Block Transitive $2 - (v, k, 1)$ Designs and Two-dimensional Projective Linear Groups

Shaojun Dai
Department of Mathematics, Tianjin Polytechnic University
No.399 Binshuixi Road, Xiqing District
Tianjin, 300387, P. R. China

Jing Chen
Mathematics and Computational Science
Hunan First Normal College
Changsha, Hunan, 410205, P. R. China

Zhenzhen Song
Department of Mathematics, Tianjin Polytechnic University
No.399 Binshuixi Road, Xiqing District
Tianjin, 300387, P. R. China

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Abstract

This article is a contribution to the study of the automorphism groups of $2 - (v, k, 1)$ designs. Let $G$ act as a block-transitive and point-primitive automorphism group of a non-trivial design $D$ with $v$ points and blocks of size $k$. Set $k_2 = (k, v - 1)$. Assume $q = p^f$ for some prime $p$ and positive integer $f$. If $q \geq [(k_2k - k_2 + 1)f]^2$, then $Soc(G)$, the socle of $G$, is not $PSL(2, q)$.

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1 Introduction

A $2-(v,k,1)$ design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a pair consisting of a finite set $\mathcal{P}$ of $v$ points and a collection $\mathcal{B}$ of $k-$subsets of $\mathcal{P}$, called blocks, such that any 2-subsets of $\mathcal{P}$ is contained in exactly one block. We will always assume that $2 < k < v$.

Let $G \leq Aut(\mathcal{D})$ be a group of automorphisms of a $2-(v,k,1)$ design $\mathcal{D}$. Then $G$ is said to be block transitive on $\mathcal{D}$ if $G$ is transitive on $\mathcal{B}$ and is said to be point transitive(point primitive on $\mathcal{D}$ if $G$ is transitive (primitive) on $\mathcal{P}$. A flag of $\mathcal{D}$ is a pair consisting of a point and a block through that point. Then $G$ is flag transitive on $\mathcal{D}$ if $G$ is transitive on the set of flags.

The classification of block transitive $2-(v,3,1)$ designs was completed about thirty years ago (see [1]). In [2], Camina and Siemons classified $2-(v,4,1)$ designs with a block transitive, solvable group of automorphisms. Li classified $2-(v,4,1)$ designs admitting a block transitive, unsolvable group of automorphisms (see [3]). Tong and Li [4] classified $2-(v,5,1)$ designs with a block transitive, solvable group of automorphisms. Han and Li [5] classified $2-(v,5,1)$ designs with a block transitive, unsolvable group of automorphisms. Liu [6] classified $2-(v,6,1)$ designs with a block transitive classical simple groups of automorphisms. In Series of papers (see [8, 9, 10]), Han considered $G$ is block transitive and point primitive with $Soc(G)$ are $E_6(q), E_7(q)$ and $E_8(q)$.

This article is a contribution to the study of the automorphism groups of $2-(v,k,1)$ designs. We prove that following theorem.

**Main Theorem.** Let $\mathcal{D}$ be $2-(v,k,1)$ design and suppose that $G$ is a group of automorphisms of $\mathcal{D}$ which is block transitive and point primitive. Set $k_2 = (k,v-1)$. Assume $q = p^f$ for some prime $p$ and positive integer $f$. If $q \geq [(k_2k-k_2+1)f]^2$, then $Soc(G)$, the socle of $G$, is not $PSL(2,q)$.

2 Preliminary Results

Let $\mathcal{D}$ be a $2-(v,k,1)$ design defined on the point set $\mathcal{P}$ and suppose that $G$ is an automorphism group of $\mathcal{D}$ that acts transitively on blocks. For a $2-(v,k,1)$ design, as usual, $b$ denotes the number of blocks and $r$ denotes the number of blocks through a given point. If $B$ is a block, $G_B$ denotes the setwise stabilizer of $B$ in $G$ and $G_{(B)}$ is the pointwise stabilizer of $B$ in $G$. Also, $G^B$ denotes the permutation group induced by the action of $G_B$ on the points of $B$, and so $G^B \cong G_B/G_{(B)}$. Let $n$ be a positive integer. We use the symbol $p^i \parallel n$ to denote $p^i \mid n$ but $p^{i+1} \not\mid n$. The symbol $|n|_p$ denotes the $p-$part of $n$ and $|n|_{p'}$ denotes the $p'$-part of $n$. In other words, $|n|_p = p^f$ where $p^f \parallel n$ and $|n|_{p'} = n/|n|_p$. 
Suppose that \(G\) is a block-transitive automorphism group of a \(2 - (v, k, 1)\) design. Recall the basic counting lemmas for \(2 - (v, k, 1)\) designs:

\[
\begin{align*}
v &= r(k - 1) + 1 \\
v(v - 1) &= bk(k - 1)
\end{align*}
\]

Then we have \(r = (v - 1)/(k - 1)\). We can show that \(b \geq v\) and \(k \leq r\).

