Witt Groups of Smooth Toric Surfaces

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Abstract

In this paper, we calculate the Witt groups of smooth projective toric surfaces over a field of characteristic different from 2. Such a surface is described combinatorially by a fan in the plan. The result is a direct sum of several copies of the Witt group of the basic field, their number depends on the line bundle used in the definition of Witt groups. The technique of the proofs is to filter the derived category of the surface by subcategories with support in unions of orbits closures whose dimension is 1. Then, by excision, we obtain long exact sequences including copies of Witt groups of the basic field.

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1. Background

Let $X$ be a smooth projective toric surface over a field of characteristic different to 2. We denote $\mathbb{T} = \text{Spec}(k[x^\pm, y^\pm])$ the open orbit of toric action on $X$. The fan associated to $X$ contains $m \geq 3$ 1-dimension faces of the form $Nv_i$ when $i \in \mathbb{Z}/m\mathbb{Z}$ and $v_i$ are primitive. We can assume that the numbering is such that $m$ 2-dimensional faces of the fan are $Nv_i + Nv_{i+1}$. Write $v_i = (s_i, t_i)$. The smoothness implies that $s_i t_{i+1} - s_{i+1} t_i = \pm 1, \forall i$. As the sign is the same for all $i$, then by reversing the order of $v_i$ we can assume that $s_i t_{i+1} - s_{i+1} t_i = 1, \forall i$. For each $i \in \mathbb{Z}/m\mathbb{Z}$, consider the monomial $u_i = x^{s_i} y^{-s_i}$.

Then the 1-dimensional faces of the fan correspond to the affine opens

$U_i = \text{Spec}(k[u_i^\pm, u_{i+1}]) \cong \mathbb{G}_m \times \mathbb{A}^1$. 

This $U_i$ is the union of the open orbit $T$ and an orbit of dimension 1 given by the equation $u_{i+1} = 0$. Call this orbit $C_i^0$ and its adherence $C_i \cong \mathbb{P}^1$.

The 2-dimensional faces of the fan correspond to affine opens

$$V_i = \text{Spec}(k[u^{-1}, u_{i+1}]) \cong \mathbb{A}^2.$$ 

The origin of this $\mathbb{A}^2$ is an orbit of dimension 0, we will call it $P_i$.

The curves $C_i$ and $C_{i+1}$ intersect transversely in $P_i$. We have $C_i \cap C_j = \emptyset$ for $j \notin \{i-1, i, i+1\}$.

From the two equations

$$\begin{vmatrix} s_i & s_{i+1} \\ t_i & t_{i+1} \end{vmatrix} = 1, \quad \begin{vmatrix} s_i & s_{i-1} \\ t_i & t_{i-1} \end{vmatrix} = -1,$$

We can write $v_{i+1} = -v_{i-1} - e_i v_i$ with $e_i \in \mathbb{Z}$.

**Proposition 1.1.**

1. The number of self-intersection of the curve $C_i$ on the surface $X$ is $C_i^2 = e_i$.

2. The canonical class of $X$ is $K_X = - \sum_{i \in \mathbb{Z}/m\mathbb{Z}} C_i$.

3. $\text{Pic}(X)$ is a free abelian group of rank $m - 2$. For all $i \in \mathbb{Z}/m\mathbb{Z}$ the set

$$\{C_j \mid j \in \mathbb{Z}/m\mathbb{Z}\} \setminus \{C_i, C_{i+1}\}$$

is a basis of $\text{Pic}(X)$.

4. $[K_X] = 0$ in $\text{Pic}(X)/2$ if and only if the number of self-intersection $C_i^2 = e_i$ are all even.

**Démonstration.** See ([15] chapter 10) for the proof of the first three assertions. For the last one, we have: $[K_X] = 0$ in $\text{Pic}(X)/2$ if and only if $K_X \cdot C_i$ is even for all $i$ (third assertion). Then the adjonction formula $C_i \cdot (C_i + K_X) = 2g(C_i) - 2 = -2$.

We have also the exact sequence:

$$0 \rightarrow \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} s_1 & t_1 \\ \vdots & \vdots \\ s_m & t_m \end{pmatrix}} \mathbb{Z}^m \xrightarrow{\begin{pmatrix} s_i & t_i \\ s_{i+1} & t_{i+1} \end{pmatrix}} \text{Pic}(X) \rightarrow 0,$$

when $\mathbb{Z}^2 \cong \{x^j y^i / i, j \in \mathbb{Z}\}$ and $\mathbb{Z}^2 \cong \bigoplus_{i=1}^m C_i$, and the matrix is the application $f \mapsto \text{div}(f)$.

