Dualities in Koszul Graded Artin-Schelter Gorenstein Algebras

R. Martínez-Villa

Centro de Ciencias Matemáticas UNAM, Morelia, Mexico
http://www.matmor.unam.mx

Abstract

The paper is dedicated to the study of certain non commutative graded Artin-Schelter Gorenstein algebras Λ.

The main result of the paper is that for Koszul algebras Λ with Yoneda algebra Γ, such that both Λ and Γ are graded Artin-Schelter Gorenstein noetherian of finite local cohomology dimension on both sides, there are dualities of triangulated categories:

\[ \text{gr}_\Lambda[\Omega^{-1}] \cong \text{D}^b(Q_{\text{gr} \Gamma}) \text{ and } \text{gr}_\Gamma[\Omega^{-1}] \cong \text{D}^b(Q_{\text{gr} \Lambda}) \]

where, \(Q_{\text{gr} \Gamma}\) is the category of tails, this is: the category of finitely generated graded modules \(\text{gr} \Gamma\) divided by the modules of finite length, and \(\text{D}^b(Q_{\text{gr} \Gamma})\) the corresponding derived category and \(\text{gr}_\Lambda[\Omega^{-1}]\) the stabilization of the category of finetely generated graded \(\Lambda\)-modules, module the finetely generated projective modules.

Mathematics Subject Classification: Primary 16E65, Secondary 16E40

Keywords: As Gorenstein, local cohomology

1 Introduction

The paper is dedicated to the study of certain non commutative graded Artin-Schelter Gorenstein algebras (AS Gorenstein, for short) Λ, [11],[14], [15], those which are noetherian of finite local cohomology dimension on both sides, and
Koszul. We proved in [14] that the Yoneda algebra $\Gamma$ of a Koszul graded AS Gorenstein algebra is again graded AS Gorenstein. We will assume in addition $\Lambda$ and $\Gamma$ are both noetherian and of finite local cohomology dimension on both sides.

For such algebras we can generalize the classical Bernstein-Gelfand-Gelfand [4] theorem, which says that there is an equivalence of triangulated categories: $\text{gr}_{\Lambda} \cong D^b(\text{CohP}_n)$, where $\text{gr}_{\Lambda}$ is the stable category of the finitely generated $\Lambda$-modules over the exterior algebra in $n$ variables and $D^b(\text{CohP}_n)$ is the derived category of bounded complexes of coherent sheaves on $n$-dimensional projective space.

This theorem was generalized in [16] and [17] as follows:

Let $\Lambda$ be a finite dimensional Koszul algebra with noetherian Yoneda algebra $\Gamma$. Then there is a duality of triangulated categories: $\text{gr}_{\Lambda}[\Omega^{-1}] \cong D^b(\text{Qgr}_{\Gamma})$, where $\text{gr}_{\Lambda}[\Omega^{-1}]$, is the stabilization of $\text{gr}_{\Lambda}$ (in the sense of [2],[3]) and $\text{Qgr}_{\Gamma}$ is the category of tails, this is: the category of finitely generated graded modules $\text{gr}_{\Gamma}$ divided by the modules of finite length, and $D^b(\text{Qgr}_{\Gamma})$ the corresponding derived category.

The main result of the paper is that for Koszul algebras $\Lambda$ with Yoneda algebra $\Gamma$, such that both $\Lambda$ and $\Gamma$ are graded AS Gorenstein noetherian of finite local cohomology dimension on both sides, there are dualities of triangulated categories:

$$\text{gr}_{\Lambda}[\Omega^{-1}] \cong D^b(\text{Qgr}_{\Gamma}) \quad \text{and} \quad \text{gr}_{\Gamma}[\Omega^{-1}] \cong D^b(\text{Qgr}_{\Lambda}).$$

2 Castelnuovo-Mumford Regularity

This section is dedicated to review the concepts and results developed by P. Jørgensen in [9],[10] and to check they apply to the algebras considered in the paper, for completeness we reproduce his proofs here. The main result is the following:

**Theorem 2.1.** Let $\Lambda$ be a noetherian Koszul AS Gorenstein algebra of finite local cohomology dimension. Then for any finitely generated graded module $M$ there is a truncation $M_{\geq k}$ such that $M_{\geq k}[k]$ is Koszul.

To prove it we use the line of arguments given in [9] and [10] for connected graded algebras, checking that they easily extend to positively graded locally finite algebras $A$ over a field $k$. This is $A = \bigoplus_{i \geq 0} A_i$, where $A_0 = k \times k \times \ldots \times k$ and for each $i \geq 0 \dim_k A_i < \infty$.

We assume the reader is familiar with basic results on triangulated and derived categories, in particular with the derived functors of Hom and $\otimes$. For details we refer the reader to [8],[20],[21],[26],[27].
We use the following notation: Given a complex $Y$ of graded left $\Lambda$-modules we will denote by $Y'$ the dual complex $Y' = \text{Hom}_\mathbb{k}(Y, \mathbb{k})$.

Given graded $\Lambda$-modules $Y$, $Z$, the degree zero maps will be denoted by $\text{Hom}_{\text{Gr}}(Y, Z)$, $Z[i] = Z[i + j]$ is the shift and $\text{Hom}_\Lambda(Y, Z) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\text{Gr}}(Y, Z[i])$.

**Proposition 2.2.** Let $A$ be a positively graded $\mathbb{k}$-algebra, $A^\text{op}$ the opposite algebra and $X$, $Y$ complexes, $X \in D^+(\text{Gr}_{A^\text{op}})$ and $Y \in D^-(\text{Gr}_A)$. Then $(X \otimes_A Y') = R\text{Hom}(Y, X')$.

**Proof.** Let $F \to Y$ be a quasi-isomorphism from a complex of free modules $F$. Then $X \otimes_A Y \cong X \otimes_A F$ and $(X \otimes_A F)^q = \bigoplus_{p+q=n} X^p \otimes F^q$, where $F^q = \oplus A^q$, hence, $(X \otimes_A F)^n = \bigoplus_{p+q=n} X^p \otimes \oplus A^q = \bigoplus_{p+q=n} X^p$.

Therefore: $\text{Hom}_\mathbb{k}(X \otimes_A F^n, \mathbb{k}) = \text{Hom}_\mathbb{k}(\bigoplus_{p+q=n} X^p, \mathbb{k}) = \prod \text{Hom}_\mathbb{k}(X^p, \mathbb{k})$.

In the other hand, $R\text{Hom}_A(Y, X')^{-n} = \text{Hom}^\circ(F, X')^{-n} = \prod \text{Hom}_A(F^q, (X')^{q-n}) = \prod \text{Hom}_A(\oplus A, (X')^{q-n}) = \prod \text{Hom}((X')^{q-n}, (X \otimes_A F)^{-n})$.

Let’s recall the definition of local cohomology dimension.

**Definition 2.3.** Let $A = \bigoplus_{i \geq 0} A_i$, be a positively graded $\mathbb{k}$-algebra with graded Jacobson radical $\mathfrak{m} = \bigoplus_{i \geq 1} A_i$, define a left exact endo functor $\Gamma_m: \text{Gr}^+_A \to \text{Gr}^+_A$ in the category of bounded above graded $\Lambda$-modules $\text{Gr}_A$, by

$$\Gamma_m(M) = \lim_{\text{proj}} \text{Hom}_A(A/A_{\geq k}, M).$$

Denote by $\Gamma_m^n(-)$, the $n$-th derived functor. It is clear that $\Gamma_m^n(M) = \lim_k \text{Ext}^n_{A}(A/A_{\geq k}, M)$. We say that $A$ has finite local cohomology dimension, if there exist a non negative integer $d$ such that for all $M \in \text{Gr}_A^+$ and $n \geq d$, $\Gamma_m^n(M) = 0$.

We refer to [6] IX Corollary 2.4 a for the proof of the following:

**Lemma 2.4.** Let $A$ be a $\mathbb{k}$-algebra and $I$ an injective $A$- $A$ bimodule. The $I$ is injective both as left and as a right $A$-module.

In order to prove next proposition we need the following:

**Lemma 2.5.** Let $A$ be a positively graded left noetherian $\mathbb{k}$-algebra of finite local cohomology dimension on the left, and $\{Z_i\}_{i \in K}$ a family of $\Gamma_m$-acyclic graded modules. Then $\bigoplus_{i \in K} Z_i$ is $\Gamma_m$-acyclic.
Proof. Let \( \{Z_i\}_{i \in K} \) be a family of \( \Gamma_m \)-acyclic graded modules, this is: each \( Z_i \) has an injective resolution:

\[
0 \to Z_i \to I_0^{I_i} \to I_1^{I_i} \to I_2^{I_i} \to \ldots I_k^{I_i} \to I_{k+1}^{I_i} \to \ldots \text{ such that } 0 \to \Gamma_m(I_0^{I_i}) \to \\
\Gamma_m(I_1^{I_i}) \to \ldots \Gamma_m(I_k^{I_i}) \to \Gamma_m(I_{k+1}^{I_i}) \to \ldots \text{ has homology zero except at degree zero.}
\]

Since \( A \) is noetherian the exact sequence:

\[
0 \to \bigoplus_{i \in K} (Z_i) \to \bigoplus_{i \in K} (I_0^{I_i}) \to \bigoplus_{i \in K} (I_1^{I_i}) \to \ldots \bigoplus_{i \in K} (I_k^{I_i}) \to \bigoplus_{i \in K} (I_{k+1}^{I_i}) \to \ldots
\]

is an injective resolution of \( \bigoplus_{i \in K} (Z_i) \), and \( \Gamma_m\left( \bigoplus_{i \in K} (I_k^{I_i}) \right) = \lim_{s} \text{Hom}_A(A/A_{\geq s}, \bigoplus_{i \in K} (I_k^{I_i})) \), and \( A/A_{\geq s} \) finitely presented implies \( \lim_{s} \text{Hom}_A(A/A_{\geq s}, \bigoplus_{i \in K} (I_k^{I_i})) = \)

\[
\lim_{s} \bigoplus_{i \in K} \text{Hom}_A(A/A_{\geq s}, (I_k^{I_i})) = \bigoplus_{i \in K} \Gamma_m(I_k^{I_i}).
\]

In fact: \( 0 \to \Gamma_m\left( \bigoplus_{i \in K} (Z_i) \right) \to \Gamma_m\left( \bigoplus_{i \in K} (I_0^{I_i}) \right) \to \Gamma_m\left( \bigoplus_{i \in K} (I_1^{I_i}) \right) \to \Gamma_m\left( \bigoplus_{i \in K} (I_k^{I_i}) \right) \to \ldots \Gamma_m\left( \bigoplus_{i \in K} (I_k^{I_i}) \right) \to \ldots
\]

\[
\Gamma_m\left( \bigoplus_{i \in K} (I_k^{I_i}) \right) \to \Gamma_m\left( \bigoplus_{i \in K} (I_k^{I_i}) \right)
\]

is isomorphic to \( 0 \to \bigoplus_{i \in K} \Gamma_m(Z_i) \to \bigoplus_{i \in K} \Gamma_m(I_0^{I_i}) \to \bigoplus_{i \in K} \Gamma_m(I_1^{I_i}) \to \ldots \bigoplus_{i \in K} \Gamma_m(I_k^{I_i}) \to \ldots
\]

The claim follows. \( \Box \)

**Proposition 2.6.** Let \( A \) be a positively graded left noetherian \( k \)-algebra of finite local cohomology dimension on the left. Then for any \( X \in D^+(\text{Gr}_A^e) \), \( Y \in D^-(\text{Gr}_A) \), there is an isomorphism \( R\Gamma_m(X \otimes_A Y) \cong R\Gamma_m(X) \otimes_A Y \).

