International Journal of Algebra, Vol. 8, 2014, no. 9, 429 - 432
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ija.2014.4438

On SCS-Duo-Rings

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Abstract.

Let R be a duo ring. An unital R-module M is said to have property (S) if every injective endomorphism of M is an automorphism. The ring R is called SCS-duo-ring if every R-module with property(S) is finitley cogenerated. In this note we show that the following conditions are equivalent: (i) R is an Artinian principal ideal duo ring; (ii) R is a SCS-duo-ring.

Keywords: duo-rings, property (I), property (S), finitly cogenerated, SCS-ring

INTRODUCTION

Let R be an associative ring with unity $1 \neq 0$. A left R-module M is said to have property (I) (resp.property (S)), if every injective (resp. surjective) endomorphism of M is an automorphism of M. It is well known that an Artinian (resp. Noetherian) left module satisfies property (I) (resp. (S)) and the converse is not true. A ring R is called a left I-ring (resp. S-ring) if every left R-module with property (I) (resp. property (S)) is Artinian (resp. Noetherian). If R is a commutative ring (resp. commutative ring whose prime ideals are maximal) every finitely generated module satisfies property (S) ([9]) (resp. (I) ([10])) and the converse is not true. A ring R is called a left FGI-ring (resp. left FGS-ring) if every left R-module with property (I) (resp.property (S) is finitely generated. R is a duo ring if every one sided ideal is two sided. It has been proved that if R is a duo ring, then the classes of S-rings, I-rings, FGI-rings and FGS-rings are exactly the class of Artinian principal ideal rings ([2], [3], [4], and [6]). An R-module M is said to be finitely cogenerated if its socle is essential in M and finitely generated; the ring R is called SCS-ring if property (S)) characterizes finitely cogenerated modules. In this paper, we show that for duo-ring R the following conditions are equivalent: (i) R is an Artinian principal ideal ring; (ii) R is a SCS-ring. All rings are associative with $1 \neq 0$ and all modules are unitary. If R is a ring, we note by $J(R)$ or simply by J the Jacobson radical of R and by $radR$ the prime radical of R. The socle of a R-module M will be noted $soc(M)$. The reader may refer to [1] for any notions not defined in this paper.

1. CONSTRUCTION OF A NON FINITELY COGENERATED MODULE WITH PROPERTY (S) OVER A LOCAL ARTINIAN RING WHOSE MAXIMAL IDEAL IS NOT PRINCIPAL

Let R be a commutative local Artinian ring which is a non principal ideal ring. We may suppose without loss of generalities that the ring R is local Artinian with Jacobson radical $J = aR + bR$ where $a^2 = b^2 = ab = 0$, $a \neq 0$ and $b \neq 0$. Following [6] lemma 5 we may write $R = C \oplus bC$ where C is an Artinian local subring of R with maximal ideal $J(C) = aC \neq 0$. Let M be the total ring of fractions of the polynomial ring $C[X]$, σ the endomorphism of the C-module M defined by $\sigma(m) = aXm$ for $m \in M$; $\varphi : R \longrightarrow End_{\,C}M$ the homomorphism of rings defined by $\varphi(\alpha+\beta b) = \alpha 1_M + \beta\sigma$ for $\alpha+\beta b \in R$ where $\alpha \in C$, $\beta \in C$ and 1_M is the identity endomorphism of M. We consider on M the R-module structure defined by $(\alpha+\beta b)m = \varphi(\alpha+\beta b)(m) = \alpha m + \beta aXm$, for $\alpha + \beta b \in R$, $(\alpha \text{ } and \text{ } \beta \in C)$ and for $m \in M$.
If f is a R-endomorphism of the R-module M, then for all $m \in M$ we have:

$$\sigma.f(m) = bf(m) = f(bm) = f(\sigma(m)).$$

Thus, the R-endomorphisms of the R-module M are the C-endomorphisms of M commuting with σ.
Following [7] proposition 2.3 the R-module M satisfies properties (S).
For $d = \lambda a + \gamma b \in J = Ra + Rb$ and for $am \in aC[X]$ we have:
$d.(am) = (\lambda a + \gamma b)am = \varphi(\lambda a + \gamma b)(am) = (\lambda a 1_M + \gamma\sigma)(am) = \lambda a^2 m + \gamma a^2 Xm = 0$
Therefore, the submodule $aC[X] = \bigoplus_{n \geq 0} aCX^n$ of M is annihilated by J, then $aC[X]$ is semisimple as R-module. Since $aC[X]$ is not finite length as R-module, then M is not finitely cogenerated.

