Construction of Units in $\mathbb{Z}C_{24}$

Ömer Küsmüş and I. Hakki Denizler

Yuzuncu Yil University, Faculty of Science, Van, Turkey

Abstract

Torsion-free part of the unit group of $\mathbb{Z}C_{24}$ is generated by 5 units.

In this paper, we firstly will give a parametrization of torsion-free units in $\mathbb{Z}C_{24}$ and later we shall characterize the unit group of $\mathbb{Z}C_{24}$ by using some ring extensions.

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1 Introduction

Let $C_n = \langle a : a^n = 1 \rangle$ be a cyclic group of order $n$. One can construct the integral group ring $\mathbb{Z}C_n$. Let $U_1(\mathbb{Z}C_n)$ denote the group of units of augmentation 1 in $\mathbb{Z}C_n$. At first, let us modify the results of Higman on the unit group of integral group ring of a finite abelian group[1].

Theorem 1.1 Let $C_n = \langle a : a^n = 1 \rangle$ be a cyclic group of order $n$. Then, $U_1(\mathbb{Z}C_n) = C_n \times F$ where $F$ is a torsion-free abelian group of finite rank $\rho = \frac{1}{2}(n + n_2 + 1 - 2l)$ where $n$ is order of $C_n$, $n_2$ is the number of elements of order 2 in $C_n$ and $l$ is the number of all cyclic subgroups of $C_n$.

Definition 1.2 The group of symmetric units in $\mathbb{Z}C_n$ is defined by the set

$$U(\mathbb{Z}C_n^+) = \{ \gamma \in U(\mathbb{Z}C_n) : \gamma = \sum_{i=0}^{n-1} \gamma_i a^i, \gamma_i = \gamma_{n-i} \}$$
Theorem 1.3 \( U_1(\mathbb{Z}C_n^+) = C_n \times F^+ \) where \( F^+ \) is a free abelian group of rank \( \rho^+ = \frac{1}{2}\phi(n) - 1 \)

Corollary 1.4 The torsion-free part of unit group of \( \mathbb{Z}C_{24} \) is generated by 3 symmetric and 2 non-symmetric units.

Proof. \( n = 24, n_2 = 1 \) and \( l = 8 \). Then, \( \rho = 5 \). Also, since \( \phi(24) = 8 \), \( \rho^+ = \frac{1}{2}\phi(24) - 1 = 3 \). □

Theorem 1.1 and 1.3 reduces characterization of \( U_1(\mathbb{Z}C_n) \) to construction of generators of \( F \) and \( F^+ \) respectively. We can easily conclude that \( U_1(\mathbb{Z}C_n) \) is trivial if \( n = 1, 2, 3, 4 \) or 6. The torsion-free part of \( U_1(\mathbb{Z}C_n) \) has only one generator for \( n = 5, 8, 12 \). For \( n = 7, 9, 10 \), the torsion-free part of \( U_1(\mathbb{Z}C_n) \) is generated by two generators. Karpilovsky[3] was constructed the unit group of \( U_1(\mathbb{Z}C_n) \) for \( n = 8 \). The torsion-free part of \( U_1(\mathbb{Z}C_n) \) was characterized by Aleev for \( n = 7 \) and 9. Sehgal gave another description of \( U_1(\mathbb{Z}C_n) \) for \( n = 8 \) by using fibre product diagram. Ari constructed the unit group of \( U_1(\mathbb{Z}C_n) \) for \( n = 8 \) by using some classical methods of ring theory. Bilgin[4] gave a structure of the unit group \( U_1(\mathbb{Z}C_n) \) for \( n = 12 \). Also, Kokluce and Kelebek gave a description of \( U_1(\mathbb{Z}C_n) \) for \( n = 7 \) and 9 by using PARI software.

In this paper, we will start with giving a parametrization of torsion-free units in \( U_1(\mathbb{Z}C_n) \) for \( n = 24 \) and later we shall construct the generators of torsion-free part of \( U_1(\mathbb{Z}C_n) \) by using some ring extensions.

2 Parametrization of Torsion-Free Units

Theorem 2.1 Let \( \gamma \) be a unit in \( U_1(\mathbb{Z}C_{24}) \). Then, all parameters of \( \gamma \) can be expressed by 8 free parameters.

Proof. Let \( C_{24} = \langle a \rangle \) be a cyclic group of order 24, \( H_1 = \langle a^{12} \rangle \) and \( H_2 = \langle a^8 \rangle \) be its subgroups of prime order. Consider the following natural group homomorphisms:

\[
\varphi_1 : C_{24} \rightarrow C_{24}/H_1 \quad \text{and} \quad \varphi_2 : C_{24} \rightarrow C_{24}/H_2
\]