Proof. The complex \( X \) is in \( D^+ \), hence, it has an injective resolution with objects in \( \text{Gr}_A^e \), \( X \to I \) and \( X \in D^b(\text{Gr}_A^e) \) implies \( H^i(X) = 0 \) for almost all \( i \).

Assume \( H^i(X) = 0 \) for \( i > s \) and let \( Z = \text{Ker}d_s \), where \( d_s : I^s \to I^{s+1} \) is the differential. Hence, \( 0 \to Z \to I^s \to I^{s+1} \to I^{s+2} \to \ldots \to I^{s+k} \to 0 \) is an injective resolution of \( Z \) as \( A \)-bimodule.

Since \( A \) has finite local cohomology dimension, there exists an integer \( t \) such that \( \Gamma_m^j(Z) = 0 \) for \( j > t \). If \( Z' = \text{Im} d_t \), \( d_t : I^t \to I^{t+1} \) is the differential, then \( \Gamma_m^j(Z') = 0 \) for \( j > 0 \), this is \( Z' \) is \( \Gamma_m \)-acyclic.

The complex \( Q : 0 \to I^0 \to I^1 \to \ldots I^t \to Z' \to 0 \) is a complex \( \Gamma_m \)-acyclic which is quasi-isomorphic to \( I \).

The \( \Gamma_m \)-acyclic complexes form an adapted class (See [8], [20]).

Let \( L \to Y \) be a free resolution of \( Y \). Then we have isomorphisms:

\[
X \otimes_A Y \cong X \otimes_A L \cong Q \otimes_A L.
\]

The module \( (Q \otimes_A L)^n \) is a direct sum of objects in the complex \( Q \) and \( A \) noetherian implies sums of injective is injective, therefore \( Q \otimes_A L \) is \( \Gamma_m \)-acyclic.

It follows \( R\Gamma_m(X \otimes_A Y) \cong R\Gamma_m(Q \otimes_A L) \). But we have isomorphisms:

\[
\text{Hom}_A(A/A_{\geq k}, (Q \otimes_A L)^n) = \text{Hom}_A(A/A_{\geq k}, Q^p \otimes_A \oplus A) = \\
\oplus \text{Hom}_A(A/A_{\geq k}, Q^p) = \text{Hom}_A(A/A_{\geq k}, Q^p \otimes_A \oplus A) = \\
\text{Hom}_A(A/A_{\geq k}, Q^p) \otimes_A L^{n-p}.
\]
Therefore:  \( \lim_{k} \text{Hom}_A(A_{\geq k},(Q \otimes A L)^n)=(\lim_{k} \text{Hom}_A(A_{\geq k}, Q^p)) \otimes_AL^{n-p} \).

We are using the fact that \( A \) is noetherian, hence \( A_{\geq k} \) is finitely presented.

We have proved:

\[ \Gamma_m(Q \otimes_A L) \cong \Gamma_m(Q) \otimes_A L. \]

Then \( R\Gamma_m(X \otimes_A Y) \cong R\Gamma_m(X) \otimes_A Y. \)

The proof of the following lemma was given in [9] and reproduced in [15], we will not give it here.

**Proposition 2.7.** Let \( \Lambda \) be a positively graded \( k \)-algebra such that the graded simple have projective resolutions consisting of finitely generated projective modules, \( m \) the graded radical of \( \Lambda \) and \( m^{op} \) the graded radical of \( \Lambda^{op} \). Then for any integer \( k \), \( \Gamma_m^k(\Lambda) = \Gamma_m^k(\Lambda) \).

We can prove now the following:

**Proposition 2.8.** Let \( A \) be a positively graded locally finite noetherian \( k \)-algebra of finite local cohomology dimension on both sides. Let \( X \), \( Y \) be bounded complexes of finitely generated graded \( A \)-modules. Then there exists a natural isomorphism: \( R\text{Hom}_A(R\Gamma_m(X), Y) \cong R\text{Hom}_A(X, Y') \).

**Proof.** Letting \( Y' = \text{Hom}_k(Y, k) \), there is an isomorphism \( R\text{Hom}_A(R\Gamma_m(X), Y) \cong \text{Hom}_A(R\Gamma_m(X), Y') \).

By Proposition 2.2, \( R\text{Hom}_A(R\Gamma_m^{op}(A), Y') \cong (Y' \otimes_A R\Gamma_m^{op}(A))' \).

By Proposition 2.6, \( Y' \otimes_A R\Gamma_m^{op}(A) \cong R\Gamma_m^{op}(Y' \otimes_A A) \cong R\Gamma_m^{op}(Y') \).

Let \( F \) be a free resolution of \( Y \), it consists of finitely generated \( A \)-modules. Hence \( Y' \) consists of finitely cogenerated injective \( A \)-modules, then of torsion modules, and \( \Gamma_m^{op}(Y') \cong \Gamma_m^{op}(F') = F' \cong Y'' \).

Therefore: \( R\text{Hom}_A(R\Gamma_m^{op}(A), Y) \cong Y'' \cong Y \).

Now, there are isomorphisms:

\[ R\text{Hom}_A(R\Gamma_m(X), Y) \cong R\text{Hom}_A(R\Gamma_m(A \otimes_A X), Y) \cong R\text{Hom}_A(R\Gamma_m(A) \otimes_A X), Y) \cong R\text{Hom}_A(X, R\text{Hom}(R\Gamma_m(A), Y)). \]

The last isomorphism is by adjunction and the previous one is by Proposition 2.6.

By Proposition 2.7, \( R\text{Hom}_A(R\Gamma_m(X), Y) \cong R\text{Hom}_A(X, R\text{Hom}(R\Gamma_m^{op}(A), Y)). \)

It follows: \( R\text{Hom}_A(R\Gamma_m(X), Y) \cong R\text{Hom}_A(X, Y) \).

Next we have:

**Lemma 2.9.** For complexes \( X \in D_-(Gr_A) \), \( Y \in D_+(Gr_A) \), there exists a spectral sequence \( E_2^{m,n} = \text{Ext}_A^n(h^{-n}X, Y) \) converging to \( \text{Ext}_A^{n+m}(X, Y) \).
Proof. Let $Y \to J$ be an injective resolution. The complex $X$ is of the form:

$$X : \ldots \to X^{-m} \to \ldots \to X^{-k} \to X^{-k+1} \to \ldots X^{-\ell} \to 0.$$  

For each $n$, there is a complex:

$$\text{Hom}_A(X, J^n): 0 \to \text{Hom}_A(X^{-\ell}, J^n) \to \text{Hom}_A(X^{-\ell-1}, J^n) \to \text{Hom}_A(X^{-k+1}, J^n) \to \ldots \to \text{Hom}_A(X^{-m}, J^n) \to \ldots$$  

Since $J^n$ is injective, $H^n(\text{Hom}_A(X, J^n)) \cong \text{Hom}_A(H^n(X), J^n)$.

If $M^{m,n} = \text{Hom}_A(X^{-m}, J^n)$, then $M = (M^{m,n})$ is a complex in the third quadrant.

Taking first the horizontal homology, then the vertical homology, we obtain the spectral sequence $E_2^{m,n} = \text{Ext}_A^{m+n}(h^{-n}X, Y)$ which converges to the homology of the total complex, which by definition, is $\text{Ext}_A^{m+n}(X, Y)$ [27].

We say that a ring $A$ is Gorenstein if $A$ has finite injective dimension, both as left and as right $A$-module. For the next lemma we need to assume either $A$ is either Gorenstein or it is of finite local cohomology dimension.

**Lemma 2.10.** For $X \in D^{-}(\text{Gr}_A)$, there is a spectral sequence $E_2^{m,n} = \text{Tor}_m^{\Gamma^{n}_{m\text{op}}}(A, X)$ converging to $\Gamma^{m+n}_{m\text{op}}(X)$.

**Proof.** By definition, $\Gamma^{m}_{m\text{op}} = h^m R \Gamma_m$. Let $F$ be a free resolution of $X$.

Then we have a double complex $M^{m,n} = (R \Gamma^{n}_{m\text{op}} A)^{m} \otimes F^n$.

The complex $R \Gamma^{n}_{m\text{op}} A$ is bounded in the Gorenstein case. If $A$ is of finite local cohomology dimension $R \Gamma^{n}_{m\text{op}} A$, can be truncated to a bounded complex of $\Gamma^{m\text{op}}$-acyclic modules.

Taking the second filtration, we obtain a spectral sequence $E_2^{m,n} = \text{Tor}_m^{\Gamma^{n}_{m\text{op}}}(A, X)$ converging to the total complex of $M$.

We have isomorphisms $\text{Tot} M \cong (R \Gamma^{n}_{m\text{op}} A)^{L}_{A} X \cong (R \Gamma_{m} A)^{L}_{A} X \cong R \Gamma_{m} X$.  

**Definition 2.11.** (Castelnovo-Mumford) A complex $X \in D(\text{Gr}_A)$ is called $p$-regular if $\Gamma^{m}_{m}(X)_{\geq p+1-m} = 0$ for all $m$.  

\[ \square \]
If $X$ is $p$-regular but not $p-1$-regular, then we say it has Cohen Macaulay regularity $p$ and write $CMreg X = p$. If $X$ is not $p$-regular for any $p$, we say $CMreg X = \infty$. If $X$ is $p$-regular for all $p$, this is $R_{\Gamma_m} X = 0$, then $CMreg X = -\infty$.

