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Block Transitive $2-(v,k,1)$ Designs and Two-dimensional Projective Linear Groups

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Abstract

This article is a contribution to the study of the automorphism groups of $2-(v,k,1)$ designs. Let G act as a block-transitive and point-primitive automorphism group of a non-trivial design $\mathcal{D}$ with v points and blocks of size k. Set $k_2 = (k, v-1)$. Assume $q = p^f$ for some prime p and positive integer f. If $q \geq [(k_2k - k_2 + 1)f]^2$, then $Soc(G)$, the socle of G, is not $PSL(2,q)$.

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1 Introduction

A $2-(v,k,1)$ design $\mathcal{D}=(\mathcal{P},\mathcal{B})$ is a pair consisting of a finite set $\mathcal{P}$ of v points and a collection $\mathcal{B}$ of $k-$subsets of $\mathcal{P}$, called blocks, such that any 2-subsets of $\mathcal{P}$ is contained in exactly one block. We will always assume that $2<k<v$.

Let $G \leq Aut(\mathcal{D})$ be a group of automorphisms of a $2-(v,k,1)$ design $\mathcal{D}$. Then G is said to be block transitive on $\mathcal{D}$ if G is transitive on $\mathcal{B}$ and is said to be point transitive(point primitive on $\mathcal{D}$ if G is transitive (primitive) on $\mathcal{P}$. A flag of $\mathcal{D}$ is a pair consisting of a point and a block through that point. Then G is flag transitive on $\mathcal{D}$ if G is transitive on the set of flags.

The classification of block transitive $2-(v,3,1)$ designs was completed about thirty years ago (see [1]). In [2], Camina and Siemons classified $2-(v,4,1)$ designs with a block transitive, solvable group of automorphisms. Li classified $2-(v,4,1)$ designs admitting a block transitive, unsolvable group of automorphisms (see [3]). Tong and Li [4] classified $2-(v,5,1)$ designs with a block transitive, solvable group of automorphisms. Han and Li [5] classified $2-(v,5,1)$ designs with a block transitive, unsolvable group of automorphisms. Liu [6] classified $2-(v,k,1)$(where $k=6,7,8,9,10$) designs with a block transitive, solvable group of automorphisms. In [7], Han and Ma classified $2-(v,11,1)$ designs with a block transitive classical simple groups of automorphisms. In Series of papers (see [8, 9, 10]), Han considered G is block transitive and point primitive with $Soc(G)$ are $E_6(q)$, $E_7(q)$ and $E_8(q)$.

This article is a contribution to the study of the automorphism groups of $2-(v,k,1)$ designs. We prove that following theorem.

Main Theorem. Let $\mathcal{D}$ be $2-(v,k,1)$ design and suppose that G is a group of automorphisms of $\mathcal{D}$ which is block transitive and point primitive. Set $k_2=(k,v-1)$. Assume $q=p^f$ for some prime p and positive integer f. If $q \geq [(k_2k-k_2+1)f]^2$, then $Soc(G)$, the socle of G, is not $PSL(2,q)$.

2 Preliminary Results

Let $\mathcal{D}$ be a $2-(v,k,1)$ design defined on the point set $\mathcal{P}$ and suppose that G is an automorphism group of $\mathcal{D}$ that acts transitively on blocks. For a $2-(v,k,1)$ design, as usual, b denotes the number of blocks and r denotes the number of blocks through a given point. If B is a block, G_B denotes the setwise stabilizer of B in G and $G_{(B)}$ is the pointwise stabilizer of B in G. Also, G^B denotes the permutation group induced by the action of G_B on the points of B, and so $G^B \cong G_B/G_{(B)}$. Let n be a positive integer. We use the symbol $p^i \parallel n$ to denote $p^i \mid n$ but $p^{i+1} \not| n$. The symbol $|n|_p$ denotes the $p-$part of n and $|n|_{p'}$ denotes the $p'-$part of n. In other words, $|n|_p=p^t$ where $p^t \parallel n$ and $|n|_{p'}=n/|n|_p$.

