Arithmetical Condition for Normality and Subnormality in Finite Groups

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Abstract

Let $H$ be a subgroup of finite group $G$, with $|G : H| = m$. We prove that $H$ is subnormal in $G$ if $m$ satisfies certain arithmetical condition that is related to the nilpotency of a group. We also give a generalization of the well-known theorem about normality of subgroups with the prime index.

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1. INTRODUCTION

Let us define arithmetical function $\psi$ in the following way. For a prime $p$ we define $\psi(p^n) = (p^n - 1)(p^{n-1} - 1)\ldots(p - 1)$. If $(n_1, n_2) = 1$, then $\psi(n_1n_2) = \psi(n_1)\psi(n_2)$. For example, we have that $|Gl_n(p)| = p^{n(n-1)/2}\psi(p^n)$, for any prime $p$. In [3] Pazderski proved that every group of order $n$ is nilpotent iff $(n, \psi(n)) = 1$.

We recall that non-empty class of groups $F$ is called formation if it is closed for homomorphic images, and if $G/H \in F$ and $G/K \in F$ imply $G/H \cap K \in F$. Every finite group $G$ possesses a unique minimal subgroup $G_F$, that has the quotient group in $F$. Subgroup $G_F$ is also a characteristic subgroup in $G$. For example, all finite nilpotent groups make a formation.

Subgroup $H$ of a group $G$ is subnormal in $G$, if in $G$ exists a chain of subgroups $H = H_1, H_2, \ldots, H_m = G$ such that $H_i$ is normal in $H_{i+1}$. 
2. Results

Our main result is the following:

Theorem 2.1. Let $G$ be a finite group, and $H \leq G$ such that $|G : H| = m$. If $(|G|, \psi(m)) = 1$, then $H$ is subnormal in $G$.

We shall first prove the following lemma.

Lemma 2.2. Let $G$ be a finite group, and $H \leq G$ such that $|G : H| = m$. If $(|G|, \psi(m)) = 1$, then every maximal subgroup of $G$ that contains $H$ is normal in $G$ and has a prime index.

Proof. Let $M$ be a maximal subgroup in $G$, containing $H$. We argue by induction on the order of $G$, assuming that $G$ is non-cyclic. First we want to show that $G$ cannot be non-cyclic simple group. Suppose the opposite. Then, if $m$ is divisible by odd prime, then $G$ is a group of odd order, and is solvable. It follows that $m = 2^k$. If $k = 1$, then $M = H$ is normal in $G$ as a subgroup of index 2. If $k \geq 2$, then the order of $G$ is not divisible by 3. It means that $G$ has to be one of the Suzuki’s simple groups of order $q^2(q^2 + 1)(q - 1)$, where $q = 2^{2n+1}$, $n > 1$. Hence, $|G|$ is divisible by $2^{10}$ and 5. If $k > 3$, then $2^4 - 1$ is not coprime with $|G|$, contradicting our initial assumption. It follows that $k = 3$ or $k = 2$. But then $G$ can be embedded into symmetric group $S_8$, and this is impossible since $|G|$ is divisible by $2^{10}$. This shows that $G$ is not simple.

Let $K$ be a minimal normal subgroup in $G$. If $K \leq M$, then we can apply induction hypothesis on $G/K$, and therefore $M/K$ is a maximal and normal subgroup in $G/K$, of prime index. Therefore, $M$ is also a maximal and normal subgroup in $G$ of prime index. Suppose now that $K$ is not contained in $M$. Then, since $M$ is maximal, we have that $MK = G$. Suppose first that $K$ and $M$ have non-trivial intersection, and let $L = K \cap M$. Then, by the induction hypothesis applied on $K$, if $Q$ is maximal subgroup in $K$ containing $L$, it follows that $Q$ is normal in $K$. By the mentioned Pazderski’s theorem, factor group $K/Q$ is nilpotent. If $F$ is a formation of all finite nilpotent groups, it follows that either $K_F$ is non-trivial, or $K_F = \{1\}$ and $K \in F$. But since $K_F$ is characteristic in $K$, and therefore normal in $G$, the minimality of $K$ implies that $K_F = \{1\}$ and $K \in F$. Again, because of the minimality of $K$, we have that $K$ is a $p$-group, and moreover $K \cong (C_p)^n$ for some prime $p$. Since $\text{Aut}(C_p)^n = \text{Gl}_n(p)$, and $|\text{Gl}_n(p)| = p^{\frac{n(n-1)}{2}}$, our initial assumption yields that $\psi(p^n)$ and $|G|$ are coprime. Therefore, every Sylow $q$-subgroup of $G$, for $p \neq q$, centralizes $K$. Subgroup $K$ is also a normal subgroup of some Sylow $p$-subgroup of $G$. Being normal subgroup of a Sylow $p$-subgroup, $K$ must have non-trivial intersection with it’s center. This implies that $K$ has a non-trivial intersection with the center of $G$. The minimality of $K$ implies that $K \cong C_p \leq Z(G)$, and so $M < G$, which completes the proof. □
Proof of the Theorem 2.1. Let $H = H_0, H_1, \ldots, H_t = G$, be the chain of subgroups in $G$, such that $H_i$ is a maximal subgroup in $H_{i+1}$. Then, by the previous lemma we have that $H_i$ is also normal in $H_{i+1}$, so $H$ is subnormal in $G$. \hfill \Box

One of the basic theorems in the finite group theory states that subgroup of a group $G$ of a prime index $p$ is normal if that prime is the smallest prime divisor of the order of $G$. In [2] it is proved that the same is true under a weaker condition $(|G|, p - 1) = 1$. Let us note that the previous theorem is a generalization. From our previous considerations we can deduce immediately the following corollary

**Corollary 2.3.** Let $H$ be a maximal subgroup of a finite group $G$, such that $|G : H| = m$. If $(|G|, \psi(m)) = 1$, then $H$ is normal in $G$, and $m$ is a prime.

Finally, setting $H = \{1\}$ in Lemma 2.2, we see that our result generalizes the mentioned Pazderski’s theorem. Indeed, finite nilpotent groups are characterized by condition that all its maximal subgroups are normal subgroups.

**References**

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