Artin and Schelter introduced in [1] a notion of a non commutative regular algebra that has been very important. We will use here a generalization of non commutative Gorenstein that extends the notion of Artin-Schelter regular. This is a variation of the definition given for connected algebras in [11].

**Definition 2.12.** Let $k$ be a field and $\Lambda$ a locally finite positively graded $k$-algebra. Then we say that $\Lambda$ is graded Artin-Schelter Gorenstein (AS Gorenstein, for short) if the following conditions are satisfied:

i) For all graded simple $S_i$ concentrated in degree zero and non negative integers $j \neq n$, $\text{Ext}^j_{\Lambda}(S_i, \Lambda) = 0$.

ii) We have an isomorphism $\text{Ext}^n_{\Lambda}(S_i, \Lambda \cong S'_i[-n])$, with $S'_i$ a graded $\Lambda^{op}$-simple.

iii) For a non negative integer $k \neq n$, $\text{Ext}^k_{\Lambda^{op}}(\text{Ext}^n_{\Lambda}(S_i, \Lambda), \Lambda) = 0$ and $\text{Ext}^n_{\Lambda^{op}}(\text{Ext}^n_{\Lambda}(S_i, \Lambda), \Lambda \cong S_i)$.

We need to assume now $A$ is graded AS Gorenstein noetherian of finite local cohomology dimension. Under this conditions the following was proved in [15].

**Theorem 2.13.** Let $\Lambda$ be a graded AS Gorenstein algebra of graded injective dimension $n$ and such that all graded simple modules have projective resolutions consisting of finitely generated projective modules and assume $\Lambda$ has finite local cohomology dimension. Then for any graded left module $M$ there is a natural isomorphism: $D(\text{lim}_{k} \text{Ext}^n_{\Lambda}(\Lambda/\Lambda_{\geq k}, M)) \cong \text{Ext}^n_{\Lambda}(M, D(\Gamma_m^\Lambda(\Lambda)))$, for $0 \leq i \leq n$.

Let $D^b_{fg}(\text{Gr}_A)$ be the subcategory of $D^b(\text{Gr}_A)$ of all bounded complexes with finitely generated homology.

Let $X \in D^b_{fg}(\text{Gr}_A)$ and $X \rightarrow I$ an injective resolution. Since $X$ is bounded, there is an integer $t$ such that $H^k(X) = H^k(I) = 0$ for $k > t$.

As above, we can truncate $I$ to obtain a complex $I_>$ consisting of $\Gamma_m$-acyclic modules, $I_> \cong X$ and $I_> \in D^b_{fg}(\text{Gr}_A)$.

We want to prove $R\Gamma_m(X)_' \in D^b_{fg}(\text{Gr}_A)$.

$$X : 0 \rightarrow X_{s_1} \xrightarrow{d_1} X_{s_2} \rightarrow ... X_{s_{t-1}} \xrightarrow{d_{t-1}} X_{s_t} \rightarrow 0.$$ 

We apply induction on $t$.

If $t = 1$, then $X$ is concentrated in degree $s_1$ and $X$ of finitely generated homology means $X$ is finitely generated and it has a projective resolution:
\[ \ldots \to P_k \to P_{k-1} \to \ldots \to P_1 \to P_0 \to X \to 0 \] with each \( P_i \) finitely generated.

Dualizing with respect to the ring we obtain a complex:
\[ P^* : 0 \to P_0^* \to P_1^* \to \ldots \to P_k^* \to P_{k+1}^* \to \ldots \]

with homology \( H^i(P^*) = \text{Ext}^i_A(X, A) \).

Since \( A^{op} \) is noetherian, each \( \text{Ext}^i_A(X, A) \) is finitely generated.

But it was proved in Theorem 2.13:
\[ \text{Ext}^i_A(X, D(\Gamma_n^m(A))) \cong D(\lim_{\to} \text{Ext}^i_A(A/A_{\geq k}, X)) = (\Gamma^i_n(A))', \]
where, according to [14], \( D(\Gamma_n^m(A)) \cong \bigoplus_{i=1}^k Q_i[n_i] \) is the \( A \)-\( A \) bimodule obtained by shifting the indecomposable projective in the decomposition \( A = \bigoplus_{i=1}^k Q_i \).

Therefore:
\[ \text{Ext}^i_A(X, D(\Gamma_n^m(A))) \cong \text{Ext}^i_A(X, A) \otimes_A D(\Gamma_n^m(A)). \]

Since \( A \) is noetherian, \( \text{Ext}^i_A(X, A) \) is a right finitely generated \( A \) module, hence \( \text{Ext}^i_A(X, D(\Gamma_n^m(A))) \) is a right finitely generated \( A \) module. This implies \( R\Gamma_n^m(X)^i \in D_{fg}^b(Gr_A) \).

Let \( C = \text{Coker}d_{\ell-1} = H_{\ell}(X) \) and \( B_{\ell} = \text{Im}d_{\ell-1} \).

Then there is an exact sequence of complexes:
\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 \to X_{s_1} & \to X_{s_2} & \ldots & X_{s_{\ell-1}} & d_{\ell-1} \to B_{\ell} & \to 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 \to X_{s_1} & \to X_{s_2} & \ldots & X_{s_{\ell-2}} & \to X_{s_{\ell-1}} & \to 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0 & \to C & \to 0 \\
\downarrow & & & & \downarrow & \\
0 & & & & & \\
\end{array}
\]

The complex:
\[ Y : 0 \to X_{s_1} \xrightarrow{d_1} X_{s_2} \to \ldots X_{s_{\ell-1}} \xrightarrow{d_{\ell-1}} B_{\ell} \to 0 \]
is quasi- isomorphic to the complex:
\[ 0 \to X_{s_1} \xrightarrow{d_1} X_{s_2} \to \ldots X_{s_{\ell-2}} \xrightarrow{d_{\ell-2}} Z_{s_{\ell-1}} \to 0 \] with \( Z_{s_{\ell-1}} = \text{Ker}d_{\ell-1} \).

By induction hypothesis \( R\Gamma_m^i(Y)^i \in D_{fg}^b(Gr_A) \).

We have a triangle \( Y \to X \to C \to Y[1] \) which induces a triangle:
\[ R\Gamma_m^i(Y) \to R\Gamma_m^i(X) \to R\Gamma_m^i(C) \to R\Gamma_m^i(Y)[1] \]

By the long homology sequence, there is an exact sequence:
\[ \Gamma_m^{i-1}(C) \to \Gamma_m^i(Y) \to \Gamma_m^i(X) \to \Gamma_m^i(C) \to \Gamma_m^{i+1}(Y) \]

Dualizing with respect to \( k \), there is an exact sequence:
\[ (\Gamma_m^i(C))^i \to (\Gamma_m^i(X))^i \to (\Gamma_m^i(Y))^i. \]
Using $A$ is noetherian and induction, it follows $(\Gamma_m^j(X))'$ is finitely generated. Since for any complex $Z$ and any $i$ there is an isomorphism $H^i(Z) \cong H^i(Z')$. It follows $R\Gamma_m^j(X)' \in D^{fg}_{\mathcal{Q}} (\mathcal{G} A)$. Therefore $R\Gamma_m^j(X)$ is a complex with finitely cogenerated homology and each $\Gamma_m^j(X)$ is finitely cogenerated hence $\text{CMreg} X \neq \infty$ and $\text{CMreg} X \neq -\infty$. In the graded AS Gorenstein case, there is an integer $n$ such that $\Gamma_m^j(A)=\Gamma_m^j(A)=0$ for $j \neq n$. According to [15], $I_n'=\Gamma_m^j(A)=\Gamma_m^j(A)=J_n'$, where $I_n'=\oplus D(P^n_j)[-n_j]$ and $J_n'=\oplus D(P_j)[-n_j]$. $I_n'$ is cogenerated as left module in the same degrees as $J_n'$ is cogenerated as right module and $\text{CMreg}(A\mathcal{A})=\text{CMreg}(A\mathcal{A})$.

**Definition 2.14.** (Ext-regularity) The complex $X \in D(\mathcal{G} A)$ is $r$-Ext-regular if $\text{Ext}_A^n(X,A_0) \leq -r-1-m=0$ for all $m$. If $X$ is $r$-Ext-regular and is not $(r-1)$-Ext-regular we say $\text{Ext}-\text{regular}(X)=r$. If $X$ is not $r$-Ext-regular for any $r$, then $\text{Ext}-\text{regular}(X)=\infty$ and if for all $r$ the complex $X$ is $r$-Ext-regular, this is $\text{Ext}_A(X,A_0)=0$, then $\text{Ext}-\text{regular}(X)=-\infty$.

In [16] we gave the following definition:

**Definition 2.15.** A complex of graded modules over a graded algebra is subdiagonal if for each $i$ the $i$-th module is generated in degrees at least $i$, provided is not zero.

We will make use of the following:

**Lemma 2.16.** Let $A$ be a locally finite graded noetherian algebra over a field $k$ and $X$ a complex in $D^R_{\mathcal{Q}} (\mathcal{G} A)$. Then $X$ has a projective resolution $P \rightarrow X$ consisting of finitely generated graded projective modules such that a shift $P[k]$ is subdiagonal.

**Proof.** Since $X$ has a graded projective resolution $P$ we may consider $P$ instead of $X$ and prove that $P=P' \oplus P''$ where $P'$ is a up to shift subdiagonal complex of finitely generated projective graded modules and $H^i(P'')=0$ for all $i$.

Given the complex:

$$P : \ldots \rightarrow P_{n+1} \rightarrow P_n \rightarrow P_{n-1} \rightarrow \ldots P_1 \rightarrow P_0 \rightarrow 0$$

there is an exact sequence: $0 \rightarrow B_1 \rightarrow P_0 \rightarrow C \rightarrow 0$ with $H^0(P)=C$ finitely generated.

Since $C$ has a finitely generated projective cover $P_0'$, there is an exact commutative diagram:

$$\begin{array}{ccc}
\text{Ext}_A^n(X,A_0) & \cong & \text{Ext}_A^n(P_0',A_0)
\end{array}$$
Hence $B_1 \cong B'_1 \oplus P''_0$ and $B'_1$ has a finitely generated projective cover $P'_1$ and there is an exact sequence: $0 \to Z'_1 \to P'_1 \to B'_1 \to 0$.

