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Galois Extensions Induced by a Central Idempotent in a Partial Galois Extension

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Abstract

Let (R, α) be a partial Galois extension of R^{α_G} with a partial action of a finite group G, e a non-zero central idempotent in R, 1_g the central idempotent associated with $g \in G$, and $E = e(\Pi_{g \in G} 1_g) \neq 0$ with a maximal number of factors 1_g for $g \in G$. A sufficient condition for a Galois extension Re with Galois group $H(e)$ and for a Galois extension RE with Galois group $N(e)$ is given respectively, where $H(e) = \{g \in G | e1_g = e\}$ and $N(e) = \{g \in G | e(\Pi_{g \in G} 1_g) \neq 0\}$ with a maximal number of factors 1_g for $g \in G$. This leads to a structure of Re as a direct sum of Galois extensions.

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1 Introduction

Galois theory for fields has been generalized for rings in [1, 3, 4, 8]. Recently, a partial action on a ring of a finite group had many applications in operator algebra, ring theory and other areas of research [2, 6, 7, 9, 10]. A lot of properties of a partial Galois extension of a ring with a partial action of a finite group have been given ([6, 9]). Let (R, α_G) be a partial Galois extension with a partial action of a finite group G. Denote the Boolean semi-group generated by $\{1_g | g \in G\}$ under the multiplication of R by $B(R)$, where 1_g is the central idempotent associated with $g \in G$. In [9], a Galois extension Rf is characterized for an $f \in B(R)$. For any non-zero central idempotent $e \in R$, not necessarily in $B(R)$, the purpose of the present paper is to give a sufficient condition for three subsets $G(e)$, $N(e)$, and $H(e)$ of G induced by e to be subgroups of G respectively, where $G(e) = \{g \in G | e1_g \neq 0\}$, $N(e) = \{g \in G | e(\Pi_{g \in G} 1_g) \neq 0\}$ with a maximal number of factors 1_g and $H(e) = \{g \in G | e1_g = e\}$. Thus we can show that Re is a Galois extension with Galois groups $H(e)$ and $N(e)$, and obtain an expression for Re as a direct sum of Galois extensions.

2 Preliminary

Let R be a ring with 1, G a finite automorphism group of R, and $R^G = \{r \in R | g(r) = r\}$ for each $g \in G$. As defined in [4], if there exist $\{a_i,\ b_i \in R | \sum_{i=1}^{n} a_i g(b_i) = \delta_{1,g}$ for some integer $n\}$, then R is called a Galois extension of R^G with Galois group G and $\{a_i,\ b_i\}$ is called a G-Galois system for R. As given in [6], let G be a finite group, (R, α_G) is called a ring with a partial action α_G of G if $\alpha_g : D_{g^{-1}} \to D_g$ is a ring isomorphism where $D_{g^{-1}}$ and D_g are ideals of R for all $g \in G$ such that (1) $D_1 = R$ and α_1 is the identity automorphism of R; (2) $\alpha_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh}$ for all $g, h \in G$; (3) $(\alpha_g \alpha_h)(r) = \alpha_{gh}(r)$ for every $r \in (D_{h^{-1}} \cap D_{(gh)^{-1}})$. Assume that $D_g = R1_g$ where 1_g is a central idempotent in R for each $g \in G$. Denote $\{r \in R | \alpha_g(r1_{g^{-1}}) = r1_g$ for all $g \in G\}$ by R^{α_G}. Then (R, α_G) is called a partial Galois extension of R^{α_G} if there exist $\{x_i,\ y_i \in R | i = 1, \cdots, n\}$ for some integer n such that $\sum_{i=1}^{n} x_i \alpha_g(y_i 1_{g^{-1}}) = \delta_{1,g} 1_R$ for $g \in G$, where $\{x_i,\ y_i\}$ is called a partial Galois system for R. In particular, if R^{α_G} is contained in the center of R, then (R, α_G) is called a partial Galois algebra. We shall employ the following identity $\alpha_g(1_h 1_{g^{-1}}) = 1_{gh} 1_g$ for all $g, h \in G$ ([6], page 79).

3 Galois Extensions

In this section, by keeping the definitions and notations in Section 2, let (R, α_G) be a partial Galois extension of R^{α_G} with a partial action of a finite group G, and e a non-zero central idempotent in R. There are three subsets of G associated with $e:$ (1) $G(e) = \{g \in G | e1_g \neq 0\}$, (2) $N(e) = \{g \in G | e(\Pi_g 1_g) \neq 0\}$ with a maximal number of factors 1_g, and (3) $H(e) = \{g \in G | e1_g = e\}$. We shall show when these subsets are subgroups of G so that Re is a Galois extension with each of these groups as Galois group. The following identity is useful: $\alpha_g(1_h 1_{g^{-1}}) = 1_{gh}1_g$ for all $g, h \in G$ ([6], p. 79).