Suppose that G is a block-transitive automorphism group of a $2-(v,k,1)$ design. Recall the basic counting lemmas for $2-(v,k,1)$designs:

$$v = r(k-1)+1 \tag{1}$$
$$v(v-1) = bk(k-1) \tag{2}$$

Then we have $r=(v-1)/(k-1)$. We can show that $b \geq v$ and $k \leq r$.

By [11], let $b_1=(b,v)$, $b_2=(b,v-1)$, $k_1=(k,v)$, $k_2=(k,v-1)$. Then

$$k=k_1k_2,\ b=b_1b_2,\ r=b_2k_2,\ \ v=b_1k_1.$$

Lemma 2.1 ([12]) Let $G=T:\langle x\rangle$ and act block-transitively on a $2-(v,k,1)$ design $\mathcal{D}=(\mathcal{P},\mathcal{B})$. Then T acts transitively on $\mathcal{P}$.

Lemma 2.2 ([7]) Let G and $\mathcal{D}=(\mathcal{P},\mathcal{B})$ be a group and a design, and $G\leq Aut(\mathcal{D})$ be block transitive, point-primitive but not flag-transitive. Let $Soc(G)=T$. Then

$$|T| \leq \frac{v}{\lambda}\cdot|T_\alpha|^2\cdot|G:T|,$$

where $\alpha\in\mathcal{P}$, λ is the length of the longest suborbit of G on $\mathcal{P}$.

Lemma 2.3 ([5]) Let G be a transitive group on the point set $\mathcal{P}$ and $T=Soc(G)$. Let $\alpha\in\mathcal{P}$ and let Γ be a G_α orbit in $\mathcal{P}\backslash\{\alpha\}$. Then Γ is a union of orbits of T_α, all having the same size.

Lemma 2.4 ([8]) Let G, $\mathcal{D}$ be as in the Main Theorem, $T=Soc(G)$ is a classical simple group and $T_\alpha=T\cap G_\alpha$, where $\alpha\in\mathcal{P}$. Then

(1) $v=k_2(k-1)b_2+1$;

(2) $\frac{v}{x}<(k_2(k-1)+1)|G:T|$, or $\frac{v-1}{x}\leq k_2(k-1)|G:T|$where x is the size of a $T_\alpha-$orbit in $\mathcal{P}\setminus\{\alpha\}$;

(3) $\frac{|T|}{|T_\alpha|^2}\leq\frac{(k_2(k-1)+1)}{2}|G:T|$;

(4) If $(v-1,q)=1$, then there exists a $T_\alpha-$ orbit with size y in $\mathcal{P}\setminus\{\alpha\}$ such that $y\mid|T_\alpha|_{p'}$.

Lemma 2.5 ([13]) Let $T=PSL(2,q)=Soc(G)$, where $q=p^f$. Suppose that M is a maximal subgroup of G not containing T. Then q is odd, and

(a) $G_\alpha=N_G(PSL(2,q_0))$, where $q=q_0^c$ and c is an odd prime;

(b) $T\cap G_\alpha$ is D_4, A_4, S_4, A_5 or $PGL(2,q^{\frac{1}{2}})$.

3 Proof of the Main Theorem

Suppose that $T = Soc(G) = PSL(2,q)$, where $q = p^f$. Then we have $G = T : \langle x\rangle$ and $Out(T) = df$, where $x \in Out(T)$ and $d = (2, q-1)$. Let $\circ(x) = m$ and $m \mid df$, then $|G| = q(q^2-1)m/d$. By Lemma 2.1, T is point-transitive and $v = |T : T_\alpha|$.

In order to prove the Main Theorem, by Lemma 2.5, we will rule out these case one by one.

Case (a). $G_\alpha = N_G(PSL(2, q_0))$, where $q = q_0^c$ and c is an odd prime.

We have

$$|T| = \frac{1}{2}q(q^2-1), |T_\alpha| = \frac{1}{2}q_0(q_0^2-1), v = \frac{q_0^{c-1}(q_0^{2c}-1)}{q_0^2-1}.$$

Since $(v-1, q) = 1$, by Lemma 2.4(4), there exists in $\mathcal{P}\backslash\{\alpha\}$ a T_α-orbit of size x such that

$$x||T_\alpha|_{p'} \leq \frac{1}{2}(q_0^2-1).$$

Thus $x \leq \frac{1}{2}(q_0^2-1)$. It follows that

$$\frac{v}{x} > \frac{2q_0^{c-1}(q_0^{2c}-1)}{(q_0^2-1)^2} > 2q_0^{3c-5} > 2q > \frac{(k_2(k-1)+1)}{2}|G:T|,$$

contradicting Lemma 2.4(2).