As the minors verify:

$$\begin{vmatrix} s_i & s_{i+1} \\ t_i & t_{i+1} \end{vmatrix} = 1,$$
then we can resolve the equation $[K_X] = 0$ for two classes of non-disjoint invariant curves $[C_i]$ and $[C_{i+1}]$ in terms of other $[C_j]$. We obtain:

$$[C_i] = \sum_{j \in \mathbb{Z}/m\mathbb{Z}, j \notin \{i, i+1\}} \frac{s_{i+1}}{s_j} \frac{t_{i+1}}{t_j} [C_j],$$

and

$$[C_{i+1}] = \sum_{j \in \mathbb{Z}/m\mathbb{Z}, j \notin \{i, i+1\}} \frac{s_j}{s_i} \frac{t_j}{t_i} [C_j].$$

\[\square\]

For a smooth projective toric surface $X$ with $[K_X] = 0$ in $\text{Pic}(X)/2$ we can not obtain $C_i^2 = -1$. Then a such surface is minimal (without exceptional $E \cong \mathbb{P}^1$ curve with $E^2 = -1$). Minimal rational surfaces are $\mathbb{P}^2$, see ([20],2.5), with a fan generated by

$$v_1 = (1, 0), \quad v_2 = (0, 1), \quad v_3 = (-1, -1),$$

Let a ruled rational surface (called Hirzebruch) $F_e$ with $e = 0$ or $e \geq 2$, and its fan is generated by

$$v_1 = (1, 0), \quad v_2 = (0, 1), \quad v_3 = (-1, 0), \quad v_4 = (-e, -1).$$

We can calculate $C_i^2$ and prove that:

**Theorem 1.2.** $[K_X] = 0$ in $\text{Pic}(X)/2$ if and only if $X = F_{2d}$ is a Hirzebruch surface with an even index.

**Démonstration.** See [20] \[\square\]

Denote that $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and $F_1$ is isomorphic to $\mathbb{P}^2$ blowing up at a rational point.

### 2. The intersection product

**Proposition 2.1.** Let $X$ be a smooth projective toric surface. Then $\text{Pic}(X)$ is a free finite rank abelian group, and that bilinear symmetric form

$$\text{Pic}(X) \times \text{Pic}(X) \longrightarrow \mathbb{Z}$$

given by the intersection product is unimodular i.e. with discriminant equal $\pm 1$.

**Démonstration.** $\text{Pic}(X)$ can be calculated combinatorially by the fan. It is independent from the basic field $k$. Then we can suppose that $k$ is algebraically close or simply $k = \mathbb{C}$. Hence $X$ is a blowing up of $\mathbb{P}^2$ or of a ruled rational surface at a finite number of points.

$$\mathbb{P}^2 \longrightarrow (1) : \text{unimodular}$$

$$F_e \longrightarrow \left( \begin{array}{cc} 0 & 1 \\ 1 & -e \end{array} \right) : \text{unimodular}$$
Lemma 2.3. Let $\text{det}$ be induced by the intersection product is nondegenerate. 

Corollary 2.2. Let $X$ be a smooth projective toric surface. Then $\text{Pic}(X)/2$ is a $\mathbb{Z}/2$-vector space with finite dimension, and the bilinear symmetric form $\text{Pic}(X) \times \text{Pic}(X) \rightarrow \mathbb{Z}/2$ induced by the intersection product is nondegenerate.

Lemma 2.3. Let $D$ be a divisor such that $D \cdot C_i$ is even for all $i = 1,..,m$. Then $[D] = 0$ in $\text{Pic}(X)/2$.

Démonstration. $[D]$ is orthogonal, with respect to a nondegenerate symmetric bilinear form, to a generator family of $\text{Pic}(X)/2$. So $[D] = 0$.

Lemma 2.4. Let $D$ be a divisor. Then $(D + C_i) \cdot C_i$ is even for all $i = 1,..,m$ if and only if $[D] = K_X$ (the canonical class) in $\text{Pic}(X)/2$.

Démonstration. By the adjonction formula for curves on a surface, we have $(K_X + C_i) \cdot C_i = 2g(C_i) - 2 = -2$ for all $i$.

