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Rank and Subdegrees of the Symmetric Group, S_n, $n \leq 7$ Acting on Ordered Triples

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Abstract

The main aim of this paper is to determine the ranks and subdegrees of the symmetric group S_n acting on $X^{(3)}$, the set of all ordered triples from the set $X = \{1, 2, \cdots, 7\}$. We show that S_n acts transitively on $X^{(3)}$. We also compute the ranks and subdegrees corresponding to these actions. Finally we show that the rank of S_n, $n \geq 6$ acting on $X^{(3)}$ is 34.

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1. INTRODUCTION

In 1970, Higman[2] calculated the rank and the subdegrees of the symmetric group S_n acting on 2-element subsets from the set $X = \{1, 2, \cdots, n\}$. He showed that the rank is 3 and the subdegrees are 1,2(n-2), $\binom{n-2}{2}$. In 1990, Faradzev and Ivanov[1] calculated the subdegrees of primitive permutation representations of $PSL(2, q)$. They showed that if $PSL(2, q)$ acts on the cosets of its maximal subgroup H, then the rank is at least $|G|/|H|^2$ and if $q > 100$, the rank is more than 5. Kamuti[3] computed the subdegrees of primitive permutation representations of $PGL(2, q)$. He showed that when $PGL(2, q)$ acts on the cosets of its maximal dihedral subgroup of order $2(q-1)$, then its rank is $\frac{1}{2}(q+3)$ if q is odd, and $\frac{1}{2}(q+2)$ if q is even. In this paper we investigate the action of the symmetric group S_n, $n \leq 7$ on $X^{(3)}$.

2. PRELIMINARIES

Notation 2.1.

- S_n - *Symmetric group of degree* n *and order* $n!$.
- $|G|$ - *the order of a group* G.
- $[a, b, c]$ - *an ordered triple.*
- $Stab_G(x)$ *or* G_x - *the stabilizer of* x *in* G.
- $H \leq G$ - H *is a subgroup of* G.
- $[G : H]$ - *the index of* H *in* G.
- $|Fix(g)|$ - *the number of elements in the fixed point set of* g.
- $\binom{\alpha}{\beta}$ - α *combination* β.

Definition 2.2. Let X be the set $X = \{1, 2, \cdots, n\}$ then the symmetric group of degree n is the group of all permutations of X under the binary operation of composition of maps. It is denoted by S_n and has order $n!$.

Definition 2.3. If a finite group G acts on a set X with n elements, each $g \in G$ corresponds to a permutation σ of X which can be written uniquely as a product of disjoint cycles. If σ has α_1 cycles of length 1, α_2 cycles of length 2,..., α_n cycles of length n, we say that σ and hence g has cycle type $(\alpha_1, \cdots, \alpha_n)$.

Theorem 2.4. *(Krishnamurthy[4]) Two permutations in* S_n *are conjugate if and only if they have the same cycle type and if* $g \in S_n$ *has cycle type* $(\alpha_1, \alpha_2, \cdots, \alpha_n)$, *then the number of permutations in* S_n *conjugate to* g *is* $\frac{n!}{\prod_{i=1}^{n} \alpha_i i^{\alpha_i}}$.

Theorem 2.5 (Orbit-Stabilizer Theorem Rose[6])**.** *Let* G *be a group acting on a finite set* X *and* $x \in X$. *Then* $|Orb_G(x)| = [G : Stab_G(x)]$.

Definition 2.6 (Cauchy-Frobenius Lemma-Rotman[7])**.** Let G be a group acting on a finite set X. Then the number of G Orbits in X is $\frac{1}{|G|}\Sigma_{g \in G}|Fix(g)|$.

Lemma 2.7. *Let the cycle type of* $g \in S_n$ *be* $(\alpha_1, \alpha_2, \cdots, \alpha_n)$. *Then the number of elements in* $X^{(3)}$ *fixed by* g *is* $|Fix(g)| = 3!\binom{\alpha_1}{3}$.

Lemma 2.8. *Let $g \in S_n$ have cycle type $(\alpha_1, \alpha_2, \cdots, \alpha_n)$. Then the number of permutations in S_n fixing $[a, b, c]$ and having the same cycle type as g is given by* $\frac{(n-3)!}{1^{\alpha_1-3}(\alpha_1-3)!\prod_{i=2}^{n}\alpha_i!i^{\alpha_i}}$

3. RANKS AND SUBDEGREES OF THE SYMMETRIC GROUP S_n, $n \leq 7$ ACTING ON ORDERED TRIPLES

Theorem 3.1. *The rank of S_3 on $X^{(3)}$ is* 6.

Proof. Let the cycle type of $g \in G$ be $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $[1, 2, 3] \in X^{(3)}$. Then the number of permutations in $G_{[1,2,3]}$ fixing $[1, 2, 3]$ and having the same cycle type as g is given by Lemma 2.8. The number of elements in $X^{(3)}$ fixed by each $g \in G$ is given by Lemma 2.7. We have a permutation $g \in G_{[1,2,3]}$ and $|Fix(g)|$ in $X^{(3)}$ is six and the number of permutations $|G_{[1,2,3]|}|$ is 1. By Cauchy-Frobenius Lemma, we have $|Orb_{G_{[1,2,3]}}[1, 2, 3]| = \frac{1}{|G_{[1,2,3]}|}\Sigma_{g\in G_{[1,2,3]}}|Fix(g)| = 6$, thus the rank of S_3 on $X^{(3)}$ is 6 and there are six suborbits of length 1. □

Theorem 3.2. *The rank of S_4 on $X^{(3)}$ is* 24.

