Galois Extensions Induced by a Central Idempotent in a Partial Galois Extension

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Abstract

Let \((R, \alpha)\) be a partial Galois extension of \(R^{\alpha G}\) with a partial action of a finite group \(G\), \(e\) a non-zero central idempotent in \(R\), \(1_g\) the central idempotent associated with \(g \in G\), and \(E = e(\Pi_{g \in G} 1_g) \neq 0\) with a maximal number of factors \(1_g\) for \(g \in G\). A sufficient condition for a Galois extension \(Re\) with Galois group \(H(e)\) and for a Galois extension \(RE\) with Galois group \(N(e)\) is given respectively, where \(H(e) = \{g \in G | e1_g = e\}\) and \(N(e) = \{g \in G | e(\Pi_{g \in G} 1_g) \neq 0\}\) with a maximal number of factors \(1_g\) for \(g \in G\). This leads to a structure of \(Re\) as a direct sum of Galois extensions.

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1 Introduction

Galois theory for fields has been generalized for rings in [1, 3, 4, 8]. Recently, a partial action on a ring of a finite group had many applications in operator algebra, ring theory and other areas of research [2, 6, 7, 9, 10]. A lot of properties of a partial Galois extension of a ring with a partial action of a finite group have been given ([6, 9]). Let \((R, \alpha_G)\) be a partial Galois extension with a partial action of a finite group \(G\). Denote the Boolean semi-group generated by \(\{1_g | g \in G\}\) under the multiplication of \(R\) by \(B(R)\), where \(1_g\) is the central idempotent associated with \(g \in G\). In [9], a Galois extension \(Rf\) is characterized for an \(f \in B(R)\). For any non-zero central idempotent \(e \in R\), not necessarily in \(B(R)\), the purpose of the present paper is to give a sufficient condition for three subsets \(G(e), N(e), H(e)\) of \(G\) induced by \(e\) to be subgroups of \(G\) respectively, where \(G(e) = \{g \in G | e1_g \neq 0\}, N(e) = \{g \in G | e(\Pi_{g \in G}1_g) \neq 0\}\) with a maximal number of factors \(1_g\) and \(H(e) = \{g \in G | e1_g = e\}\). Thus we can show that \(Re\) is a Galois extension with Galois groups \(H(e)\) and \(N(e)\), and obtain an expression for \(Re\) as a direct sum of Galois extensions.

2 Preliminary

Let \(R\) be a ring with 1, \(G\) a finite automorphism group of \(R\), and \(R^G = \{r \in R | g(r) = r\}\) for each \(g \in G\). As defined in [4], if there exist \(\{a_i, b_i \in R | \sum_{i=1}^{n} a_i g(b_i) = \delta_{1,g}\}\) for some integer \(n\), then \(R\) is called a Galois extension of \(R^G\) with Galois group \(G\) and \(\{a_i, b_i\}\) is called a \(G\)-Galois system for \(R\). As given in [6], let \(G\) be a finite group, \((R, \alpha_G)\) is called a ring with a partial action \(\alpha_G\) of \(G\) if \(\alpha_g : D_{g^{-1}} \rightarrow D_g\) is a ring isomorphism where \(D_{g^{-1}}\) and \(D_g\) are ideals of \(R\) for all \(g \in G\) such that (1) \(D_1 = R\) and \(\alpha_1\) is the identity automorphism of \(R\); (2) \(\alpha_g(D_{g^{-1}} \cap D_h) = D_{g} \cap D_{gh}\) for all \(g, h \in G\); (3) \(\alpha_g(\alpha_h(r)) = \alpha_{gh}(r)\) for every \(r \in (D_{h^{-1}} \cap D_{(gh)^{-1}})\). Assume that \(D_g = R1_g\) where \(1_g\) is a central idempotent in \(R\) for each \(g \in G\). Denote \(\{r \in R | \alpha_g(r1_{g^{-1}}) = r1_g\}\) for all \(g \in G\) by \(R^{\alpha_G}\). Then \((R, \alpha_G)\) is called a partial Galois extension of \(R^{\alpha_G}\) if there exist \(\{x_i, y_i \in R | i = 1, \ldots, n\}\) for some integer \(n\) such that \(\sum_{i=1}^{n} x_i \alpha_g(y_i 1_{g^{-1}}) = \delta_{1,g}1_R\) for \(g \in G\), where \(\{x_i, y_i\}\) is called a partial Galois system for \(R\). In particular, if \(R^{\alpha_G}\) is contained in the center of \(R\), then \((R, \alpha_G)\) is called a partial Galois algebra. We shall employ the following identity \(\alpha_g(1_h 1_{g^{-1}}) = 1_{gh}1_g\) for all \(g, h \in G\) ([6], page 79).
3 Galois Extensions

In this section, by keeping the definitions and notations in Section 2, let \((R, \alpha_G)\) be a partial Galois extension of \(R^{\alpha_G}\) with a partial action of a finite group \(G\), and \(e\) a non-zero central idempotent in \(R\). There are three subsets of \(G\) associated with \(e: (1) G(e) = \{g \in G | e1_g \neq 0\}\), \(2) N(e) = \{g \in G | e(\Pi_g1_g) \neq 0\}\) with a maximal number of factors \(1_g\), and \(3) H(e) = \{g \in G | e1_g = e\}\). We shall show when these subsets are subgroups of \(G\) so that \(Re\) is a Galois extension with each of these groups as Galois group. The following identity is useful: \(\alpha_g(1_h1_g^{-1}) = 1_{gh}1_g\) for all \(g, h \in G\) ([6], p. 79).