Conversely, a such $D$ verifies $(D - K_X) \cdot C_i = 0$ for all $i$. Then $D - K_X = 0$ in $\text{Pic}(X)/2$.

3. Calculation of $W^2(X, \mathcal{O}_X(D))$ for X toric

Proposition 3.1. If $D \neq K_X$ in $\text{Pic}(X)/2$, then we have $W^2(X, \mathcal{O}_X(D)) = 0$ and an exact sequence

$$0 \rightarrow W^0(X, \mathcal{O}_X(D)) \overset{i^*}{\rightarrow} W(k) \overset{d}{\rightarrow} W(k)^{m-3} \rightarrow W^1(X, \mathcal{O}_X(D)) \rightarrow 0.$$

Démonstration. Using the lemma 2.4, there exists a $C_i$ such that $(D + C_i) \cdot C_i$ is odd. Filter now the category $D^b(X)$ as :

$$0 \subset D_1 \subset D_2 \subset ... \subset D_m \subset D^b(X)$$

with $D_j = D^b_{C_i, ..., C_{i+j-1}}(X)$.

According to excision, we have :

$$W^n(D_1, \mathcal{O}_X(D)) = W^n_{C_i}(X, \mathcal{O}_X(D))$$

and the direct image gives isomorphisms :

$$W^{n-1}(C_i, \mathcal{O}_{C_i}(C_i + D)) \cong W^n_{C_i}(X, \mathcal{O}_X(D))$$

when $W^{n-1}(C_i, \mathcal{O}_{C_i}(C_i + D)) \cong W^{n-1}((\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(C_i \cdot (C_i + D))))$. As $C_i \cdot (C_i + D)$ is odd, then, for all $n$, we have

$$W^n(D_1, \mathcal{O}_X(D)) = 0$$

(1)
Let $V_j$ the open $X \setminus (C_i \cup \cdots \cup C_{j+i+1})$ of $X$. A restriction to this open gives an equivalence

\[ D^b(X)/\mathbb{D}_j \cong D^b(V_j). \]

Then we deduce that $\mathbb{D}_{j+1}/\mathbb{D}_j \cong D^b_{C_i+j \cap V_j}(V_j)$.

Denote that $C_{i+j} \cap V_j = C_{i+j} \setminus \{P_{i+j-1}\} \cong \mathbb{A}^1$. Then by excision, we have isomorphisms:

\[ W^{n-1}(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}) \cong W^n_{C_{i+j} \cap V_j}(V_j) \cong W^n(\mathbb{D}_{j+1}/\mathbb{D}_j). \]

We find

\[ W^n(\mathbb{D}_{j+1}/\mathbb{D}_j) = \begin{cases} W(k) & \text{if } n \equiv 1 \mod 4 \\ 0 & \text{otherwise.} \end{cases} \tag{2} \]

The open $V_{m-2}$ is isomorphic to $\mathbb{A}^2$. So we have

\[ W^n(D^b/\mathbb{D}_{m-2}) = \begin{cases} W(k) & \text{if } n \equiv 1 \mod 4 \\ 0 & \text{otherwise.} \end{cases} \tag{3} \]

\[ \square \]

**Proposition 3.2.** Let $i : \text{Spec}(k) \to X$ be the inclusion associated to the point $P_i$. Then $i_* : W(k) \to W^2(X, \mathcal{O}_X(K_X))$ is an isomorphism and we have an exact sequence

\[ 0 \to W^0(X, \mathcal{O}_X(D)) \xrightarrow{i_*} W(k) \xrightarrow{\partial} W(k)^{m-2} \to W^1(X, \mathcal{O}_X(D)) \to 0. \]

**Démonstration.** $i_*$ is clearly a split monomorphism, because if $\pi : X \to \text{Spec}(k)$ is the structural morphism, then $\pi_* : W^2(X, \mathcal{O}_X(K_X)) \to W(k)$ verifies $\pi_* \circ i_* = 1_{W(k)}$.