We have an exact commutative diagram:

\[
\begin{array}{cccccc}
0 & 0 & 0 & \downarrow & \downarrow & \downarrow \\
0 \to B'_1 & \to P'_0 & \to C & \to 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 \to B_1 & \to P_0 & \to C & \to 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 \to P''_0 & \to P''_0 & \to 0 \\
\end{array}
\]

Therefore: $P$ is isomorphic to the complex:

\[
\begin{array}{cccccc}
0 & 0 & 0 & \downarrow & \downarrow & \downarrow \\
0 \to Z'_1 & \to P'_1 \oplus P''_0 & \to B'_1 \oplus P''_0 & \to 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 \to Z_1 & \to P_1 & \to B'_1 \oplus P''_0 & \to 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 \to P''_1 & \to P''_0 & \to 0 \\
\end{array}
\]

with $\text{Im}d_2 \subseteq Z'_1 \oplus P''_1$.

It follows $P$ decomposes as $P = P' \oplus P''$ with:

$P'$ : $\ldots \to P_n \to P_{n-1} \to \ldots P_2 \xrightarrow{d_2} P'_1 \oplus P''_0 \oplus P''_1 \xrightarrow{d_1} P'_0 \oplus P''_0 \to 0$

$P'' : 0 \to P''_0 \to P''_0 \to 0$

The projective $P'_0$ is finitely generated.

Assume now $P = P' \oplus P''$, where $H^i(P'') = 0$ for all $i$ and

$P' : \ldots \to P_{n+1} \to P_n \to \ldots P_1 \to P_0 \to 0$ with $P_1$ finitely generated for $0 \leq i \leq n - 2$.

Hence $B_{n-2} = \text{Im}d_{n-1}$ is finitely generated, therefore it has finitely generated projective cover $P'_{n-1}$ and as before, there is a commutative exact diagram:
\[
\begin{array}{c}
0 \\
\downarrow \\
0 \to Z'_{n-1} \to P'_{n-1} \to B_{n-2} \to 0 \\
\downarrow \\
0 \to Z_{n-1} \to P_{n-1} \to B_{n-2} \to 0 \\
\downarrow \\
0 \to P''_{n-1} \to P''_{n-1} \to 0 \\
\downarrow \\
0 \\
\end{array}
\]

Therefore: \( Z_{n-1} \cong Z'_{n-1} \oplus P''_{n-1} \).

Letting \( B_{n-1} \) be the image of \( d_n \) and \( H_{n-1} \) the homology \( H^{n-1}(P) \), which we assume finitely generated, there is an exact sequence: \( 0 \to B_{n-1} \to Z'_{n-1} \oplus P''_{n-1} \to H_{n-1} \to 0 \) and an induced commutative, exact diagram:

\[
\begin{array}{c}
0 \\
\downarrow \\
0 \to \overline{B}_{n-1} \to Z'_{n-1} \oplus P''_{n-1} \to H'_{n-1} \to 0 \\
\downarrow \\
0 \to B_{n-1} \to Z_{n-1} \oplus P''_{n-1} \to H_{n-1} \to 0 \\
\downarrow \\
0 \to B''_{n-1} \to P''_{n-1} \to H''_{n-1} \to 0 \\
\downarrow \\
0 \\
\end{array}
\]

with \( \overline{B}_{n-1} = B_{n-1} \cap Z'_{n-1} \) and \( H''_{n-1} \) is finitely generated.

Therefore: the exact sequence: \( 0 \to B''_{n-1} \to P''_{n-1} \to H''_{n-1} \to 0 \) is isomorphic to the direct sum of the exact sequences:

\( 0 \to L_{n-1} \to Q''_{n-1} \to H''_{n-1} \to 0 \) and \( 0 \to Q'_{n-1} \to Q'_{n-1} \to 0 \to 0 \), with \( Q''_{n-1} \) the projective cover of \( H''_{n-1} \), hence finitely generated. Then \( B''_{n-1} \cong L_{n-1} \oplus Q'_{n-1} \).

There is a commutative exact diagram:

\[
\begin{array}{c}
0 \\
\downarrow \\
0 \to \overline{B}_{n-1} \to B'_{n-1} \to L_{n-1} \to 0 \\
\downarrow \\
0 \to \overline{B}_{n-1} \to B_{n-1} \to L_{n-1} \oplus Q'_{n-1} \to 0 \\
\downarrow \\
0 \to Q'_{n-1} \to Q'_{n-1} \to 0 \\
\downarrow \\
0 \\
\end{array}
\]

where \( \overline{B}_{n-1} \) and \( L_{n-1} \) are finitely generated. It follows \( B_{n-1} \cong B'_{n-1} \oplus Q'_{n-1} \) with \( B'_{n-1} \) finitely generated.
We have an exact sequence: $0 \to B'_{n-1} \oplus Q'_{n-1} \to P'_{n-1} \oplus Q''_{n-1} \to P_{n-2}$.

Taking the projective cover of $B'_{n-1}$ we obtain an exact sequence: $0 \to Z'_{n} \to P'_{n} \to B'_{n-1} \to 0$. Therefore: $0 \to Z'_{n} \to P'_{n} \oplus Q'_{n-1} \to B'_{n-1} \oplus Q'_{n-1} \to 0$ is exact.

As above, $P_{n}$ decomposes $P'_{n} \oplus Q'_{n-1} \oplus P''_{n}$.

We have proved that $P$ decomposes in the direct sum of the complexes:

$\cdots \to P_{n+1} \to P_{n} \oplus P''_{n} \to P'_{n-1} \oplus Q''_{n-1} \to P_{n-2} \cdots P_{1} \to P_{0} \to 0$

and $0 \to Q'_{n-1} \to Q'_{n-1} \to 0 \to 0$, where $P'_{n-1} \oplus Q''_{n-1}$ is finitely generated.

\[ \square \]

We proved in the previous lemma that for $X \in D^b_{fg}(Gr_{A})$, we can choose a projective resolution of finitely generated projective graded modules: $P \to X$ such that the differential map $d_{j} : P_{j} \to P_{j-1}$ has image contained in the radical of $P_{j-1}$.

Hence the complex $\hom_{A}(P, A_{0})$:

$0 \to \hom_{A}(P_{0}, A_{0}) \to \hom_{A}(P_{1}, A_{0}) \to \cdots \hom_{A}(P_{n}, A_{0}) \to \cdots$ has zero differential.

It follows $\ext^{k}_{A}(X, A_{0})=\hom_{A}(P_{k}, A_{0}) \neq 0$ and $\ext_{A}(X, A_{0}) \neq 0$.

It follows $\ext-\regular(X) \neq -\infty$, but $\ext-\regular(X)=\infty$ is possible.

Assume $\ext-\regular(X)=r$ is finite.

Each projective $P_{j}$ has a decomposition in indecomposable summands:

$P_{j}=\bigoplus_{i=1}^{m} Q_{j}[-n_{j}^{i}]$, with $n_{j}^{i}$ integers.

Then $\ext^{i}_{A}(X, A_{0})=\hom_{A}(P_{j}, A_{0})=\bigoplus_{i=1}^{m} \hom_{A}(Q_{j}/mQ_{j}[-n_{j}^{i}], A_{0})=
\bigoplus_{i=1}^{m} \hom_{A}(Q_{j}/mQ_{j}[-n_{j}^{i}], A_{0}[k])_{0}$

Therefore: $\hom_{A}(P_{j}, A_{0})_{k} \neq 0$ if and only if for some $i$, $-n_{j}^{i}=k$.

By definition $\ext^{i}_{A}(X, A_{0})_{k} \leq -r-1-j=0$, this means $-r-j \leq -n_{i}^{j}$ or $r \geq n_{i}^{j}-j$, for all $i$ and $r'=\max\{n_{j}^{i}-j\}$ exists.

Then $\ext^{j}_{A}(X, A_{0})_{k} \leq -r-1-j=0$ and $\ext^{j}_{A}(X, A_{0})_{-(n_{i}^{j}-j)} \neq 0$.

We have proved $\ext-\regular(X)=r=\max\{n_{j}^{i}-j\}$.

Let $P : \cdots \to P_{n+1} \to P_{n} \to P_{n-1} \to \cdots P_{1} \to P_{0} \to A_{0} \to 0$ and $P' : \cdots \to P'_{n+1} \to P'_{n} \to P'_{n-1} \to \cdots P'_{1} \to P'_{0} \to A_{0} \to 0$ be minimal projective resolutions of $A_{0}$ as left and as right module, respectively.

Each $P_{j}$ has a decomposition $P_{j}=\bigoplus_{i=1}^{m} Q_{j}[-n_{j}^{i}]$ and $\tor^{A}_{n}(A_{0}, A_{0})$ is computed as the nth-homology of the complex $A_{0} \otimes A P :$

$\cdots \to A_{0} \otimes A P_{n+1} \to A_{0} \otimes A P_{n} \to A_{0} \otimes A P_{n-1} \to \cdots A_{0} \otimes A P_{1} \to A_{0} \otimes A P_{0} \to 0$ and $A_{0} \otimes A P_{n}=A_{0} \otimes A \oplus Q_{j}[-n_{j}^{m}]=A/m \otimes A \oplus Q_{j}[-n_{j}^{m}] \cong m \bigoplus_{i=1}^{m} Q_{j}/m Q_{j}[-n_{j}^{m}]
\cong \bigoplus_{i=1}^{m} S_{j}[-n_{j}^{m}]$ and the differential of $A_{0} \otimes A P$ is zero.
Using the second resolution $\Tor^A_n(A_0, A_0)$ is the $n$th-homology of the complex $P' \otimes_A A_0$:

$$
\cdots \rightarrow P'_{n+1} \otimes_A A_0 \rightarrow P'_n \otimes_A A_0 \rightarrow P'_{n-1} \otimes_A A_0 \rightarrow \cdots P'_1 \otimes_A A_0 \rightarrow P'_0 \otimes_A A_0 \rightarrow 0
$$

Each $P'_i$ has a decomposition $P'_i = \bigoplus Q'_i[-n'_i]$ and $P''_n \otimes A_0 = \bigoplus Q'_i[-n'_i] \otimes A_0 = \bigoplus \frac{Q'_i}{Q'_i m[-n'_i]} \cong \bigoplus S'_i[-n'_i]$ and the differential of $P' \otimes A_0$ is zero.

It follows $n'_1 = n'_i$ for all $i$.

By the above remark, $\Ext-reg A_0 = \Ext-reg A_{0A} = \Ext-reg A_0$.

We write this as a theorem.

**Theorem 2.17.** Let $A$ be a locally finite $k$-algebra. Then

$$
\Ext - reg A_0 = \Ext-reg A_{0A} = \Ext-reg A_0.
$$

We next have:

**Theorem 2.18.** Let $A$ be a noetherian graded AS Gorenstein algebra of finite local cohomology dimension.

**Proof.** We proved above $CMreg(X) \neq -\infty$. If $Ext-reg A_0 = \infty$, then the inequality is trivially satisfied.

