Unit Group of $Z_2 D_{10}$

Parvesh Kumari

Department of Mathematics
Indian Institute of Technology
Hauz Khas, New Delhi 110016, India

R. K. Sharma

Department of Mathematics
Indian Institute of Technology
Hauz Khas, New Delhi 110016, India

Copyright © 2015 Parvesh Kumari and R. K. Sharma. This article is distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this paper we have determined presentation of unit group of group algebra $Z_2 D_{10}$ of the dihedral group $D_{10}$ over the Galois field of 2 elements.

Mathematics Subject Classification: 20F05

Keywords: Unit group; Group Algebra

1 Introduction

Let $FG$ denotes the group algebra of a group $G$ over a field $F$. The homomorphism $\varepsilon : FG \to F$ given by $\varepsilon \left( \sum_{g \in G} \alpha_g g \right) = \sum_{g \in G} \alpha_g$ is called the augmentation mapping of $FG$ and its Kernel, denoted by $\Delta(G) = \left\{ \sum_{g \in G} \alpha_g g \mid \alpha_g \in F, \sum_{g \in G} \alpha_g = 0 \right\}$ is called the augmentation ideal of $FG$.

For $g_1, g_2 \in G$, the commutator is defined as $(g_1, g_2) = g_1^{-1} g_2^{-1} g_1 g_2$. 
The upper central series of a group $G$ is given by
\[ \{1\} = Z_0 \leq Z_1 \leq Z_2 \leq ... \leq Z_n \leq ... \]
where $Z_1$ is the center of $G$. For $n > 1$, $Z_n$ is the unique subgroup of $G$ such that $Z_n/Z_{n-1}$ is center of $G/Z_{n-1}$. If the upper central series of a group terminates with $Z_n = G$ for some $n$, then $G$ is called nilpotent group.

A group $G$ is called solvable group if it has a normal series with abelian factor groups, that is, if there are subgroups
\[ \{1\} = G_0 \leq G_1 \leq G_2 \leq ... \leq G_k = G \]
such that $G_{j-1}$ is normal in $G_j$ and $G_j/G_{j-1}$ is an abelian group for $j = 1, 2, ..., k$.

The order of $U(F_{2^k}D_{10})$ is determined in [2] which is $2^{3k}(2^k - 1)^5(2^k + 1)^2$ when $5|(2^k - 1)$. The structure of $U(FD_{10})$, $F$ is a finite field, is established in [3], especially when $char(F) = 2$ and it contains a primitive 5th root of unity. The structure of $Z_2D_8$ and $Z_2Q_8$ have been obtained in [4]. In this paper we have determined complete structure of $U(Z_2D_{10})$.

2 Presentation of Unit Group $Z_2D_{10}$

**Theorem:** Let the dihedral group $D_{10}$ of order 10 be given by $D_{10} = \langle x, y \mid x^5 = y^2 = 1, xy = x^4y \rangle$. Then the unit group $U(Z_2D_{10})$ (= $G$ say) of the group algebra $Z_2D_{10}$ of the dihedral group $D_{10}$ over the Galois field of 2 elements has the following properties:

1. Order of $G$ is 360.

2. Center $Z(G)$ of $G$ is a cyclic group of order 6 generated by $b^5$, where $b = x^2 + x^4 + y$. And $G/Z(G) = \langle \alpha, \beta \mid \alpha^5, \beta^5, (\alpha\beta)^2, (\alpha^4\beta)^3 \rangle$, where $\alpha = a^3Z, \beta = b^5Z$ and $a = 1 + x + y + x^2y + x^4y$.

3. $G = \langle r, s, t \mid r^5, s^5, t^6, (rs)^2, (r^4s)^3, rtr^4t^5, sts^4t^5 \rangle$, where $r = a^2, s = b^6$ and $t = b^5$.

4. The unit group $G$ is neither nilpotent nor solvable.

**Proof:**

1. The group algebra $Z_2D_{10}$ contains 1024 elements and 512 elements of even length. Any element of even length belongs to the augmentation ideal $\Delta(Z_2D_{10})$, therefore any element of even length in $Z_2D_{10}$ cannot be a unit. Therefore $U(Z_2D_{10})$ will contain at most 512 elements. 152 elements of length 5 listed below
\[ 1 + x + x^2 + x^3 + x^4, \{1 + x + x^2 + x^3 + x^4y \mid 0 \leq i \leq 4\}, \{1 + x + x^2 + x^4 + x^iy \mid 0 \leq i \leq 4\}, \{1 + x + x^2 + y + x^iy \mid i = 1, 4\}, \]
cannot be unit, because these elements satisfy one of the following conditions:

\( \alpha^k = \alpha \) for some \( k \geq 2 \), where \( \alpha \) is some element listed above or \( \alpha^2 = 1 + x + x^2 + x^3 + x^4 \), where \( 1 + x + x^2 + x^3 + x^4 \) is an idempotent. Therefore

\( o(U(Z_2D_{10})) \leq 360 \).