By [11], let \(b_1 = (b, v), b_2 = (b, v - 1), k_1 = (k, v), k_2 = (k, v - 1)\). Then

\[
k = k_1k_2, \quad b = b_1b_2, \quad r = b_2k_2, \quad v = b_1k_1.
\]

**Lemma 2.1** ([12]) Let \(G = T : \langle x \rangle\) and act block-transitively on a \(2 - (v, k, 1)\) design \(D = (\mathcal{P}, \mathcal{B})\). Then \(T\) acts transitively on \(\mathcal{P}\).

**Lemma 2.2** ([7]) Let \(G\) and \(D = (\mathcal{P}, \mathcal{B})\) be a group and a design, and \(G \leq Aut(D)\) be block transitive, point-primitive but not flag-transitive. Let \(Soc(G) = T\). Then

\[
|T| \leq \frac{v}{\lambda} \cdot |T_\alpha|^2 \cdot |G : T|,
\]

where \(\alpha \in \mathcal{P}\), \(\lambda\) is the length of the longest suborbit of \(G\) on \(\mathcal{P}\).

**Lemma 2.3** ([5]) Let \(G\) be a transitive group on the point set \(\mathcal{P}\) and \(T = Soc(G)\). Let \(\alpha \in \mathcal{P}\) and let \(\Gamma\) be a \(G_\alpha\) orbit in \(\mathcal{P}\{\alpha}\). Then \(\Gamma\) is a union of orbits of \(T_\alpha\), all having the same size.

**Lemma 2.4** ([8]) Let \(G, D\) be as in the Main Theorem, \(T = Soc(G)\) is a classical simple group and \(T_\alpha = T \cap G_\alpha\), where \(\alpha \in \mathcal{P}\). Then

1. \(v = k_2(k - 1)b_2 + 1\);
2. \(\frac{v}{x} < (k_2(k - 1) + 1)|G : T|\), or \(\frac{v - 1}{x} \leq k_2(k - 1)|G : T|\) where \(x\) is the size of a \(T_\alpha\)-orbit in \(\mathcal{P}\{\alpha}\);
3. \(\frac{|T|}{|T_\alpha|^2} \leq \frac{(k_2(k - 1) + 1)}{2}|G : T|\);
4. If \((v - 1, q) = 1\), then there exists a \(T_\alpha\)-orbit with size \(y\) in \(\mathcal{P}\{\alpha}\) such that \(y \mid |T_\alpha|_{p'}\).

**Lemma 2.5** ([13]) Let \(T = PSL(2, q) = Soc(G)\), where \(q = p^f\). Suppose that \(M\) is a maximal subgroup of \(G\) not containing \(T\). Then \(q\) is odd, and

\[
\begin{align*}
(a) \quad G_\alpha &= N_G(PSL(2, q_0)), \quad \text{where } q = q_0^c \text{ and } c \text{ is an odd prime;}
(b) \quad T \cap G_\alpha \text{ is } D_4, A_4, S_4, A_5 \text{ or PGL}(2, q^2).
\end{align*}
\]
3 Proof of the Main Theorem

Suppose that $T = \text{Soc}(G) = \text{PSL}(2, q)$, where $q = p^f$. Then we have $G = T : \langle x \rangle$ and $\text{Out}(T) = df$, where $x \in \text{Out}(T)$ and $d = (2, q - 1)$. Let $\circ(x) = m$ and $m \mid df$, then $|G| = q(q^2 - 1)m/d$. By Lemma 2.1, $T$ is point-transitive and $v = [T : T_{\alpha}]$.

In order to prove the Main Theorem, by Lemma 2.5, we will rule out these case one by one.

**Case (a).** $G_{\alpha} = N_{G}(\text{PSL}(2, q_0))$, where $q = q_0$ and $c$ is an odd prime.

We have

$$|T| = \frac{1}{2}q(q^2 - 1), \quad |T_{\alpha}| = \frac{1}{2}q_0(q_0^2 - 1), \quad v = \frac{q_0 - 1}{q_0^2 - 1}.$$

Since $(v - 1, q) = 1$, by Lemma 2.4(4), there exists in $P \setminus \{\alpha\}$ a $T_{\alpha}$-orbit of size $x$ such that

$$x||T_{\alpha}|v' \leq \frac{1}{2}(q_0^2 - 1).$$

Thus $x \leq \frac{1}{2}(q_0^2 - 1)$. It follows that

$$\frac{v}{x} > \frac{2q_0 - 1}{(q_0^2 - 1)^2} > 2q_0 > \frac{(k_2(k - 1) + 1)}{2}G : T|,$$

contradicting Lemma 2.4(2).

