part of the Theorem 2.2 that $O_{p'}(H) = H'$.

Let $B = B(e) = FHe$ be a block of FH of p-defect $d = 1$, that is, with P as defect group. It follows by [8, Theorem 6.4, (i)] that $J(FHe) = J(FH)e \neq 0$. Since H is p-nilpotent, we have by [12, Corollary 3.6] that B contains a unique irreducible FH-module W. Thus $\mathrm{Ker}B = \mathrm{Ker}W$.

Claim: $\mathrm{Ker}B = H$.

Suppose that $\mathrm{Ker}B \neq H$. By [13, Theorem 1], $\mathrm{Ker}B$ is a p-nilpotent normal subgroup of H. Then $\mathrm{Ker}B$ has a normal p'-subgroup U such that $\mathrm{Ker}B/U$ is a p-group. Since the Sylow p-subgroups of H have order 2 or 3, we may write $\mathrm{Ker}B = UP$, $U \cap P = \{1\}$. Then p divides $|\mathrm{Ker}B| = |U||P|$ and it follows by Lemma 2.1 that FH is not strongly associative, a contradiction. Thus $\mathrm{Ker}B = H$ and the claim follows.

Since the 1-representation of H, $h \mapsto (1)$, $\forall\ h \in H$, is the only irreducible representation φ of H such that $\mathrm{Ker}(\varphi) = H$, it follows from $\mathrm{Ker}B = H$ that W is the trivial FH-module and consequently B is the principal block of FH. Since B is an arbitrary block of FH of p-defect 1, we have shown that the principal block is the only block of FH of p-defect 1. Now, since H is p-constrained and $H' = O_{p'}(H)$, $e_1 = |H'|^{-1} \sum_{h' \in H'} h'$, it follows by [8, Proposition 1.20] that $FHe_1 = B_1$ is the principal block of FH. Hence $B = B_1 = FHe_1$.

Now, from $(**)$ and by [3, Theorem 55.2], we have that the ideals FHe_1 and $FH(1 - e_1) = \Delta(H, H')$ of FH are direct sums of blocks. Since FHe_1 is the principal block of FH, we conclude that

$$FH = FHe_1 \oplus FHe_2 \oplus \cdots \oplus FHe_k$$

is the direct decomposition of FH into sum of blocks for some $k \geq 2$, where $e_1, \ldots, e_k$ are mutually ortogonal centrally primitive idempotents. Indeed, since P is not normal in H, we have by [7, Theorem 7.12 and Observation 7.13] that p divides $dim_F V$, for some irreducible FH-module V. Now, since the principal block $B_1 = FHe_1$ contains only the trivial FH-module 1_H (up to FH-module isomorphism) and $dim_F 1_H = 1$, it follows that $V \notin FHe_1$. Therefore, $k \geq 2$ and $\Delta(H, H')$ contains at least one block of p-defect zero. Then, by [8, Theorem 6.4, (i)], $FHe_2, \ldots, FHe_k$ are simple F-algebras, so that $J(FHe_i) = 0$ for every $i = 2, \ldots, k$. Hence

$$J(\Delta(H, H')) = J(FHe_2) \oplus \cdots \oplus J(FHe_k) = 0,$$

that is, $\Delta(H, H')$ is a semisimple F-algebra. Thus

$$J(FH) = J(FHe_1) \oplus J(\Delta(H, H')) = J(FHe_1) \simeq J(FP).$$

This implies that $dim_F J(FH) = |P| - 1$. Since $H' = O_{p'}(H) \neq \{1\}$, we have $H \neq P$, and by a theorem of Wallace (1958), [8, Theorem 8.11], it follows that H is a *Frobenius group* with complement P and kernel H' (by the uniqueness of the Frobenius kernel). Therefore, H is of type (2).

Conversely, if $|H| = 2$ or $|H| = 3$, we have by Examples 2.1 and 2.2 that FH is strongly associative. Now, if H is of type (2), it follows by Theorem 2.2 that FH is strongly associative. This completes the proof. □

The next result is the Corollary 2.3 of [11]. The proof given here is different and F is an arbitrary field of characteristic $p > 0$.

Corollary 2.1. *Let H be a finite group of even order $2m$ with m odd and let F be a field whose characteristic p divides the order of H. Then the group algebra FH is strongly associative if and only if H is one of the following two types:*

(3) H is cyclic of order 2;

(4) H is a Frobenius group with complement P and kernel H', where P is a Sylow 2-subgroup of H, $|P| = p = 2$.