Theorem 3.1 *Let e be a non-zero central idempotent in R and $G(e) = \{g \in G | e1_g \neq 0\}$. If $e1_g 1_h \neq 0$ and e is in $R^{\alpha_{G(e)}}$ for all $g, h \in G(e)$, then $G(e)$ is a subgroup of G and Re is a partial Galois extension with a partial action of $G(e)$.*

Proof. For any $g \in G$, $\alpha_g(e1_{g^{-1}}) = e1_g \neq 0$, so $e1_{g^{-1}} \neq 0$. Hence g^{-1} is in $G(e)$. Next for any $g, h \in G(e)$, $e1_h 1_g \neq 0$ by hypothesis, we have $0 \neq \alpha_g(e1_h 1_{g^{-1}}) = \alpha_g(e1_{g^{-1}})\alpha_g(1_h 1_{g^{-1}}) = e1_g 1_{gh} 1_g = e1_{gh}1_g$; and so $e1_{gh} \neq 0$. Thus gh is in $G(e)$. This implies that $G(e)$ is a subgroup of G. Noting that e is in $R^{\alpha_{G(e)}}$ and (R, α_G) is a partial Galois extension, we conclude that $(Re, \alpha_{G(e)})$ is a partial Galois extension.

Recall that $N(e) = \{g \in G | e(\Pi_g 1_g) \neq 0$ with a maximal number of factors $1_g\}$. Since G is finite, there are finite number of subsets $\{N(e) : N_1(e), \cdots, N_k(e)$ for some integer $k\}$. Denote $(\Pi_g 1_g)$ for $g \in N_i(e)$ by E_i for each $i = 1, \cdots, k$.

Proposition 3.2 *Let $G(e), N_i(e) = \{g \in G | e(\Pi_g 1_g) \neq 0$ with a maximal number of factors $1_g\}$ for each i and E_i be given above. Then $G(e) = \cup_{i=1}^k N_i(e)$ and $E_i E_j = \delta_{i,j} E_i$ for $i, j = 1, \cdots, k$.*

Proof. For any $g \in G(e), e1_g \neq 0$, so $e1_g \neq 0$ can extend to $e1_g(\Pi_{h \in G} 1_h) \neq 0$ with a maximal number of factors 1_h for $h \in G$. Hence g is in $N_i(e)$ for some $i = 1, \cdots, k$; and so $G(e) \subset \cup_{i=1}^k N_i(e)$. Also clearly $\cup_{i=1}^k N_i(e) \subset G(e)$. Thus $G(e) = \cup_{i=1}^k N_i(e)$. Moreover, since $E_i \neq E_j$ for $i \neq j$, there exists a factor 1_g of E_i which is not a factor of E_j. Then we have $E_i E_j = 0$ by the maximality of the number of factors 1_h of E_j. This implies $E_i E_j = \delta_{i,j} E_i$.

Next we show a sufficient condition under which $N_i(e)$ is a subgroup of G for each i.

Theorem 3.3 *By keeping the notations in Proposition 3.2, if E_i is in $R^{\alpha_{N_i(e)}}$, then $N_i(e)$ is a subgroup of G and RE_i is a Galois extension of $(RE_i)^{\alpha_{N_i(e)}}$ with Galois group $N_i(e)$.*

Proof. For any $g \in N_i(e), \alpha_g(E_i 1_{g^{-1}}) = E_i 1_g \neq 0$ because E_i is in $R^{\alpha_{N_i(e)}}$ by hypothesis. Hence $E_i 1_{g^{-1}} \neq 0$. Thus $g^{-1} \in N_i(e)$. Next for any $h, g \in N_i(e)$, $0 \neq \alpha_g(E_i 1_h 1_{g^{-1}}) = E_i 1_{hg} 1_g = E_i 1_{hg}$, so $hg \in N_i(e)$. Therefore $N_i(e)$ is a subgroup of G. Moreover, noting that E_i is in $R^{\alpha_{N_i(e)}}$ and $E_i 1_g = E_i$ for each $g \in N_i(e)$, we have that RE_i is a Galois extension of $(RE_i)^{\alpha_{N_i(e)}}$ with Galois group $N_i(e)$.