Let \( H = \langle a, b \rangle \), where \( a = 1 + x + y + x^2y + x^4y \), \( b = x^2 + x^4 + y \) are invertible elements of \( Z_2D_{10} \) and their inverse respectively are given by \( 1 + x^4 + y + x^2y + x^4y \) and \( 1 + x + x^2 + x^3 + y + xy + x^4y \). Therefore
$H \leq G$. Here $bab^1 = x$, $a^3b^6a^4 = y$, therefore $x$, $y$ belongs to $H$ and therefore $x^2$, $x^3$, $x^4$, $xy$, $x^2y$, $x^3y$, $x^4y$ all belongs to $H$. Further

\[
1 + x + x^2 = b^4a^4b^12, \quad 1 + x + x^3 = b^2a^2, \quad 1 + x + y = ab^8, \quad 1 + x + xy = ab^{26}, \\
1 + x + x^2y = a^3b^{26}, \quad 1 + x + x^3y = b^3a^3b^{23}, \quad 1 + x + x^4y = b^3ab^{23}, \\
1 + x^2 + y = b^7a^8, \quad 1 + x^2 + xy = a^4b^{19}, \quad 1 + x^2 + x^2y = b^{13}a^2, \quad 1 + x^2 + x^3y = ba^8, \\
1 + x^2 + x^4y = b^{13}a^6, \quad 1 + x + x^2 + y + x^2y = a^4b^{24}, \quad 1 + x + x^2 + y + x^3y = b^3a, \\
1 + x + x^2 + xy + x^3y = b^3a^7, \quad 1 + x + x^2 + xy + x^4y = b^{12}a^8, \quad 1 + x + x^2 + x^2y + x^4y = b^6a^8, \\
1 + x + x^3 + y + xy = b^4a^2b^{14}, \quad 1 + x + x^3 + y + x^4y = a^3b^{21}, \quad 1 + x + x^3 + xy + x^2y = b^5ab^{18}, \quad 1 + x + x^3 + x^2y + x^3y = ab^3, \\
1 + x + x^3 + x^3y + x^4y = ab^21, \quad 1 + x + x^2 + x^3 + y + xy = ab^7, \\
1 + x + x^2 + x^3 + x^4 + y + x^2y = b^2a^3b^{27}, \quad 1 + x + x^2 + x^3 + y + xy + x^2y = b^{11}a^6, \\
1 + x + x^2 + x^3 + y + xy + x^3y = ba^3b^{27}, \quad 1 + x + x^2 + x^3 + y + xy + x^4y = a^2b^{23}, \\
1 + x + x^2 + x^3 + y + x^2y + x^3y = a^4b^7, \quad 1 + x + x^2 + x^3 + y + x^2y + x^4y = a^2b^{25}, \\
1 + x + x^2 + x^3 + y + x^3y + x^4y = a^4b^{17}, \quad 1 + x + x^2 + x^3 + y + x^2y + x^3y = a^2b^2, \\
1 + x + x^2 + x^3 + xy + x^2y + x^3y = b^5a^3b^7, \quad 1 + x + x^2 + x^3 + xy + x^3y + x^4y = a^2b^7, \\
1 + x + x^2 + x^3 + x^2y + x^3y + x^4y = b^2a^3b^7, \quad 1 + x + x^2 + x^3 + xy + x^3y + x^4y = a^2b^{29}, \quad 1 + x + x^2 + x^3 + x^4 + y + xy + x^2y + x^3y = a^2b^5.
\]

Every left multiplication of these 35 elements of $H$ respectively by $x$, $x^2$, $x^3$, $x^4$, $y$, $xy$, $x^2y$, $x^3y$, $x^4y$ gives more 315 elements of $H$. Now we have 360 elements in $H$. Since $H \leq G$, $o(G) \leq 360$ and $o(H) \geq 360$, which implies that $o(G) = o(H) = 360$.

2. Now $G = \langle a, b \rangle$. Here $o(a) = 10$, $o(b) = 30$, $a^5 = b^{15}$, and $(a^i, b^5) = 1$ for all $i$, $1 \leq i \leq 10$, which implies that $b^5 \in Z(G)$, center of G. Let $K = \langle c \mid c^6 = 1 \rangle$, where $c = b^5$, $K$ is normal in $G$, because $K$ is a central subgroup of $G$. Now $G/K = \langle aK, bK \rangle$, so $o(G/K) = 60$, and $o(aK) = 0(bK) = 5$. Thus $G/K$ is a group of order 60 containing more than one element of order 5, so $G/K$ is a simple group isomorphic to $A_5$ (Alternating group of degree 5).

Now $K$ is normal in $Z(G)$ and $Z(G)$ is normal in $G$, therefore

\[
\frac{G/K}{Z(G)K} \cong G/Z(G)
\]

But $G/K \cong A_5$, implying that $Z(G) = K$, since $Z(G) \neq G$. And $G/Z(G)$ is presented by $\langle \alpha, \beta \mid \alpha^5, \beta^5, (\alpha \beta)^2, (\alpha^4 \beta^3) \rangle$, where $\alpha = a^3Z, \beta = b^4Z$, as $aZ = \alpha^2$ and $bZ = \beta^4$.

3. By (2) group $G$ is presented by

\[
G = \langle U(Z_5D_{10}) \rangle = \langle r, s, t \mid r^5, s^5, t^6, (rs)^2, (r^4s)^3, rtr^4t^5, sts^4t^5 \rangle,
\]

where $r = a^2$, $s = b^5$, and $t = b^5$, as $a = r^3t^3$ and $b = s^5$. 

---

Parveen Kumar and R. K. Sharma
4. Suppose $G$ is solvable, then $G/Z(G)$ is also solvable being a factor group of a solvable group. But $G/Z(G)$ is isomorphic to $A_5$ which is not solvable, so $G/Z(G)$ is not solvable. This is a contradiction hence $G$ is not solvable. Also, since $G/Z(G)$ is simple, therefore its center will be trivial and in upper central series for $G$, we have

$$Z_n = Z_1$$

for all $n \geq 1$.

Hence $G$ is not nilpotent.

References


Received: April 15, 2015; Published: May 4, 2015