Proof. Since the Sylow 2-subgroups of H are cyclic, we obtain by [8, Lemma 12.1] that H is 2-nilpotent. Suppose that FH is strongly associative. If $p \neq 2$, then p divides $|O_{2'}(H)|$, and by Lemma 2.1 FH is not strongly associative, a contradiction. Thus $p = 2$ and it follows by Theorem 2.3 that H is of type (3) or of type (4).

Conversely if H is of type (3) or of type (4), we know that FH is strongly associative. □

REMARK 1. If $|H| = 2k$ with k even, we may write $|H| = 2^l m$, $(m, 2) = 1$, $l > 1$.

REMARK 2. If H is of type (1) or of type (2), note that $J(B_1)$ is commutative, where B_1 is the principal block of FH. Now, if $J(B_1)$ is commutative and FH is strongly associative, we obtain the following result.

Corollary 2.2. *Let H be a nonabelian group of order $2^l m$, $(m, 2) = 1$, $l > 1$, and let F be a field whose characteristic p divides the order of H. Assume that $J(B_1)$ is commutative. Then the group algebra FH is strongly associative if and only if H is a Frobenius group with complement P and kernel H', where P is a Sylow 3-subgroup of H, $|P| = p = 3$.*

Proof. Since $J(B_1)$ is commutative, it follows by [8, Theorem 14.7] that H is p-nilpotent whit abelian Sylow p-subgroups. Using that H is nonabelian and $l > 1$, the result follows by Theorem 2.3. □

In the next result, $N_H(P)$ denotes the normalizer of P in H.

Theorem 2.4. *Let F be a field of characteristic $p > 0$ and let H be a finite group descriped by*

$$H = O_{p'}(H)N_H(P), O_{p'}(H) \cap N_H(P) = \{1\},$$

where $O_{p'}(H) \neq \{1\}$ and P is a Sylow p-subgroup of H. Then the group algebra FH is strongly associative if and only if H is a Frobenius group with complement P and kernel H', $|P| = 2$ or $|P| = 3$. In particular, $N_H(P) = P$.

Proof. Suppose that FH is strongly associative. Since $O_{p'}(H) \neq \{1\}$ we have $|H| \geq 6$. Let $e = |O_{p'}(H)|^{-1} \sum_{x \in O_{p'}(H)} x.$. Then

$$FH = FHe \oplus FH(1 - e),$$

where $FHe \simeq F(H/O_{p'}(H)) \simeq FN_H(P)$ and $FH(1 - e) = \triangle(H, O_{p'}(H))$. Since FH is strongly associative, it folloes by [10, Lemma 2.3] that FHe is strongly associative. Now, $FHe \simeq F(H/O_{p'}(H)) \simeq FN_H(P)$ implies that $FN_H(P)$ is strongly associative.

Claim. $N_H(P) = P$

Indeed, if $P \neq N_H(P)$, since p divides $|P|$ and P is a normal subgroup of $N_H(P)$, we obtain by Lemma 2.1 that $FN_H(P)$ is not strongly associative, a contradiction. Hence $N_H(P) = P$ and the claim follows.

Thus $H = O_{p'}(H)P$ is p-nilpotent. From $|H| \geq 6$, we conclude by Theorem 2.3 that H is a Frobenius group with complement P and kernel H', $|P| = 2$ or $|P| = 3$.

Conversely if H is a Frobenius group with conplement P and kernel H', $|P| = 2$ or $|P| = 3$, it follows by Theorem 2.2 that FH is strongly associative. □

Example 2.3. *Let $H = S_4$ be the symmetric group on 4 symbols and let F be a field of characteristic $p = 3$. We have*

$$N_H(P) = \{(1), (12), (13), (23), (123), (132)\},$$

$|N_H(P)| = 6$ *and* $N_H(P) \simeq S_3$;

$$O_{p'}(H) = \{(1), (12)(34), (14)(23), (13)(24)\}$$

and

$$S_4 = O_{p'}(H)N_H(P), O_{p'}(H) \cap N_H(P) = \{1\}.$$

Since $N_H(P) \neq P$, it follows by Theorem 2.4 that FS_4 is not strongly associative.