Now, using the same method at 3.1 but with $D = K_X$, then the calculations (2) and (3) still true. But by the lemma 2.4 we have $(K_X + C_i) \cdot C_i = -2$ for all $i$, then we get

\[ W^n(\mathbb{D}_1, \mathcal{O}_X(K_X)) \cong W^n_{C_i}(X, \mathcal{O}_X(K_X)) \cong W^{n-1}(C_i, \mathcal{O}_{C_i}(-2)) \]

\[ \cong \begin{cases} W(k) & \text{if } n \equiv 1, 2 \mod 4 \\ 0 & \text{if } n \equiv 0, 3 \mod 4. \end{cases} \tag{4} \]

Moreover, $P_i \in C_i \subseteq X$ then $i_*$ can be factorized as

\[ W(k) \xrightarrow{i_*} W^2(\mathbb{D}_1, \mathcal{O}_X(K_X)) \xrightarrow{\partial} W^2(X, \mathcal{O}_X(K_X)). \tag{5} \]

Next, applying the localisation exact sequences, we can deduce that each application

\[ W^2(\mathbb{D}_1, \mathcal{O}_X(K_X)) \to W^2(\mathbb{D}_2, \mathcal{O}_X(K_X)) \to \cdots \to W^2(X, \mathcal{O}_X(K_X)) \tag{6} \]

is surjective. So the application $g$ from (5) is a surjection. Then $i_*$ is surjective and so it is an isomorphism. The exact sequence (6) is analogous to the proposition 3.1. \[ \square \]
4. Calculation of $W^0(X, \mathcal{O}_X(D))$ for $X \neq \mathbb{P}^2$ Toric

**Proposition 4.1.** Suppose that $D \neq 0$ in $\text{Pic}(X)/2$. Then $W^0(X, \mathcal{O}_X(D)) = 0$.

Démonstration. If we retake the filtration used in the proof of 3.1, but by eliminating $D_m^{-2}$:

$$0 \subset D_1 \subset \cdots \subset D_{m-3},$$

we obtain the equivalence

$$D^b/\mathbb{D}_{m-3} \simeq D^b(V_{m-3}),$$

where $V_{m-3}$ is the open formed by adding the open orbit, and others which have adherences $C_{i-3}$, $C_{i-2}$, $C_{i-1}$ and the points $P_{i-3}$, $P_{i-2}$. It is the toric variety corresponding to the subfan of the fan $X$, composed by 2 neighbour-faces of dimension 2 which are duals of $P_{i-3}$, $P_{i-2}$, adding to the faces of theirs edges. We can remark that $V_{m-3}$ is the total space of a line bundle on $\mathbb{P}^1$, where $C_{i-2}$ is the zero section, and $C_{i-3} \cap V_{m-3}$ and $C_{i-1} \cap V_{m-3}$ are fibres at 0 and $\infty$. According to homotopic invariance, the restriction to $C_{i-2} \cong \mathbb{P}^1$ give isomorphisms

$$W^n(V_{m-3}, \mathcal{O}_{V_{m-3}}) \simeq W^n(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(D \cdot C_{i-2})).$$

By lemma 2.3, there is $i$ such that $D \cdot C_{i-2}$ is odd.

So we can suppose, for all $n$, that

$$W^n(D^b(X)/\mathbb{D}_{m-3}, \mathcal{O}_X(D)) = 0. \quad (7)$$

The calculations (1), (3) and (2) still true. So, when we add (7), we obtain localisation exact sequence and finally $W^0(X, \mathcal{O}_X(D)) = 0$. □

**Proposition 4.2.** Let $\pi : X \to \text{Spec}(k)$ be the structural morphism. Then $\pi^* : W(k) \to W^0(X, \mathcal{O}_X)$ is an isomorphism.

Démonstration. When we retake the exact sequence from 3.1, the application $i^*$ is, by construction, the composition of the restriction to the open $V_{m-2} \cong \mathbb{A}^2$, $W^0(X, \mathcal{O}_X(D)) \to W^0(\mathbb{A}^2)$ and the homotopic invariance $W^0(\mathbb{A}^2) \cong W(k)$ for all $D$. Then we can identify it to the pullback of the inclusion $i : \text{Spec}(k) \to X$ corresponding to the origin of $\mathbb{A}^2 \cong V_{m-2} \subset X$. Now, using long exact sequences and (1), (3) and (2) which still true for all $D$, $i^* : W^0(X, \mathcal{O}_X) \to W(k)$ is injective. As $i^* \circ \pi^* = 1_{W(k)}$, $i^*$ is also surjective, and finally $\pi^*$ is the inverse isomorphism. □

5. Calculation of $W^n(X, \mathcal{O}_X(D))$

**Proposition 5.1.** Let $D = 0 \neq K_X$ in $\text{Pic}(X)/2$. Then

$$W^n(X, \mathcal{O}_X(D)) \cong \begin{cases} W(k) & \text{if } n \equiv 0 \ [4] \\ W(k)^{m-3} & \text{if } n \equiv 1 \ [4] \\ 0 & \text{if } n \equiv 2, 3 \ [4]. \end{cases} \quad (8)$$
Let Proposition 5.2. Démonstration. It results from propositions 3.1 and 4.2.