When e is in $R^{\alpha}_{N_i(e)}$, the converse of $Theorem$ 3.3 holds.

Theorem 3.4 *Assume e is in $R^{\alpha_{N_i(e)}}$. If $N_i(e)$ is a subgroup of G, then E_i is in $R^{\alpha_{N_i(e)}}$.*

Proof. Since $E_i = e(\Pi_g 1_g) \neq 0$ with a maximal number of factors 1_g, for any $h \in N_i(e)$, $\alpha_h(E_i 1_{h^{-1}}) = \alpha_h(e \Pi_g 1_g 1_{h^{-1}}) = \alpha_h(e 1_{h^{-1}}) \alpha_h((\Pi_g 1_g) 1_{h^{-1}}) = (e 1_h)(\Pi_g 1_{hg} 1_h) = e(\Pi_g 1_g) 1_h = E_i 1_h$. Thus E_i is in $R^{\alpha_{N_i(e)}}$.

Keeping the notations of $Theorem$ 3.3, we have an expression of *Re*.

Theorem 3.5 *Let E_i be in $R^{\alpha_{N_i(e)}}$ as given in $Theorem$ 3.3. If $e = \sum_{i=1}^{k} E_i$, then $Re = \oplus \sum_{i=1}^{k} RE_i$ such that RE_i is a Galois extension with Galois group $N_i(e)$ for each i.*

Proof. Since $E_i(e) = e(\Pi_g 1_g) \neq 0$ with a maximal number of factors 1_g for $g \in G$, $E_i E_j = \delta_{i,j} E_i$ by $Proposition$ 3.2. Since $e = \sum_{i=1}^{k} E_i$ by hypothesis, $Re = \oplus \sum_{i=1}^{k} RE_i$ such that RE_i is a Galois extension with Galois group $N_i(e)$ for each i by $Theorem$ 3.3.

Recall that $H(e) = \{g \in G | e1_g = e\}$. Next we show a sufficient condition for $H(e)$ to be a Galois group for the Galois extension *Re*.

Theorem 3.6 *If e is in $R^{\alpha_{H(e)}}$, then $H(e)$ is a subgroup of G and Re is a Galois extension of $R^{\alpha_{H(e)}}$ with Galois group $H(e)$.*

Proof. For any $h, g \in H(e)$, $\alpha_g(e 1_h 1_{g^{-1}}) = \alpha_g(e 1_{g^{-1}}) = e 1_g = e$ by hypothesis. Also, $\alpha_g(e 1_h 1_{g^{-1}}) = \alpha_g(e 1_{g^{-1}}) \alpha_g(1_h 1_{g^{-1}}) = (e 1_g)(1_{gh} 1_g) = e 1_{gh}$, so $e 1_{gh} = e$. Thus gh is in $H(e)$. But $H(e)$ is finite, so $H(e)$ is a subgroup of G. Moreover, since $e 1_g = e$ for each $g \in H(e)$, the action of $H(e)$ on *Re* is global; that is, $H(e)$ is an automorphism group of *Re*. Noting that e is in $R^{\alpha_{H(e)}}$, we conclude that *Re* is a Galois extension of $R^{\alpha_{H(e)}}$ with Galois group $H(e)$.

When e is a minimal central idempotent in $R^{\alpha_{G(e)}}$, all subsets of G induced by e are the same.

Corollary 3.7 *If e is a minimal central idempotent in $R^{\alpha_{G(e)}}$, then $G(e) = H(e) = N_i(e)$ for each i and Re is a Galois extension of $R^{\alpha_{G(e)}}$ with Galois group $G(e)$.*

Proof. Since e is minimal, $G(e) = H(e) = N_i(e)$ is clear by the definitions of these subsets of G. Also e is in $R^{\alpha_{G(e)}}$, so Re is a Galois extension of $R^{\alpha_{G(e)}}$ with Galois group $G(e)$ by *Theorem* 3.6.

Corollary 3.8 *If (R, α_G) is a partial Galois algebra with finitely many central idempotents $\{e_i | i = 1, \cdots, n\}$ for some integer n, then $R = \oplus \sum_i (Re_i)$ where Re_i is a Galois extension of $R^{\alpha_{G(e_i)}}$ for each i.*

Proof. Since (R, α_G) is a partial Galois algebra with finitely many central idempotents $\{e_i | i = 1, \cdots, n\}$, we can assume that $\{e_i\}$ are minimal such that $1 = \sum_i e_i$ and e_i is in $R^{\alpha_{G(e_i)}}$ for each i. Thus the *Corollary* holds by *Corollary* 3.7.

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