The following theorem was proved by Lopes [11, Theorem 2.1] for group algebras over algebraically closed fields. Here F is an arbitrary field of characteristic $p > 0$.

Theorem 2.5. *Let H be a finite p-solvable group and let F be a field of characteristic $p > 0$. Then the group algebra FH is strongly associative if and only if H is of type (1) or of type (2).*

Proof. Supopse that FH is strongly associative. Because H is p-solvable, there is a normal subgroup M of H such that either H/M is a p'-group or $|H/M| = p$. If H/M is a p'-group, then P is a Sylow p-subgroup of M, so that p divides $|M|$. It follows by Lemma 2.1 that FH is not strongly associative, a contradiction. Thus $|H/M| = p$ and p does not divide $|M|$, that is, M is a normal p'-subgroup of H. Hence H is p-nilpotent. It follows by Theorem 2.3 that H is of type (1) or of type (2).

Conversely if H is of type (1) or of type (2), we know that FH is strongly associative. □

Corollary 2.3. *Let H be a finite group of odd order and let F be a field whose caharacteristic p divides the order of H. Then the group algebra FH is strongly associative if and only if H is one of the following two types;*

(5) H is cyclic of order 3;

(6) H is a Frobenius group with complement P and kernel H', where P is a Sylow 3-subgroup of H, $|P| = p = 3$.

Proof. Since H is finite of odd order, it follows that H is solvable (see [5]), thus H is p-solvable. Using that H has odd order, the result follows by Theorem 2.5. □

The next result obtained applies to some non-p-solvable groups.

Theorem 2.6. *Let H be a finite group such that $H' \neq H$ and let F be a field whose characteristic p divides the order of H. Then the group algebra FH is strongly associative if and only if H is of type (1) or of type (2).*

Proof. Suppose that FH is strongly associative. If H is abelian, then by Theorem 2.1, $|H| = 2$ or $|H| = 3$, so H is of type (1).

Now, if H is nonabelian, then $|H| \geq 6$ and $H' \neq \{1\}$. Hence $\{1\} \neq H' \neq H$. If p divides $|H'|$, it follows by Lemma 2.1 that FH is not strongly associative, a contradiction. Thus, p does not divide $|H'|$, that is, H' is a p'-group. Let $e = |H'|^{-1} \sum_{x \in H'} x$. By [1, Theorem 12.9 (a)] we have

$$FH = FHe \oplus FH(1-e),$$

where $FHe \simeq F(H/H')$ and $FH(1-e) = \triangle(H, H')$.

Since FH is strongly associative, we obtain by [10, Lemma 2.3] that FHe is strongly associative. Now, $FHe \simeq F(H/H')$ implies that $F(H/H')$ is strongly associative. Since H/H' is abelian and p divides $|H/H'|$, it follows by Theorem 2.1 that $|H/H'| = 2$ or $|H/H'| = 3$. Therefore H is p-nilpotent, $p = 2$ or $p = 3$, $|H| \geq 6$, and by Theorem 2.3, we conclude that H is of type (2).

Conversely if H is of type (1) or of type (2), we know that FH is strongly associative. □

Example 2.4. *Let $H = S_n$ be the symmetric group on n symbols. Let F be a field whose characteristic p divides the order of H. We know that $(S_n)' = A_n \neq S_n$, where A_n is the alternating group of degree n. If $n = 2$, $p = 2$, we have $|H| = 2$; now, if $n = 3$, $p = 2$, then H is a Frobenius group with complement P and kernel $H' = A_3$, $|P| = 2$. It follows by Theorem 2.6 that, in both cases, FH is strongly associative. Finally, if $n = 3$, $p = 3$; and $n \geq 4$, we conclude by Theorem 2.6 that FH is not strongly associative. Note that for $n \geq 5$, S_n is not p-solvable.*

Finally, the main result obtained by completing the characterization.

Theorem 2.7. *Let H be a finite group and let F be a field whose characteristic p divides the order of H. Then the group algebra FH is strongly associative if and only if H is one of the following two types:*

(1) H is cyclic of order 2 or 3;

(2) H is a Frobenius group with complement P and kernel H', where P is a Sylow p-subgroup of H, $|P| = p = 2$ or $|P| = p = 3$.

Proof. Suppose that FH is strongly associative. Consider $N_H(P)$, the normalizer of P in H, where P is a Sylow p-subgroup of H.