2. CHARACTERIZATION OF SCS-DUO-RINGS

Proposition 2.1. *Let R be an SCS-duo-ring. If R is an integral domain, then R is a division ring.*

Proof. Let K be the division ring of the integral domain R. The R-module ${}_RK$ satisfies property (S). Therefore, ${}_RK$ is finitely cogenerated.Thus,$soc({}_RK) \cap {}_RR \neq \{0\}$. Let $S = Ra$ $(a \in R\backslash\{0\})$ a simple submodule of $soc({}_RK) \cap {}_RR$.

The map:

$$\varphi : {}_RR \longrightarrow S = Ra$$
$$x \mapsto xa$$

is an isomorphism of R-modules. Therefore ${}_RR$ is simple. For every $b \in R\backslash\{0\}$ we have $R = Rb = Rb^2$, then $b = cb^2$ for some $c \in R$. It follows that $1 = cb$ and $1 \in Rb = bR$ which implies that $1 = bd$ for some $d \in R$. Thus b is left and right invertible, so b is invertible. □

Proposition 2.2. *Let R be a SCS-duo-ring. We have the following results:*

1. *Every prime ideal of R is a maximal ideal;*
2. *The Jacobson radical of R is nil;*
3. *The set of the maximal ideals of R is finite;*
4. *R is semiperfect;*
5. *R is a finite direct product of local SCS-duo-rings.*

Proof. (1) results from proposition 2.2.
(2) By (1) the Jacobson radical is equal to the prime radical and consequently it is nil.
(3) Let D be the set of all prime ideals of R. If P and P' are two elements of D such that $P \nsubseteq P'$, then $Hom(R/P, R/P') = \{0\}$ (see [8]) lemma 2). Therefore, the semisimple R-module $M = \bigoplus_{P\in D} R/P$ satisfies property (S), so $soc(M) = M$ is finitely cogenerated. It follows that M is finitely generated and then D is a finite set.
(4) By (3) $R/J \cong \prod_{P\in D} R/P$. This implies that R/J is a semisimple ring and since J is a two sided nilideal of R, then R is semiperfect.
(5) results from (4). □

Proposition 2.3. [1] *proposition 10.10.*
For a module M the following statements are equivalent.
(1) M is Artinan.
(2) Every factor module of M is finitely cogenerated.

Proposition 2.4. *For a duo ring R every R-module generated by one element satisfies property (S).*

Proof. Let f an endomorphism surjective of the R-module $M = Rx$ with $f(x) = ax$. There exists $b \in R$ such that $x = f(bx) = bax$. Let $z \in Rx$, we may write $z = cx$ where $c \in R$. If $0 = f(z) = cax$, then $z = cx = cbax$. Since $cb \in cR = Rc$, there exists $d \in R$ such that $cb = dc$. Then $z = dcax = 0$ and f is injective. □

Proposition 2.5. *Let R be a SCS-duo-ring. Then R is Artinian.*

Proof. Let I an ideal of R. Then the R-module R/I satisfies property (S) by prposition 2.4 . This implies that R/I is finitely cogenerated and by proposition 2.3 R is Artinian. □

Proposition 2.6. *Let R be a SCS-duo-ring. Then R is a finite direct product of local Artinian SCS-ring.*

Proof. It follows from (2.2) and (2.5).

Proposition 2.7. [1] *proposition 10.8*
For a module M the following statements are equivalent.
(1) R is left Artinian.
(2) Every finitely generated left R-module is finitely cogenerated.

Theorem 2.8. *Let R be a duo ring. Then the following conditions are equivalent:*

1. *R is a SCS-ring;*
2. *R is an Artinian principal ideal ring.*

Proof. (1) $\Rightarrow$ (2)
Following (2.6) we may suppose that R is a local Artinian SCS-duo-ring. Then by §1 R is a principal ideal ring.
(2) $\Rightarrow$ (1)
By [7], condition (2) implies that every R-module with property (S) is finitely generated and consequently is finitely cogenerated by proposition 2.7 .Thus R is a SCS-ring. □

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Received: April 25, 2014