We may assume $Ext-reg A_0 = r$ is finite.

Let $P \rightarrow A_0$ be a minimal projective resolution. Changing notation, $P$ is of the form: $\cdots P^{n+1} \rightarrow P^n \rightarrow \cdots P^1 \rightarrow P^0 \rightarrow 0$, where $P^m = \bigoplus P_j^{(m)}[-\sigma_{m,j}]$ and $\sigma_{m,j} \leq r + m$.

Dualizing, we obtain an injective resolution $I$ with $I^m = \bigoplus D(P_j^{(m)})[\sigma_{m,j}]$, of $A_0$ as right module.

Let $p$ be $p = CMreg(X)$, $Z = R\Gamma_m(X)$ and denote by $h^{-n}$ the homology. Then by definition we have:

$$
h^{-n}(Z)_{\geq p + 1 + n} = h^{-n}(R\Gamma_m(X))_{\geq p + 1 + n} = \Gamma_m^{-n}(X)_{\geq p + 1 + n} = 0 \text{ for all } n.
$$

Therefore: $(h^{-n}(Z))_{\leq -p - 1 - n} = 0$.

But $\Ext_A^m(h^{-n}(Z), A_0)$ is a subquotient of $\Hom_A(h^{-n}(Z), I^m)$

$$
\Hom_A(h^{-n}(Z), \bigoplus D(P_j^{(m)})[\sigma_{m,j}]) = \Hom_A(h^{-n}(Z), D(P_j^{(m)})[\sigma_{m,j}])
$$

$$
\cong \bigoplus \Hom_k((P_j^{(m)})^* \otimes h^{-n}(Z), k)[\sigma_{m,j}] \cong \bigoplus \Hom_k((e_j h^{-n}(Z), k)[\sigma_{m,j}]
$$

with $e_j$ the idempotent corresponding to $P_j^{(m)}$. Since $(h^{-n}(Z))_{\leq -p - 1 - n} = 0$, it follows $\Hom_k((e_j h^{-n}(Z), k)_{\leq -p - 1 - n} = 0$. 


Observe that the truncation of a shifted module $M[k]_{\leq -t-k} = M_{\leq -t}[k]$. Therefore: $\text{Ext}^m_{\mathcal{A}}(h^{-n}(Z), A_0)_{\leq -p-1-n-r-m} = 0$.

We have a converging spectral sequence:
$$E^2_{m,n} = \text{Ext}^m_{\mathcal{A}}(h^{-n}(Z), A_0) \Rightarrow \text{Ext}^{m+n}_{\mathcal{A}}(Z, A_0).$$

This means $\text{Ext}^{m+n}_{\mathcal{A}}(Z, A_0)$ is a subquotient of $E^2_{m,n} = \text{Ext}^m_{\mathcal{A}}(h^{-n}(Z), A_0)$ and $\text{Ext}_A^m(h^{-n}(Z), A_0)_{\leq -p-1-n} = 0$ implies $\text{Ext}_A^q(Z, A_0)_{\leq -p-1-r-q} = 0$ for all $q$.

We have isomorphisms:
$$\text{Ext}_A^q(Z, A_0) \cong \text{Ext}^q_{\mathcal{A}}(R\Gamma_m(X), A_0) \cong H^q(\text{RHom}(R\Gamma_m(X), A_0))$$
$$\cong H^q(\text{RHom}(X, A_0)) \cong \text{Ext}_A^q(X, A_0).$$

Therefore: $\text{Ext}^q_{\mathcal{A}}(X, A_0)_{\leq -p-1-r-q} = 0$.

This implies $\text{Ext}\text{-reg}(X)_{\leq p+r} = \text{CMreg}(X) + \text{Ext}\text{-reg}A_0$. \hfill $\square$

**Corollary 2.19.** Assume the same conditions as in the theorem and $\text{Ext}\text{-reg}A_0$ finite. Then for any $X \in \mathcal{D}^b_f(\text{Gr}_A)$, $\text{Ext}\text{-reg}(X)$ is finite.

**Proof.** This follows from the above remark that $\text{CMreg}(X)$ is finite. \hfill $\square$

Interchanging the roles of $\text{Ext}$-regular and $\text{CM}$-regular we obtain in the next result a similar inequality.

**Theorem 2.20.** Let $A$ be a noetherian AS Gorenstein algebra of finite local cohomology dimension.

Given $X \in \mathcal{D}^b_f(\text{Gr}_A), X \neq 0$. Then $\text{CMreg}(X) \leq \text{Ext}\text{-reg}(X) + \text{CMreg}A$.

**Proof.** Since we know $\text{CMreg}A \neq -\infty$, the assumption $\text{Ext}\text{-reg}(X) = \infty$ gives the inequality and we can assume $\text{Ext}(X) = r$ is finite.

As before, there is a projective resolution $P \to X$ of $X$ with $P^{(m)} = \oplus P_j^{(m)}[-\sigma_{m,j}]$ and $\sigma_{m,j} \leq r+m$.

Let $p$ be $p = \text{CMreg}_A A = \text{CMreg}A_A$. Then by definition $\Gamma_{m^{op}}^n(A)_{\geq p+1-n} = 0$ for all $n$.

Then $\text{Tor}^A_{-m}(\Gamma_{m^{op}}^n(A), X)$ is a subquotient of $\Gamma_{m^{op}}^n(A) \otimes_A P^{(-m)} = \oplus \Gamma_{m^{op}}^n(A) \otimes_A P_j^{(-m)}[-\sigma_{m,j}] = \oplus \Gamma_{m^{op}}^n(A)e_j[-\sigma_{m,j}]$, with $e_j$ the idempotent corresponding to $P_j^{(-m)}$ and $\sigma_{m,j} \leq r-m$.

Therefore: $\Gamma_{m^{op}}^n(A)[-\sigma_{m,j}]_{\geq p+1-n+(r-m)} = 0$.

As above, it follows $\text{Tor}^A_{-m}(\Gamma_{m^{op}}^n(A), X)_{\geq p+1-n+r-m} = 0$.

The spectral sequence $E_2^{m,n} = \text{Tor}^A_{-m}(\Gamma_{m^{op}}^n(A), X) \Rightarrow \Gamma_{m^{op}}^{m+n}(X)$ converges (Lemma 2.4).

Hence $\Gamma_{m^{op}}^{m+n}(X)$ is a subquotient of $\text{Tor}^A_{-m}(\Gamma_{m^{op}}^n(A), X)$ and it follows $\Gamma_{m}^q(X)_{\geq p+1-r-q} = 0$.

We have proved $\text{CMreg}(X) \leq p+r = \text{Ext}\text{-reg}(X) + \text{CMreg}A$. \hfill $\square$
Remark 1. The algebra $A$ is Koszul if and only if $\text{Ext} - \text{reg} A_0 = 0$.

Corollary 2.21. Assume the same conditions on $A$ as in the theorem and in addition $A$ Koszul and $\text{CMreg} A = 0$. Then $\text{Ext} - \text{reg}(X) = \text{CMreg}(X)$.

Let $J_n$ the $A$- $A$ bimodule, $J_n = \bigoplus_{j=1}^{m} D(Q_j)[-n_j]$ with $Q_j$ the indecomposable summands of $A$ and the shifts are the shifts of the simple appearing in the definition of AS Gorenstein. Then we have the following:

Lemma 2.22. Let $A$ be a noetherian AS Gorenstein algebra of graded injective dimension $n$, $M$ a finitely generated left $A$-module. Then for any integer $s$, $\text{Ext}^n_A(M/M \geq s, D(J'_n)) \geq s = 0$.

Proof. Assume $M_{k_0} \neq 0$ and $M_j = 0$ for $j < k_0$ and $s = k_0 + t$, with $t$ a positive integer. Then $M/M \geq s = M_{k_0} + M_{k_0+1} + \ldots M_{k_0+t-1}$.

We apply induction on $t$. Let $S_i[-k]$ be a simple concentrated in degree $k$. Then $\text{Ext}^n_A(S_i[-k], D(J'_n)) = \text{Ext}^n_A(S_i, A) \otimes_A D(J'_n)[k] \cong S_i[-n_i] \otimes_{\oplus Q_j[n_j]}^m \cong S_i'[k].$

Since $M_{k_0} = M/M \geq k_0+1$ is a semisimple concentrated in degree $k_0$, $\text{Ext}^n_A(M_{k_0}, D(J'_n))$ is a semisimple concentrated in degree $-k_0$. By the AS Gorenstein property, the exact sequence:

$0 \to M_{k_0+1}/M \geq s \to M/M \geq s \to M_{k_0} \to 0$, induces an exact sequence:

$0 \to \text{Ext}^n_A(M_{k_0}, D(J'_n)) \to N \to \text{Ext}^n_A(M \geq k_0+1/M \geq s, D(J'_n)) \to 0$

where $\text{Ext}^n_A(M/M \geq s, D(J'_n)) = N$. It follows by induction that $N$ has a filtration: $N \supseteq N_1 \supseteq N_2 \supseteq \ldots N_{t-1} \supseteq 0$, such that for $0 \leq i \leq t-1$, $N_i/N_{i+1}$ is a semisimple concentrated in degree $-k_0+t-1-i$.

Therefore: $\text{Ext}^n_A(M/M \geq s, D(J'_n)) \geq s = 0$.

We have all the ingredients to prove the main theorem of the section.

Theorem 2.23. Let $A$ be a noetherian AS Gorenstein algebra of finite local cohomology dimension. Assume $A$ Koszul and let $M$ be a finitely generated graded $A$-module. Then for $s \geq \text{CMreg} M$, the projective resolution of $M_{\geq s}[s]$ is linear.

Proof. Assume $M_{\geq s}[s] \neq 0$ and let $P^{(n+1)} \to P^{(n)} \to \ldots P^{(1)} \to P^{(0)} \to M_{\geq s}[s] \to 0$ be the projective resolution. The module $M_{\geq s}[s]$ is generated in degree zero and $P^{(m)}$ decomposes as $P^{(m)} = \oplus P^{(m)}_j[-\sigma_{m,j}]$ and $m \leq \sigma_{m,j}$.