**Case (b).** $T \cap G_{\alpha}$ is $D_4$, $A_4$, $S_4$, $A_5$ or $\text{PGL}(2, q^3)$.

**Subcase (1).** $T \cap G_{\alpha}$ is $D_4$, then $|T| = \frac{1}{2}q(q^2 - 1), |T_{\alpha}| = 4$. Thus

$$\frac{|T|}{|T_{\alpha}|^2} \geq \frac{q(q^2 - 1)}{32} > \frac{q^{1/2}}{2} > \frac{(k_2(k - 1) + 1)}{2}|G : T| \quad (q > 3),$$

contradicting Lemma 2.4(3). If $q = 3$, then $|T| = 12$ and $v = 3 > k > 2$, which is impossible.

**Subcase (2).** $T \cap G_{\alpha}$ is $A_4$, then $|T| = \frac{1}{2}q(q^2 - 1), |T_{\alpha}| = 12$. Thus

$$\frac{|T|}{|T_{\alpha}|^2} \geq \frac{q(q^2 - 1)}{12^2} > \frac{q^{1/2}}{2} > \frac{(k_2(k - 1) + 1)}{2}|G : T| \quad (q^2(2q - 1) > 12^2),$$

contradicting Lemma 2.4(3). If $q = 3$, then $|T| = 12$ and $v = 1 > k > 2$, which is impossible. If $q = 5, 7$, then $v = 5, 14$. By equation (1), $k = 2$, which is impossible.

**Subcase (3).** $T \cap G_{\alpha}$ is $S_4$, then $|T| = \frac{1}{2}q(q^2 - 1), |T_{\alpha}| = 24$. Thus

$$\frac{|T|}{|T_{\alpha}|^2} \geq \frac{q(q^2 - 1)}{24^2} > \frac{q^{1/2}}{2} > \frac{(k_2(k - 1) + 1)}{2}|G : T| \quad (q^2(2q - 1) > 24^2),$$

contradicting Lemma 2.4(3). If $q = 3$, then $|T| = 24$ and $v = 1 > k > 2$, which is impossible.
contradicting Lemma 2.4(3). If \( f = 1 \) and \( p = 7 \), then \( v = 7 \) and \( k = 3 \), which is impossible by [2]. If \( f = 2 \) and \( p = 3 \), then \( v = 15 \). By equation (1), \( k = 3 \), which is impossible by [2].

**Subcase (4).** \( T \cap G_\alpha \) is \( A_5 \), then \(|T| = \frac{1}{2}q(q^2 - 1)\), \(|T_\alpha| = 60\). Thus

\[
\frac{|T|}{|T_\alpha|^2} \geq \frac{q(q^2 - 1)}{60^2 \cdot 2} > \frac{q^{1/2}}{2} \cdot \frac{k_2(k - 1) + 1}{2} |G : T| > (q^{1/2}(q^2 - 1) > 60^2),
\]

contradicting Lemma 2.4(3). If \( f = 1 \) and \( p = 11, 19 \), then \( v = 11, 57 \), which is impossible by [2, 4, 5]. If \( f \geq 2 \), then \( q \) is not exist.

**Subcase (5).** \( T \cap G_\alpha \) is \( PGL(2, q^{1/2}) \), then

\[
|T_\alpha| = \frac{1}{2}q^{1/2}(q - 1), \quad v = \frac{1}{2}q^{1/2}(q + 1).
\]

Since \((v - 1, q) = 1\), by Lemma 2.4(4), there exists in \( \mathcal{P}\{\alpha\} \) a \( T_\alpha \)-orbit of size \( x \) such that

\[
x|T_\alpha|_{p'} \leq (q - 1).
\]

Thus \( x \leq (q - 1) \). It follows that

\[
\frac{v - 1}{x} > \frac{1}{2}q^{1/2}(q + 1) - 1 > \frac{1}{2}q^{1/2} > \frac{k_2(k - 1) + 1}{2} |G : T|,
\]

contradicting Lemma 2.4(2).

Thus the proof of the Main Theorem is complete.

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