**Theorem 3.3** Let \(e\) be a non-zero central idempotent in \(R\) and \(G(e) = \{g \in G | e1_g \neq 0\}\). If \(e1_g1_h \neq 0\) and \(e\) is in \(R^{\alpha_G(e)}\) for all \(g, h \in G(e)\), then \(G(e)\) is a subgroup of \(G\) and \(Re\) is a partial Galois extension with a partial action of \(G(e)\).

**Proof.** For any \(g \in G\), \(\alpha_g(e1_g^{-1}) = e1_g \neq 0\), so \(e1_g^{-1} \neq 0\). Hence \(g^{-1}\) is in \(G(e)\). Next for any \(g, h \in G(e)\), \(e1_h1_g \neq 0\) by hypothesis, we have \(0 \neq \alpha_g(e1_h1_g^{-1}) = \alpha_g(e1_g^{-1})\alpha_g(1_h1_g^{-1}) = e1_g1_h1_g = e1_{gh}1_g\); and so \(e1_{gh} \neq 0\). Thus \(gh\) is in \(G(e)\). This implies that \(G(e)\) is a subgroup of \(G\). Noting that \(e\) is in \(R^{\alpha_G(e)}\) and \((R, \alpha_G)\) is a partial Galois extension, we conclude that \((Re, \alpha_{G(e)})\) is a partial Galois extension.

Recall that \(N(e) = \{g \in G | e(\Pi_g1_g) \neq 0\}\) with a maximal number of factors \(1_g\}. Since \(G\) is finite, there are finite number of subsets \(\{N(e) : N_1(e), \ldots, N_k(e)\}\) for some integer \(k\). Denote \((\Pi_g1_g)\) for \(g \in N_i(e)\) by \(E_i\) for each \(i = 1, \ldots, k\).

**Proposition 3.2** Let \(G(e), N_i(e) = \{g \in G | e(\Pi_g1_g) \neq 0\}\) with a maximal number of factors \(1_g\}\} for each \(i\) and \(E_i\) be given above. Then \(G(e) = \bigcup_{i=1}^k N_i(e)\) and \(E_iE_j = \delta_{ij}E_i\) for \(i, j = 1, \ldots, k\).

**Proof.** For any \(g \in G(e), e1_g \neq 0\), so \(e1_g \neq 0\) can extend to \(e1_g(\Pi_{h\in G}1_h) \neq 0\) with a maximal number of factors \(1_h\) for \(h \in G\). Hence \(g\) is in \(N_i(e)\) for some \(i = 1, \ldots, k\); and so \(G(e) \subseteq \bigcup_{i=1}^k N_i(e)\). Also clearly \(\bigcup_{i=1}^k N_i(e) \subseteq G(e)\). Thus \(G(e) = \bigcup_{i=1}^k N_i(e)\). Moreover, since \(E_i \neq E_j\) for \(i \neq j\), there exists a factor \(1_g\) of \(E_i\) which is not a factor of \(E_j\). Then we have \(E_iE_j = 0\) by the maximality of the number of factors \(1_h\) of \(E_j\). This implies \(E_iE_j = \delta_{i,j}E_i\).

Next we show a sufficient condition under which \(N_i(e)\) is a subgroup of \(G\) for each \(i\).

**Theorem 3.3** By keeping the notations in Proposition 3.2, if \(E_i\) is in \(R^{N_i(e)}\), then \(N_i(e)\) is a subgroup of \(G\) and \(RE_i\) is a Galois extension of \((RE_i)^{\alpha_{N_i(e)}}\) with Galois group \(N_i(e)\).
Proof. For any \( g \in N_i(e) \), \( \alpha_g(E_i1_{g^{-1}}) = E_i1_g \neq 0 \) because \( E_i \) is in \( R^{\alpha_{N_i(e)}} \) by hypothesis. Hence \( E_i1_{g^{-1}} \neq 0 \). Thus \( g^{-1} \in N_i(e) \). Next for any \( h, g \in N_i(e) \), \( 0 \neq \alpha_g(E_i1_{h^{-1}g^{-1}}) = E_i1_{hg}1_g = E_i1_{hg} \), so \( hg \in N_i(e) \). Therefore \( N_i(e) \) is a subgroup of \( G \). Moreover, noting that \( E_i \) is in \( R^{\alpha_{N_i(e)}} \) and \( E_i1_g = E_i \) for each \( g \in N_i(e) \), we have that \( RE_i \) is a Galois extension of \( (RE_i)^{\alpha_{N_i(e)}} \) with Galois group \( N_i(e) \).