Case (b). $T\cap G_\alpha$ is D_4, A_4, S_4, A_5 or $PGL(2, q^{\frac{1}{2}})$.

Subcase (1). $T \cap G_\alpha$ is D_4, then $|T| = \frac{1}{2}q(q^2-1), |T_\alpha| = 4$. Thus

$$\frac{|T|}{|T_\alpha|^2} \geq \frac{q(q^2-1)}{32} > \frac{q^{1/2}}{2} > \frac{(k_2(k-1)+1)}{2}|G:T| \quad (q>3),$$

contradicting Lemma 2.4(3). If $q = 3$, then $|T| = 12$ and $v = 3 > k > 2$, which is impossible.

Subcase (2). $T \cap G_\alpha$ is A_4, then $|T| = \frac{1}{2}q(q^2-1), |T_\alpha| = 12$. Thus

$$\frac{|T|}{|T_\alpha|^2} \geq \frac{q(q^2-1)}{12^2\cdot 2} > \frac{q^{1/2}}{2} > \frac{(k_2(k-1)+1)}{2}|G:T| \quad (q^{\frac{1}{2}}(q^2-1) > 12^2),$$

contradicting Lemma 2.4(3). If $q = 3$, then $|T| = 12$ and $v = 1 > k > 2$, which is impossible. If $q = 5, 7$, then $v = 5, 14$. By equation (1), $k = 2$, which is impossible.

Subcase (3). $T \cap G_\alpha$ is S_4, then $|T| = \frac{1}{2}q(q^2-1), |T_\alpha| = 24$. Thus

$$\frac{|T|}{|T_\alpha|^2} \geq \frac{q(q^2-1)}{24^2\cdot 2} > \frac{q^{1/2}}{2} > \frac{(k_2(k-1)+1)}{2}|G:T| \quad (q^{\frac{1}{2}}(q^2-1) > 24^2),$$

contradicting Lemma 2.4(3). If $f=1$ and $p=7$, then $v=7$ and $k=3$, which is impossible by [2]. If $f=2$ and $p=3$, then $v=15$. By equation (1), $k=3$, which is impossible by [2].

Subcase (4). $T\cap G_\alpha$ is A_5, then $|T|=\frac{1}{2}q(q^2-1), |T_\alpha|=60$. Thus

$$\frac{|T|}{|T_\alpha|^2}\geq\frac{q(q^2-1)}{60^2\cdot 2}>\frac{q^{1/2}}{2}>\frac{(k_2(k-1)+1)}{2}|G:T| \quad (q^{\frac{1}{2}}(q^2-1)>60^2),$$

contradicting Lemma 2.4(3). If $f=1$ and $p=11,19$, then $v=11,57$, which is impossible by [2,4,5]. If $f\geq 2$, then q is not exist.

Subcase (5). $T\cap G_\alpha$ is $PGL(2,q^{\frac{1}{2}})$, then

$$|T_\alpha|=\frac{1}{2}q^{1/2}(q-1), \quad v=\frac{1}{2}q^{\frac{1}{2}}(q+1).$$

Since $(v-1,q)=1$, by Lemma 2.4(4), there exists in $\mathcal{P}\backslash\{\alpha\}$ a T_α-orbit of size x such that

$$x||T_\alpha|_{p'}\leq(q-1).$$

Thus $x\leq(q-1)$. It follows that

$$\frac{v-1}{x}>\frac{\frac{1}{2}q^{\frac{1}{2}}(q+1)-1}{q-1}>\frac{1}{2}q^{\frac{1}{2}}>\frac{k_2(k-1)+1}{2}|G:T|,$$

contradicting Lemma 2.4(2).

Thus the proof of the Main Theorem is complete.

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