We must prove $P^{(m)}$ does not have generators in degrees larger than $m$, or equivalently $\text{Ext} - \text{reg}(M_{\geq s}[s]) \leq 0$, which will follow from the above inequalities once we prove $\text{CMreg}(M_{\geq s}[s]) \leq 0$ or equivalently, $\text{CMreg}(M_{\geq s}) \leq s$, this is:
\[ \Gamma_m^\ast (M_{\geq s})_{\geq s+1-m} = 0. \]

The module \( L = M/M_{\geq s} \) is of finite length. By the local cohomology formula, \( \lim_{k} \text{Ext}^j_A(A/m^k, L) = \text{D}(\text{Ext}^{n-j}_A(L, \text{D}(\Gamma_m^\ast (A))). \]

Since \( A \) is graded AS Gorenstein \( \text{Ext}^{n-j}_A(L, \text{D}(\Gamma_m^\ast (A))) = 0 \) for \( j \neq n \). It follows \( \Gamma_m^\ast (M/M_{\geq s}) \geq s \). It follows by Lemma 2.22 that \( \Gamma_m^\ast (M/M_{\geq s})_{\geq s+1-m} = 0 \) for all \( m \).

The exact sequence: \( 0 \rightarrow M_{\geq s} \rightarrow M \rightarrow M/M_{\geq s} \rightarrow 0 \) induces a triangle \( M_{\geq s} \rightarrow M \rightarrow M/M_{\geq s} \rightarrow M_{\geq s}[1] \), hence a triangle \( R\Gamma_m^\ast (M_{\geq s}) \rightarrow R\Gamma_m^\ast (M) \rightarrow R\Gamma_m^\ast (M/M_{\geq s}) \rightarrow R\Gamma_m^\ast (M_{\geq s})[1] \), by the long homology sequence we obtain an exact sequence:

\[ \rightarrow \Gamma_m^{m-1}(M/M_{\geq s}) \rightarrow \Gamma_m^\ast (M) \rightarrow \Gamma_m^\ast (M/M_{\geq s}) \]

The inequality \( s \geq \text{CMreg}(M) \) implies \( \Gamma_m^\ast (M)_{\geq s+1-m} = 0 \) for all \( m \).

Therefore: \( \Gamma_m^\ast (M_{\geq s})_{\geq s+1-m} = 0 \) for all \( m \).

\[ \square \]

3 Algebras AS Gorenstein and Koszul

In this section we will use the main theorem of the last section in order to extend a theorem by Bernstein-Gelfand-Gelfand, [4] which claims that for the exterior algebra in \( n \)-variables \( \Lambda \) there is an equivalence of triangulated categories \( \text{gr}_A \cong \text{D}^b(\text{CohP}_n) \) from the stable category of finitely generated graded modules to the category of bounded complexes of coherent sheaves on projective space \( P_n \). The theorem was extended to finite dimensional Koszul algebras in [16],[17] see also [22]. We want to prove here a version of this theorem for AS Gorenstein algebras of finite cohomological dimension. We will show that the arguments used in [16] can be easily extended to this situation. We will assume the reader is familiar with the results of [14], [16] and [18] and the bibliography given there.

It was proved in [25] and [13] that a finite dimensional Koszul algebra \( \Lambda \) is selfinjective if and only if its Yoneda algebra \( \Gamma \) is Artin Schelter regular [1]. The following generalization was proved in [14] and [23].

**Theorem 3.1.** A Koszul algebra \( \Lambda \) is graded AS Gorenstein if and only if its Yoneda algebra \( \Gamma \) is graded AS Gorenstein.

**Remark 2.** Observe the following:

i) The algebra \( \Lambda \) can be noetherian with non noetherian Yoneda algebra \( \Gamma \).

ii) The algebra \( \Lambda \) could be Gorenstein and \( \Gamma \) only weakly Gorenstein this is: there exists an integer \( n \) such that for all \( \Gamma \)-modules left (right) of finite length \( \text{Ext}^j_A(M, \Gamma) = 0 \) for all \( j > n \).

iii) The algebra \( \Lambda \) could be of finite local cohomology dimension and \( \Gamma \) of infinite local cohomology dimension.
However, there are Koszul algebras $\Lambda$ with Yoneda algebra $\Gamma$ such that both $\Lambda$ and $\Gamma$ are graded AS Gorenstein, noetherian (in both sides) and of finite cohomological dimension, for example if $\Lambda$ and $\Delta$ are Koszul selfinjective algebras with noetherian Yoneda algebras, $\Gamma$ and $\Sigma$, respectively, then $\Lambda \otimes \Sigma$ is AS Gorenstein noetherian of finite local cohomology dimension on both sides and with Yoneda algebra the skew tensor product (in the sense of [6] or [19]) $\Gamma \boxtimes \Delta$ which is also AS Gorenstein noetherian and of finite cohomological dimension on both sides. A special case would be $\Lambda \otimes \Gamma$.

A concrete example of such algebras is $\Lambda$ the exterior algebra in $n$ variables and $\Gamma$ the polynomial algebra in $n$ variables, this example appears as the cohomology ring of an elementary abelian $p$-group over a field of positive characteristic $p \neq 2$. [5]

Another example is the trivial extension $\Lambda = \mathbb{k}QD(\mathbb{k}Q)$ with $Q$ an Euclidean diagram and $\Gamma$ the preprojective algebra corresponding to $Q$ [12]. We need the following definitions and results from [18]:

**Definition 3.2.** Let $\Lambda$ be a Koszul algebra with graded Jacobson radical $m$. A finitely generated graded $\Lambda$-module $M$ is weakly Koszul if it has a minimal projective resolution: $0 \to P_n \to P_{n-1} \to \ldots P_1 \to P_0 \to M \to 0$ such that $m^{k+1}P_i \cap \ker d_i = m^k \ker d_i$.

The next result characterizing weakly Koszul modules was proved in [18].

**Theorem 3.3.** Let $\Lambda$ be a Koszul algebra with Yoneda algebra and denote by $\text{gr}_{\Lambda}$, the category of finitely generated graded $\Lambda$-modules, $F : \text{gr}_{\Lambda} \to \text{Gr}_{\Gamma}$ be the exact functor $F(M) = \bigoplus_{k \geq 0} \text{Ext}^k_{\Lambda}(M, \Lambda_0)$. Then $M$ is weakly Koszul if and only if $F(M)$ is Koszul.

As a consequence of this theorem and the results of the last section we have:

**Theorem 3.4.** Let $\Lambda$ be a Koszul algebra with Yoneda algebra $\Gamma$ such that both are AS graded Gorenstein noetherian algebras of finite local cohomology dimension on both sides. Then given a finitely generated left $\Lambda$-module $M$ there is a non negative integer $k$ such that $\Omega^k(M)$ is weakly Koszul.

**Proof.** Since $\Lambda$ is Koszul AS graded Gorenstein noetherian algebras of finite local cohomology dimension on both sides, for any finitely generated graded $\Lambda$-module $M$ there is a truncation $M_{\geq s}$ such that $M_{\geq s}[s]$ is Koszul and there is an exact sequence: $0 \to M_{\geq s} \to M \to M/M_{\geq s} \to 0$ with $M/M_{\geq s}$ of finite length. Then we have an exact sequence: $F(M/M_{\geq s}) \to F(M) \to F(M_{\geq s})$. Since $F$ sends simple modules to indecomposable projective, it sends modules of finite length to finitely generated modules and $M_{\geq s}$ Koszul up to shift implies $F(M_{\geq s})$ Koszul up to shift, hence finitely generated. Since we are assuming...
Γ noetherian, it follows $F(M)$ is finitely generated. By Theorem 2.23, $F(M)$ has a truncation $F(M)_{\geq t}$ Koszul up to shift and $F(M)_{\geq t} = \bigoplus_{k \geq t} \text{Ext}^k_{\Lambda}(M, \Lambda_0) [-t]$

$\cong \bigoplus_{k \geq 0} \text{Ext}^k_{\Lambda}(\Omega^t(M), \Lambda_0) [-t] = F(\Omega^t(M))$.

By Theorem 3.3, $\Omega^t(M)$ is weakly Koszul.

**Definition 3.5.** A complex of graded $\Lambda$-modules is linear if for each $i$, the $i$th module is generated in degree $i$, provided it is not zero.

Let $Q$ be a finite quiver, $\mathbb{k}Q$ the path algebra graded by path length and $\Lambda = \mathbb{k}Q/I$ be a quotient with $I$ a homogeneous ideal contained in $\mathbb{k}Q_{\geq 2}$ and $\Gamma$ the Yoneda algebra of $\Lambda$, it was shown in [17] that there is a functor

$$\Phi: \ell.f.gr_{\Lambda} \rightarrow \text{lcp}_{\Gamma^-}$$

between the category of locally finite graded $\Lambda$-modules, $\ell.f.gr_{\Lambda}$, and the category of right bounded linear complexes of finitely generated graded projective $\Gamma$-modules $\text{lcp}_{\Gamma^-}$. We recall the construction of $\Phi$.

Let $M = \{M_i\}_{i \geq n_0}$ be a finitely generated graded $\Lambda$-module and $\mu: \Lambda_1 \otimes_{\Lambda_0} M_k \rightarrow M_{k+1}$ the map of $\Lambda_0$-modules given by multiplication.

Since $M_k$ is a finitely generated $\Lambda_0$-module, we have a homomorphism of $\Lambda_0$-modules

$$\text{D}(\mu): \text{D}(M_{k+1}) \rightarrow \text{D}(M_k) \otimes_{\Lambda_0} \text{D}(\Lambda_1),$$

where $\text{D}(\cdot) = \text{Hom}_{\Lambda_0}(\cdot, \Lambda_0)$. Applying $\text{Hom}_{\Lambda}(\cdot, \Lambda_0)$ to the exact sequence

$$0 \rightarrow \mathfrak{m} \rightarrow \Lambda \rightarrow \Lambda_0 \rightarrow 0$$

By the long homology sequence, we obtain an exact sequence

$$0 \rightarrow \text{Hom}_{\Lambda}(\Lambda_0, \Lambda_0) \rightarrow \text{Hom}_{\Lambda}(\Lambda, \Lambda_0) \rightarrow \text{Hom}_{\Lambda}(\mathfrak{m}, \Lambda_0) \rightarrow \text{Ext}^1_{\Lambda}(\Lambda_0, \Lambda_0) \rightarrow 0$$

the second map is an isomorphism, which implies $\text{Hom}_{\Lambda}(\mathfrak{m}, \Lambda_0) \rightarrow \text{Ext}^1_{\Lambda}(\Lambda_0, \Lambda_0)$ is an isomorphism. Since $\Lambda_0$ is semisimple, there is an isomorphism

$$\text{Hom}_{\Lambda}(\mathfrak{m}, \Lambda_0) \cong \text{Hom}_{\Lambda}(\mathfrak{m}/\mathfrak{m}^2, \Lambda_0)$$

As a result there is an isomorphism $\text{D}(\Lambda_1) = \text{Hom}_{\Lambda_0}(\Lambda_1, \Lambda_0) \cong \Gamma_1$ and we have a $\Lambda_0$-linear map $d_{k_0}: \text{D}(M_{k+1}) \rightarrow \text{D}(M_k) \otimes_{\Lambda_0} \Gamma_1$.