Let's divide the demonstration of the theorem in the following two cases:
Case 1. $N_H(P) = H$

In this case, P is a normal subgroup of H. If $P \neq H$, then by Lemma 2.1 FH is not strongly associative, a contradiction. Thus $H = P$ and by Theorem 2.1, H is of type (1). In particular, $N_H(P) = P$, $|P| = p = 2$ or $|P| = p = 3$.
Case 2. $N_H(P) \neq H$

Here P is not normal in H, so that $P \neq H$ and $|H| \geq 6$.
Claim. $FN_H(P)$ is strongly associative.

We will show that $N_H(P) = P$, $|P| = p = 2$ or $|P| = p = 3$. For this purpose, we proceed by induction on $|N_H(P)|$. If $|N_H(P)| = 1$ there is nothing to prove. So, assume that $|N_H(P)| > 1$ and that the claim is true for all groups M of order less than $|N_H(P)|$, and such that p divides $|M|$.

Suppose that $P \neq N_H(P)$. Then by the induction hypothesis FP is strongly associative. It follows by Theorem 2.1 that $|P| = p = 2$ or $|P| = p = 3$. If $|P| = p = 2 = 2^a$, we take $D = \{1\}$, so that $|D| = 1 = 2^0 = 2^d$, $d = a - 1 = 1 - 1 = 0$, and $p = 2$ clearly is the smallest prime divisor of $|H|$. By [8, Lemma 9.22 (i)] we obtain that $\frac{N_H(D)}{D} \simeq H$ is 2-nilpotent. Since FH is strongly associative, by hypothesis, and $|H| \geq 6$, it follows by Theorem

2.3 that H is of type (2); in particular, $N_H(P) = P$, $|P| = p = 2$, a contradiction. Then $p = 2$ does not divide $|H|$, otherwise we consider a Sylow 2-subgroup P of H, thus P is also a Sylow 2-subgroup of $N_H(P)$, and by the foregoing we obtain $N_H(P) = P$, a contradiction. Hence $|P| = p = 3$, which is now the smallest prime divisor of $|H|$. We apply again the Lemma 9.22 (i) of [8], with $a = 1$, $D = \{1\}$, $d = a - 1 = 1 - 1 = 0$, and we obtain that H is 3-nilpotent. It follows by Theorem 2.3 that $N_H(P) = P$, again a contradiction. Therefore $N_H(P) = P$.

Suppose now that $N_H(P) = P$ is not of prime order. Since P is solvable, then P has at least one proper normal subgroup D. By induction FD is strongly associative, and by Theorem 2.1 we obtain that $|D| = p = 2$ or $|D| = p = 3$. Note that D is the only (up to isomorphism) proper subgroup of $N_H(P) = P$. Indeed, if D_0 is another such subgroup, then by induction FD_0 is strongly associative and by Theorem 2.1, we obtain $|D_0| = p = 2$ or $|D_0| = p = 3$. It follows that $|P| = 2^2$ or $|P| = 3^2$, and so P is abelian. Then by [6, Theorem 4.4], $P \cap H' = P \cap (N_H(P))'$. Now, as $N_H(P) = P$, we obtain $P \cap H' = P \cap P' = P \cap \{1\} = \{1\}$, so that $H' \neq H$. Since P is not normal in H, we have that H is not abelian, thus $H' \neq \{1\}$. Hence $\{1\} \neq H' \neq H$, and as FH is strongly associative, by hypothesis, and $|H| \geq 6$, it follows by Theorem 2.6 that H is of type (2). In particular, $N_H(P) = P$, $|P| = p = 2$ or $|P| = p = 3$, a contradiction.

Therefore $|N_H(P) = P| = p$. Hence $N_H(P)$ is abelian, so that $N_H(P) = Z(N_H(P))$ (center of $N_H(P)$). Since $P = N_H(P)$ it follows by a theorem of Burnside [6, Theorem 4.3] that H is p-nilpotent. Finally, as FH is strongly associative, by hypothesis, and $|H| \geq 6$, we conclude by Theorem 2.3 that H is of type (2). In particular, $p = 2$ or $p = 3$. By Examples 2.1 and 2.2, we obtain that $FN_H(P)$ is strongly associative, and the claim follows. This completes the proof of the "only if " part.

Conversely if H is of type (1) or of type (2), we know that FH is strongly associative. □

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