Proof. Let $g \in G$ have cycle type $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $[1, 2, 3] \in X^{(3)}$. Then the number of permutations in $G_{[1,2,3]}$ fixing $[1, 2, 3]$ and having the same cycle type as g is given by Lemma 2.8 and the number of elements in $X^{(3)}$ fixed by each $g \in G$ is given by Lemma 2.7. We have one permutation $g \in G_{[1,2,3]}$ and $|Fix(g)|$ in $X^{(3)}$ is 24 and the number of permutations $|G_{[1,2,3]}|$ is 1. Thus on invoking the Cauchy- Frobenius Lemma gives,
$|Orb_{G_{[1,2,3]}}[1, 2, 3]| = \frac{1}{|G_{[1,2,3]}|}\Sigma_{g\in G_{[1,2,3]}}|Fix(g)| = 24$, thus the rank of S_4 on $X^{(3)}$ is 24 and there are 24 suborbits of length 1. □

Theorem 3.3. *The rank of S_5 on $X^{(3)}$ is* 33.

Proof. Let $g \in G$ have cycle type $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $[1, 2, 3] \in X^{(3)}$. Then the number of permutations in $G_{[1,2,3]}$ fixing $[1, 2, 3]$ and having the same cycle type as g is given by Lemma 2.8 and the number of elements in $X^{(3)}$ fixed by each $g \in G$ is given by Lemma 2.7. From table 1, we see that $|G_{[1,2,3]}| = 2$ and by the Cauchy-Frobenius Lemma, we have $|Orb_{G_{[1,2,3]}}[1, 2, 3]| = \frac{1}{|G_{[1,2,3]}|}\Sigma_{g\in G_{[1,2,3]}}|Fix(g)| = 33$, thus the rank of S_5 on $X^{(3)}$ is 33 and there are 6 suborbits of length 1 and 27 of length 2. □

Theorem 3.4. *The rank of S_6 on $X^{(3)}$ is* 34.

Proof. Let $g \in G$ have cycle type $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $[1, 2, 3] \in X^{(3)}$. Then the number of permutations in $G_{[1,2,3]}$ fixing $[1, 2, 3]$ and having the same cycle type as g is given by Lemma 2.8 and the number of elements in $X^{(3)}$ fixed by each $g \in G$ is given by Lemma 2.7. From table 2, we see that $|G_{[1,2,3]}| = 6$ and by the Cauchy-Frobenius Lemma, we have $|Orb_{G_{[1,2,3]}}[1, 2, 3]| = \frac{1}{|G_{[1,2,3]}|}\Sigma_{g\in G_{[1,2,3]}}|Fix(g)| = \frac{204}{6} = 34$, thus the rank of S_6 on $X^{(3)}$ is 34 and the subdegrees of S_6 are; 6 suborbits of length 1, 18 of length 3 and 10 of length 6. □

Theorem 3.5. *The rank of S_7 on $X^{(3)}$ is 34.*

Proof. Let the cycle type of $g \in G$ be $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $[1,2,3] \in X^{(3)}$. Then the number of permutations in $G_{[1,2,3]}$ fixing $[1,2,3]$ and having the same cycle type as g is given by Lemma 2.8 and the number of elements in $X^{(3)}$ fixed by each $g \in G$ is given by Lemma 2.7. From table 3, we see that $|G_{[1,2,3]}| = 24$ and by the Cauchy-Frobenius Lemma, we have
$|Orb_{G_{[1,2,3]}}[1,2,3]| = \frac{1}{|G_{[1,2,3]}|}\Sigma_{g \in G_{[1,2,3]}}|Fix(g)| = \frac{816}{24} = 34$, thus the rank of S_7 on $X^{(3)}$ is 34 and the subdegrees of S_7 are; 6 suborbits of length 1, 18 of length 4, 12 of length 9 and 1 of length 24. □

4. LIST OF TABLES

TABLE 1

Permutation $g \in G_{\{1,2,3\}}$	Number of permutations	$\|Fix(g)\|$ in $X^{(3)}$
1	1	60
(45)	1	6

TABLE 2

Permutation $g \in G_{\{1,2,3\}}$	Number of permutations	$\|Fix(g)\|$ in $X^{(3)}$
1	1	120
(1)(2)(3)(de)(f)	3	24
(1)(2)(3)(def)	2	6

TABLE 3

Permutation $g \in G_{\{1,2,3\}}$	Number of permutations	$\|Fix(g)\|$ in $X^{(3)}$
1	1	210
(1)(2)(3)(de)(f)(g)	6	60
(1)(2)(3)(def)(g)	8	24
(1)(2)(3)(defg)	6	6
(1)(2)(3)(de)(fg)	3	6

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