When \( e \) is in \( R^{\alpha_{N_i(e)}} \), the converse of Theorem 3.3 holds.

**Theorem 3.4** Assume \( e \) is in \( R^{\alpha_{N_i(e)}} \). If \( N_i(e) \) is a subgroup of \( G \), then \( E_i \) is in \( R^{\alpha_{N_i(e)}} \).

**Proof.** Since \( E_i = e(\Pi_g1_g) \neq 0 \) with a maximal number of factors \( 1_g \), for any \( h \in N_i(e) \), \( \alpha_h(E_i1_{h^{-1}}) = \alpha_h(e\Pi_g1_g1_{h^{-1}}) = \alpha_h(e1_{h^{-1}})\alpha_h(\Pi_g1_g1_{h^{-1}}) = (e1_h)(\Pi_g1_{hg}1_h) = e(\Pi_g1_g)1_h = E_i1_h \). Thus \( E_i \) is in \( R^{\alpha_{N_i(e)}} \).

Keeping the notations of Theorem 3.3, we have an expression of \( Re \).

**Theorem 3.5** Let \( E_i \) be in \( R^{\alpha_{N_i(e)}} \) as given in Theorem 3.3. If \( e = \sum_{i=1}^{k} E_i \), then \( Re = \oplus \sum_{i=1}^{k} RE_i \) such that \( RE_i \) is a Galois extension with Galois group \( N_i(e) \) for each \( i \).

**Proof.** Since \( E_i(e) = e(\Pi_g1_g) \neq 0 \) with a maximal number of factors \( 1_g \) for \( g \in G \), \( E_iE_j = \delta_{i,j}E_i \) by Proposition 3.2. Since \( e = \sum_{i=1}^{k} E_i \) by hypothesis, \( Re = \oplus \sum_{i=1}^{k} RE_i \) such that \( RE_i \) is a Galois extension with Galois group \( N_i(e) \) for each \( i \) by Theorem 3.3.

Recall that \( H(e) = \{ g \in G | e1_g = e \} \). Next we show a sufficient condition for \( H(e) \) to be a Galois group for the Galois extension \( Re \).

**Theorem 3.6** If \( e \) is in \( R^{\alpha_{H(e)}} \), then \( H(e) \) is a subgroup of \( G \) and \( Re \) is a Galois extension of \( R^{\alpha_{H(e)}} \) with Galois group \( H(e) \).

**Proof.** For any \( h, g \in H(e) \), \( \alpha_g(e1_h1_{g^{-1}}) = \alpha_g(e1_g) = e1_g = e \) by hypothesis. Also, \( \alpha_g(e1_h1_{g^{-1}}) = \alpha_g(e1_{g^{-1}})\alpha_g(1_h1_{g^{-1}}) = (e1_g)(1_{gh}1_g) = e1_{gh} = e \). Thus \( gh \) is in \( H(e) \). But \( H(e) \) is finite, so \( H(e) \) is a subgroup of \( G \). Moreover, since \( e1_g = e \) for each \( g \in H(e) \), the action of \( H(e) \) on \( Re \) is global; that is, \( H(e) \) is an automorphism group of \( Re \). Noting that \( e \) is in \( R^{\alpha_{H(e)}} \), we conclude that \( Re \) is a Galois extension of \( R^{\alpha_{H(e)}} \) with Galois group \( H(e) \).

When \( e \) is a minimal central idempotent in \( R^{\alpha_{G(e)}} \), all subsets of \( G \) induced by \( e \) are the same.
Corollary 3.7 If \( e \) is a minimal central idempotent in \( R^{\alpha G(e)} \), then \( G(e) = H(e) = N_i(e) \) for each \( i \) and \( Re \) is a Galois extension of \( R^{\alpha G(e)} \) with Galois group \( G(e) \).

\textbf{Proof.} Since \( e \) is minimal, \( G(e) = H(e) = N_i(e) \) is clear by the definitions of these subsets of \( G \). Also \( e \) is in \( R^{\alpha G(e)} \), so \( Re \) is a Galois extension of \( R^{\alpha G(e)} \) with Galois group \( G(e) \) by Theorem 3.6.

Corollary 3.8 If \( (R, \alpha_G) \) is a partial Galois algebra with finitely many central idempotents \( \{e_i | i = 1, \ldots, n\} \) for some integer \( n \), then \( R = \oplus \sum_i(Re_i) \) where \( Re_i \) is a Galois extension of \( R^{\alpha G(e_i)} \) for each \( i \).

\textbf{Proof.} Since \( (R, \alpha_G) \) is a partial Galois algebra with finitely many central idempotents \( \{e_i | i = 1, \ldots, n\} \), we can assume that \( \{e_i\} \) are minimal such that \( 1 = \sum_i e_i \) and \( e_i \) is in \( R^{\alpha G(e_i)} \) for each \( i \). Thus the \textit{Corollary} holds by Corollary 3.7.

\textbf{References}


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