For any $\ell \geq 0$, using the fact $\Lambda_0 \cong \Gamma_0$ the multiplication map $\nu: \Gamma_1 \otimes \Gamma_0 \Gamma_\ell \rightarrow \Gamma_{\ell+1}$ induces a new map $d_{k_\ell}$, as shown in the diagram:

$$\begin{array}{ccc}
\text{D}(M_{k+1}) \otimes_{\Gamma_0} \Gamma_\ell & \rightarrow & \text{D}(M_k) \otimes_{\Gamma_0} \Gamma_1 \otimes_{\Gamma_0} \Gamma_\ell \\
\downarrow d_{k_\ell} & & \downarrow 1 \otimes \nu \\
\text{D}(M_k) \otimes_{\Gamma_0} \Gamma_{\ell+1} & & \\
\end{array}$$
Hence there is a map in degree zero
\[ d_k: D(M_{k+1}) \otimes_{\Gamma_0} \Gamma[-k-1] \to D(M_k) \otimes_{\Gamma_0} \Gamma[-k] \]

**Definition 3.6.** We call $\Phi$ the linearization functor.

**Proposition 3.7.** The sequence $\Phi(M) = \{ D(M_{k+1}) \otimes_{\Gamma_0} \Gamma[-k-1], d_k \}$ is a right bounded linear complex of finitely generated graded projective $\Gamma$-modules.

The following proposition was proved in [17].

**Proposition 3.8.** The algebra $\Lambda = \mathbb{k}Q/I$ is quadratic if and only if $\Phi: \ell.f.gr\Lambda \to \text{lcp}_\Gamma$ is a duality.

We can say more in case $\Lambda = \mathbb{k}Q/I$ is a Koszul algebra.

**Theorem 3.9.** Suppose $\Lambda = \mathbb{k}Q/I$ is a Koszul algebra and $M$ a locally finite bounded above graded $\Lambda$-module. Then $M$ is Koszul if and only if $\Phi(M)$ is exact, except at minimal degree; in that case, $\Phi(M)$ is a minimal projective resolution of the Koszul module (up to shift) $F(M) = \bigoplus_{k \geq t} \text{Ext}_\Lambda^k(M, \Lambda_0)$.

### 3.1 Approximations by linear complexes

In this section we will see that the approximations by linear complexes given in [16] can be extended to the family of AS Gorenstein Koszul algebras considered above. Let $\Lambda$ be a possibly infinite dimensional Koszul algebra with noetherian Yoneda algebra $\Gamma$. The category of complexes of finitely generated graded projective $\Gamma$-modules with bounded homology $K^{-b}(\text{gr}P\Gamma)$, module the homotopy relations, is equivalent to the derived category of bounded complexes $D_{fg}^{b}(\text{Gr}\Gamma)$.

It was proved proved in Lemma 2.16, that any complex $X$ in $D_{fg}^{b}(\text{Gr}\Gamma)$ has projective resolution $P \to X$ with $P$ consisting of finitely generated projective graded modules and subdiagonal up to a shift.

Since our interest is in Koszul algebras we need the following:

**Definition 3.10.** A complex is said to be totally linear, if it is linear and each of its terms has a linear projective resolution. Observe that this notion is a generalization of a linear complex of projective modules.

Note that, though the proposition below has been stated more generally than in [16], the proof is the same as in [16].

**Proposition 3.11.** Let $\Gamma$ be a noetherian graded algebra and $M_\bullet = \{ M_i, d_i \}_{n \geq i \geq 0}$ a bounded totally linear complex of finitely generated graded $\Gamma$-modules. Then there exists a bounded on the right linear complex of finitely generated projective graded modules $P_\bullet$ and a quasi-isomorphism $\mu: P_\bullet \to M_\bullet$ such that $\mu_i: P_i \to M_i$ is an epimorphism for each $i$. 
Proof. The approximation is constructed by induction using a variation of a lemma given by Verdier in [26]. We start with the exact sequence: \(0 \rightarrow B_0 \rightarrow M_0 \rightarrow H_0 \rightarrow 0\), take the projective cover \(P_0 \rightarrow M_0 \rightarrow 0\) and complete a commutative exact diagram:

\[
\begin{array}{ccc}
0 & 0 \\
\downarrow & \downarrow \\
0 & \Omega(M_0) & \Omega(M_0) & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \Omega(H_0) & P_0 & \rightarrow & H_0 & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & 1 \\
0 & B_0 & M_0 & \rightarrow & H_0 & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0
\end{array}
\]

Taking the pull back we obtain a commutative exact diagram:

\[
\begin{array}{ccc}
0 & 0 \\
\downarrow \\
0 & \Omega(M_0) & \Omega(M_0) & \rightarrow & 0 \\
\downarrow \\
0 & Z_1 & W_1 & \rightarrow & \Omega(H_0) & \rightarrow & 0 \\
\downarrow \\
0 & Z_1 & M_1 & \rightarrow & B_0 & \rightarrow & 0 \\
\downarrow \\
0 & 0 & 0
\end{array}
\]

Since \(M_1\) and \(\Omega(M_0)\) are both generated in degree one and have linear resolutions, the same is true for \(W_1\).

It is clear that the complex \(0 \rightarrow M_n \rightarrow ... \rightarrow M_2 \rightarrow W_1 \rightarrow P_0 \rightarrow 0\) is totally linear and quasi-isomorphic to \(M_\bullet\) and the quasi-isomorphism is an epimorphism in each degree.

Assume by induction we have constructed the totally linear complex: \(0 \rightarrow M_n \rightarrow ... \rightarrow M_{j+1} \rightarrow W_j \rightarrow P_{j-1} \rightarrow ... \rightarrow P_0 \rightarrow 0\) together with a quasi-isomorphism \(\mu\) to the complex \(M_\bullet\) which is an epimorphism in each degrees \(k\) with \(0 \leq k \leq j\) and the identity in degrees \(k\) for \(j+1 \leq k \leq n\).

We have a commutative exact diagram:
which induces by pullback the commutative exact diagram:

\[
\begin{array}{cccc}
0 & 0 & & \\
\downarrow & \downarrow & & \\
0 & \Omega(W_j) & \Omega(W_j) & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & K & P_j & W_j/B_j \\
\downarrow & \downarrow & \downarrow & \\
0 & B_j & W_j & W_j/B_j \\
\downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0
\end{array}
\]

By Verdier’s lemma [26] we have a complex: \( P^{(j)}_\bullet : 0 \to M_n \to \cdots \to M_{j+2} \to \cdots \to P_j \to \cdots \to P_0 \to 0 \) and a quasi isomorphism \( \tilde{\mu} : P^{(j)}_\bullet \to M_\bullet \) which is the identity in degrees \( k \) such that \( j+2 \leq k \leq n \) and an epimorphism in the remaining degrees.

We get by induction a totally linear complex: \( P^{(n-1)}_\bullet : 0 \to W_n \to P_{n-1} \to P_{n-2} \to \cdots \to P_0 \to 0 \) with \( P_j \) for \( 0 \leq j \leq n-1 \) finitely generated graded projective modules generated in degree \( j \). There is a quasi-isomorphism \( \mu : P^{(n-1)}_\bullet \to M_\bullet \) such that in each degree the maps are epimorphisms.

As above, we obtain the commutative exact diagram:

\[
\begin{array}{cccc}
0 & 0 & & \\
\downarrow & \downarrow & & \\
0 & \Omega(W_n) & \Omega(W_n) & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & Z'_n & P_n & B_{n-1} \\
\downarrow & \downarrow & \downarrow & \\
0 & Z_n & W_n & B_{n-1} \\
\downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0
\end{array}
\]

Since \( W_n \) has a linear resolution \( \Omega(W_n) \) has a linear resolution \( P^{(n+1)}_\bullet \to \Omega(W_n) \).
It follows $P_{n+1} \to P_n \to P_{n-1} \to P_{n-2} \to \ldots \to P_0 \to 0$ is a linear complex of finitely generated graded projective modules which is quasi-isomorphic to $M_\bullet$ and all the maps in the quasi-isomorphism are epimorphisms.

We see next that for noetherian AS Gorenstein algebras of finite local cohomology any bounded complex can be approximated by a totally linear complex.

**Proposition 3.12.** Let $\Gamma$ be a Koszul algebra AS graded Gorenstein noetherian algebras of finite local cohomology dimension on both sides. Then given a bounded complex $M_\bullet$ of finitely generated graded $\Gamma$-modules, there exists a up to shift totally linear subcomplex $L_\bullet$ such that $M_\bullet/L_\bullet$ is a complex of modules of finite length.

**Proof.** Let $M_\bullet$ be the complex $M_\bullet = \{M_j \mid 0 \leq j \leq n\}$. By Theorem 6, for each $j$ there is a truncation $(M_j)_{\geq n_j}$ such that $(M_j)_{\geq n_j}[n_j]$ is Koszul. Taking $n=\max\{n_j\}$ each $(M_j)_{\geq n}$ is Koszul. Define $L_\bullet = \{L_j \mid L_j = (M_j)_{\geq n+j}\}$. Then $L_\bullet$ is totally linear with $M_\bullet/L_\bullet$ a is a complex of modules of finite length.

We have now the following:

**Lemma 3.13.** Let $\Lambda$ be a Koszul algebra AS graded Gorenstein noetherian algebras of finite local cohomology dimension on both sides with Yoneda algebra $\Gamma$ and $\Phi:gr\Lambda \to lcp_\Gamma$ the linearization functor. Then for any finitely generated module $M$ the complex $\Phi(M)$ is contained in $lcp_\Gamma^{-b}$, this is the homology $H^i(\Phi(M))=0$ for almost all $i$.

**Proof.** According to Theorem 2.23, there is a truncation $M_{\geq s}$ which is Koszul up to shift, and the exact sequence $0 \to M_{\geq s} \to M \to M/M_{\geq s} \to 0$, which induces an exact sequence of complexes $0 \to \Phi(M/M_{\geq s}) \to \Phi(M) \to \Phi(M_{\geq s}) \to 0$ where $\Phi(M/M_{\geq s})$ is a finite complex and $\Phi(M_{\geq s})$ is exact, except at minimal degree, it follows by the long homology sequence that $H^i(\Phi(M))=0$ for almost all $i$.