Proposition 5.2. Let \( D = 0 = K_X \) in \( \text{Pic}(X)/2 \). Then
\[
W^n(X, \mathcal{O}_X(D)) \cong \begin{cases} 
W(k) & \text{if } n \equiv 0, 2 \pmod{4} \\
W(k)^{m-2} & \text{if } n \equiv 1 \pmod{4} \\
0 & \text{if } n \equiv 3 \pmod{4}.
\end{cases}
\]

Démonstration. It results from propositions 4.1 and 4.2.

Proposition 5.3. Let \( D = K_X \neq 0 \) in \( \text{Pic}(X)/2 \). Then
\[
W^n(X, \mathcal{O}_X(D)) \cong \begin{cases} 
0 & \text{if } n \equiv 0, 3 \pmod{4} \\
W(k)^{m-3} & \text{if } n \equiv 1 \pmod{4} \\
W(k) & \text{if } n \equiv 2 \pmod{4}.
\end{cases}
\]

Démonstration. We use the filtration of the proposition 4.1 and (3), (2) and (7), so the result is deduced immediately from the localisation exact sequences.

Lemma 5.4. If \( D \) is neither 0 nor \( K_X \) in \( \text{Pic}(X)/2 \), so there exist curves \( C_i \) and \( C_k \) which are closures of orbits and disconnected (i.e. \( i \notin \{k-1, k, k+1\} \)), and such that \( D \cdot C_k \) and \( (K_X + D) \cdot C_i \) are odd.

Démonstration. As \( D \neq 0 \), there exists an index \( k \) such that \( D \cdot C_k \) is odd. Suppose that the conclusion of the lemma is false, then \( (K_X + D) \cdot C_j \) must be even for all \( j \) does not belong \( \{k-1, k, k+1\} \). So these \( C_j \) constitute a part of a basis of a \( \text{Pic}(X)/2 \) subspace. Then they are linearly independent and they span a subspace of \( \text{Pic}(X)/2 \) with dimension \( m - 3 \) over \( \mathbb{Z}/2 \). But \( \text{Pic}(X)/2 \) has dimension equals to \( m - 2 \).

Thus, \( K_X + D \) is included in the orthogonal complement of a 1-dimensional subspace. This complement has dimension equals to 1. As \( C_k \) and \( C_j \) are disconnected where \( j \notin \{k-1, k, k+1\} \), so it is included in the orthogonal complement \( \{0, C_X\} \). Then we obtain \( K_X + D = 0 \) or \( K_X + D = C_X \). The first one is unacceptable, by hypothesis \( (D \neq K_X \) in \( \text{Pic}(X)/2 \). The second one is excluded because it implies \( D = C_X + K_X \) in \( \text{Pic}(X)/2 \), but according to ad jonction formula, which gives \( D \cdot C_k \) must be even, all this contradict the supposition that \( D \cdot C_k \) is odd.

Proposition 5.5. Let \( D \neq 0 \) and \( D \neq K_X \) in \( \text{Pic}(X)/2 \). Then
\[
W^n(X, \mathcal{O}_X(D)) \cong \begin{cases} 
W(k)^{m-4} & \text{if } n \equiv 1 \pmod{4} \\
0 & \text{if } n \neq 1 \pmod{4}.
\end{cases}
\]

Démonstration. By this idea, we can filter the derived category by \( \mathbb{D}_1 \) with support \( C_i \). For \( \mathbb{D}_2, \cdots, \mathbb{D}_{m-3} \), we add \( C_{i+1}, C_{i+2}, \cdots, C_{k-2}, C_{i-1}, C_{i-2}, \cdots, C_{k+2} \) progressively and with respect to this order.

Then every localisation exact sequence is trivial, because each quotient of categories have \( W^0(\bullet) = 0 = W^2(\bullet) \) with this twists, and we have also : \( W^0(X, \mathcal{O}_X(D)) = 0 \), \( W^1(X, \mathcal{O}_X(D)) = W(k)^{m-4} \) and \( W^2(X, \mathcal{O}_X(D)) = 0 \). □
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