We remarked above that the categories $D^b(gr\Gamma)$ and $K^-(gr\Gamma)$ are equivalent as triangulated categories, we have proved that the image of $\Phi$ is contained in $K^-(gr\Gamma)$. Composing with the equivalence, we obtain a functor $\Phi':gr\Lambda \to D^b(gr\Gamma)$.

Let $\mathcal{A}$ be an abelian category, a Serre subcategory $\mathcal{T}$ of $\mathcal{A}$ is a full subcategory with the property that for every short exact sequence of $\mathcal{A}$, say, $0 \to A \to B \to C \to 0$ the object $B$ is in $\mathcal{T}$ if and only if $A, C \in \mathcal{T}$. By [7], we have a quotient abelian category $\mathcal{A}/\mathcal{T}$ and an exact functor $\pi: \mathcal{A} \to \mathcal{A}/\mathcal{T}$, which induces at the level of derived categories an exact functor: $D(\pi):D(\mathcal{A}) \to D(\mathcal{A}/\mathcal{T})$. The following result is well known:
Lemma 3.14. [21] The kernel of $D(\pi)$ is the full subcategory $\mathcal{K}$ with objects the complex with homology in $\mathcal{T}$ and $D(\pi)$ induces an equivalence of categories $D^*(A)/\mathcal{K} \cong D^*(A/\mathcal{T})$ for $* = +, -, b$.

We apply the lemma in the following situation:

Let $\Gamma$ be a noetherian Koszul algebra, $\text{gr}\Gamma$ the category of finitely generated graded $\Gamma$-modules. Let $Q\text{gr}\Gamma$ be the quotient category of $\text{gr}\Gamma$ by the Serre subcategory of the modules of finite length. Let $\pi: \text{gr}\Gamma \to Q\text{gr}\Gamma$ be the natural projection and $D(\pi): D^b(\text{gr}\Gamma) \to D^b(Q\text{gr}\Gamma)$ the induced functor. Denote by $\mathcal{F}_\Gamma$ be the full subcategory of $D^b(\text{gr}\Gamma)$ consisting of bounded complexes of graded $\Gamma$-modules of finite length. Then we have:

Theorem 3.15. [17] The functor $D(\pi): D^b(\text{gr}\Gamma) \to D^b(Q\text{gr}\Gamma)$ has kernel $\mathcal{F}_\Gamma$. It induces an equivalence of triangulated categories $\sigma: D^b(\text{gr}\Gamma)/\mathcal{F}_\Gamma \to D^b(Q\text{gr}\Gamma)$.

Let $q: D^b(\text{gr}\Gamma) \to D^b(\text{gr}\Gamma)/\mathcal{F}_\Gamma$ be the quotient functor. Then $\sigma q = D(\pi)$. The functor $j: K^{-b}(\text{gr}\Gamma) \to D^b(\text{gr}\Gamma)$ is truncation, $j$ is an equivalence.

Let $\Lambda$ be a Koszul algebra with Yoneda algebra $\Gamma$ such that both are AS graded Gorenstein noetherian algebras of finite local cohomology dimension on both sides. The functor $\theta: \text{gr}\Lambda \to D^b(Q\text{gr}\Gamma)$ is the composition:

$\text{gr}\Lambda \xrightarrow{\Phi} K^{-b}(\text{gr}\Gamma) \xrightarrow{i} D^b(\text{gr}\Gamma) \xrightarrow{\pi} D^b(Q\text{gr}\Gamma)$,

where $i$ is just the inclusion.

Now let $P$ be a finitely generated projective graded $\Lambda$-module, $P = \bigoplus P_i[n_i]$, with each $P_i$ generated in degree zero. Then $\Phi(P)$ is isomorphic in the category of complexes over $\text{gr}\Gamma$ to $\bigoplus \Phi(P_i)[n_i]$ and each $\Phi(P_i)$ is a projective resolution of a semisimple $\Gamma$-module. It follows $\theta$ sends any map factoring through a graded projective module to a zero map in $D^b(Q\text{gr}\Gamma)$. Consequently, $\theta$ induces a functor $\theta: \text{gr}\Lambda \to D^b(Q\text{gr}\Gamma)$. The functor $\theta$ sends exact sequences to exact triangles, the syzygy functor $\Omega: \text{gr}\Lambda \to \text{gr}\Lambda$ is an endofunctor that makes $\text{gr}\Lambda$ "half" triangulated, given an exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ in $\text{gr}\Lambda$ and $p: P \to C$ the projective cover, there is an induced exact commutative diagram:

$\begin{array}{c}
0 & \to & \Omega(C) & \to & P & \to & C & \to & 0 \\
& & w \downarrow & & \downarrow & & \downarrow 1 \\
0 & \to & A & \to & B & \to & C & \to & 0
\end{array}$

We obtain a half triangle: $\Omega(C) \to A \to B \to C$ and $\theta$ sends the half triangle into a triangle in $D^b(Q\text{gr}\Gamma)$. We want to construct a triangulated
category $\text{gr}_A[\Omega^{-1}]$ such that $\Omega$ is an equivalence which acts as the shift and a functor of half triangulated categories $\lambda: \text{gr}_A \to \text{gr}_A[\Omega^{-1}]$ such that given any triangulated category $D$ and a functor of half triangulated categories: $\beta: \text{gr}_A \to D$ there is a unique functor of triangulated categories $\beta: \text{gr}_A[\Omega^{-1}] \to D$ such that $\hat{\beta}\lambda = \beta$.

This was the approach of Beligiannis in [2].

We recall the construction given in [3] and reproduced in [16].

Let $(A, \phi)$ be a category with endofunctor, if $(B, \psi)$ is another pair, then a functor $F: A \to B$ is said a morphism of pairs if it makes the diagram

\[ A \xrightarrow{\phi} A \\
\downarrow F \quad \downarrow F \\
B \xrightarrow{\psi} B \]

commute, this is: the functors $F\phi$ and $\psi F$ are naturally isomorphic. If $\psi$ happens to be an auto equivalence, we say that the morphism $F$ inverts $\phi$. Then there is a the following universal problem. Given a pair $(A, \phi)$, find a pair $(A[\phi^{-1}], \rho)$ and a morphism of pairs $G: (A, \phi) \to (A[\phi^{-1}], \rho)$ such that $G$ inverts $\phi$ and for any morphism of pairs $F: (A, \phi) \to (B, \psi)$ such that $F$ inverts $\phi$, there is a unique morphism of pairs $F': (A[\phi^{-1}], \rho) \to (B, \psi)$ making the diagram

\[ (A,\phi) \xrightarrow{G} (A[\phi^{-1}], \rho) \xleftarrow{F'} (B, \psi) \]

commute.

The objects of $A[\phi^{-1}]$ are the formal symbols $\phi^{-n}M$ where $M$ is an object of $A$ and $n \geq 0$, $\phi^0M=M$. If $M, N$ are objects in $A[\phi^{-1}]$, we define the morphisms by

\[ \text{Mor}_{A[\phi^{-1}]}(M,N) = \lim_{\to_k} \text{Mor}_A(\phi^kM, \phi^kN) \]

where we assume $M=\phi^{-m}M'$ and $N=\phi^{-n}N'$ and $k \geq \max\{m,n\}$. (See [16] for details)

We define the endofunctor $\rho: A[\phi^{-1}] \to A[\phi^{-1}]$ by setting $\rho(M)=\phi(M)$ and $\rho(\phi^{-n}M)=\phi^{-n+1}(M)$ for any $M$ in $A$ and any natural number $n$. If $f$ is a morphism represented by some $f_n: \phi^nM \to \phi^nN$ and $n$ sufficiently large, then $\rho(f)$ is represented by $\phi(f_n)$.

We obtain the morphism of pairs $G: (A, \phi) \to (A[\phi^{-1}], \rho)$ having the desired properties.
We apply this construction to our pair \((\text{gr}_A, \Omega)\) to obtain a pair \((\text{gr}_A[\Omega^{-1}], \Omega)\) and a map of pairs \(G : (\text{gr}_A, \Omega) \to (\text{gr}_A[\Omega^{-1}], \Omega^{-1})\).

One can check as in [16] or [3] that \((\text{gr}_A[\Omega^{-1}], \Omega^{-1})\) is a triangulated category and \(\theta : \text{gr}_A \to \text{D}^b(\text{gr}_\Gamma)\) induces an exact functor \(\overset{\wedge}{\theta} : \text{gr}_A[\Omega^{-1}] \to \text{D}^b(\text{Qgr}_\Gamma)\) such that the triangle

\[
\begin{array}{ccc}
\text{gr}_A \\
\downarrow \theta \ \\
\overset{\wedge}{\text{gr}}_A[\Omega^{-1}] \\
\end{array}
\]

We now state the main result of the paper.

**Theorem 3.16.** Let \(\Lambda\) be a Koszul algebra with Yoneda algebra \(\Gamma\) such that both are AS graded Gorenstein noetherian algebras of finite local cohomology dimension on both sides. Then the linearization functor

\[\overset{\wedge}{\theta} : \text{gr}_A[\Omega^{-1}] \to \text{D}^b(\text{Qgr}_\Gamma)\]

is a duality of triangulated categories.

**Proof.** We will only check the functor \(\overset{\wedge}{\theta}\) is dense, for the rest of the proof we proceed as in [16].

Choose any bounded complex \(B_*\) of finitely generated graded \(\Gamma\)-modules. By Proposition 3.12, the complex \(B_*\) is isomorphic in \(\text{D}^b(\text{Qgr}_\Gamma)\) to the shift of a totally linear complex, which is in turn, by Proposition 3.11, isomorphic to a linear complex \(P_*\) of finitely generated graded projective \(\Gamma\)-modules with zero homology except for a finite number of indices. By Proposition 3.8, there is a finitely generated graded \(\Lambda\)-module \(M\) such that \(\Phi(M) \cong P_*\). Therefore:

\[\overset{\wedge}{\theta}(M) \cong B_*\] in \(\text{D}^b(\text{Qgr}_\Gamma)\). \(\square\)

**Corollary 3.17.** Let \(\Lambda\) be a Koszul algebra with Yoneda algebra \(\Gamma\) such that both are AS graded Gorenstein noetherian algebras of finite local cohomology dimension on both sides. Then the linearization functor \(\overset{\wedge}{\theta} : \text{gr}_\Gamma[\Omega^{-1}] \to \text{D}^b(\text{Qgr}_\Lambda)\) is a duality of triangulated categories.

**Proof.** It follows by symmetry. \(\square\)

**Acknowledgements**

I express my gratitude to Jun-ichi Miyachi for his criticism and some helpful suggestions.
References


Received: April 28, 2014