If we extend linearly \( \varphi_1 \) and \( \varphi_2 \) over integers, we obtain the following ring homomorphisms:

\[
\tilde{\varphi}_1 : \mathbb{Z}C_{24} \rightarrow \mathbb{Z}(C_{24}/H_1) \quad \text{and} \quad \tilde{\varphi}_2 : \mathbb{Z}C_{24} \rightarrow \mathbb{Z}(C_{24}/H_2)
\]
Let us pick $\gamma \in U_1(\mathbb{Z}C_{24})$ such that $\varphi_1(\gamma) = H_1$ and $\varphi_2(\gamma) = H_2$. Then,

$$
\varphi_1(\gamma) = H_1 \implies \begin{cases}
\gamma_0 + \gamma_{12} = 1 \\
\gamma_i + \gamma_{i+12} = 0
\end{cases}
$$
and

$$
\varphi_2(\gamma) = H_2 \implies \begin{cases}
\gamma_0 + \gamma_8 + \gamma_{16} = 1 \\
\gamma_i + \gamma_{i+8} + \gamma_{i+16} = 0
\end{cases}
$$

Hence, by choosing freely the parameters

$$
\gamma_{12} = p, \gamma_1 = q, \gamma_2 = r, \gamma_3 = s, \gamma_4 = t, \gamma_5 = x, \gamma_6 = y, \gamma_7 = z
$$

we can get all the parameters as in the following table:

<table>
<thead>
<tr>
<th>$\gamma_0 = 1 - p$</th>
<th>$\gamma_6 = y$</th>
<th>$\gamma_{12} = p$</th>
<th>$\gamma_{18} = -y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_1 = q$</td>
<td>$\gamma_7 = z$</td>
<td>$\gamma_{13} = -q$</td>
<td>$\gamma_{19} = -z$</td>
</tr>
<tr>
<td>$\gamma_2 = r$</td>
<td>$\gamma_8 = p + t$</td>
<td>$\gamma_{14} = -r$</td>
<td>$\gamma_{20} = -p - t$</td>
</tr>
<tr>
<td>$\gamma_3 = s$</td>
<td>$\gamma_9 = x - q$</td>
<td>$\gamma_{15} = -s$</td>
<td>$\gamma_{21} = q - x$</td>
</tr>
<tr>
<td>$\gamma_4 = t$</td>
<td>$\gamma_{10} = y - r$</td>
<td>$\gamma_{16} = -t$</td>
<td>$\gamma_{22} = r - y$</td>
</tr>
<tr>
<td>$\gamma_5 = x$</td>
<td>$\gamma_{11} = z - s$</td>
<td>$\gamma_{17} = -x$</td>
<td>$\gamma_{23} = s - z$</td>
</tr>
</tbody>
</table>

That is, all paramaters of a torsion-free unit $\gamma = \sum_{i=0}^{23} \gamma_i a^i$ in $\mathbb{Z}C_{24}$ can be expressed in terms of free parameters: $\gamma_{12}, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7$. Q.E.D.

**Corollary 2.2** Let $C_{24} = \langle a \rangle$ be a cyclic group of order 24, $H_1 = \langle a^{12} \rangle$ and $H_2 = \langle a^8 \rangle$ be its subgroups of prime order. Then,

$$
\triangle(C_{24}, H_1) \cap \triangle(C_{24}, H_2) = \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2, \beta_3 \rangle
$$

where $\alpha_j = a^j(1 - a^{12})(1 - a^8)$ and $\beta_j = a^{j+4}(1 - a^{12})(1 + a^4)$ for $j = 0, 1, 2, 3$.

**Proof.** Let $\gamma \in U_1(\mathbb{Z}C_{24})$ such that $\varphi_1(\gamma) = H_1$ and $\varphi_2(\gamma) = H_2$. Then, $\varphi_1(\gamma - 1) = 0$ and $\varphi_2(\gamma - 1) = 0$. That is,

$$
\gamma - 1 \in \text{Ker}(\varphi_1) \cap \text{Ker}(\varphi_2) = \triangle(C_{24}, H_1) \cap \triangle(C_{24}, H_2)
$$

By the table in Theorem 2.1, we can form a unit in $\mathbb{Z}C_{24}$ as follows

$$
\gamma = 1 - p + qa + ra^2 + sa^3 + ta^4 + xa^5 + ya^6 + za^7 + (p + t)a^8 + (x - q)a^9 \\
+ (y - r)a^{10} + (z - s)a^{11} + pa^{12} - qa^{13} - ra^{14} - sa^{15} - ta^{16} - xa^{17} \\
- ya^{18} - za^{19} - (p + t)a^{20} - (x - q)a^{21} - (y - r)a^{22} - (z - s)a^{23}
$$

Then,

$$
\gamma - 1 = -p + qa + ra^2 + sa^3 + ta^4 + xa^5 + ya^6 + za^7 + (p + t)a^8 \\
+ (x - q)a^9 + (y - r)a^{10} + (z - s)a^{11} + pa^{12} - qa^{13} - ra^{14} \\
- sa^{15} - ta^{16} - xa^{17} - ya^{18} - za^{19} - (p + t)a^{20} \\
- (x - q)a^{21} - (y - r)a^{22} - (z - s)a^{23}
$$
If we arrange all the parameters with respect to \( p, q, r, s, t, x, y, z \), we get the form
\[
\gamma - 1 = -p(1 - a^8 - a^{12} + a^{20}) + q(a - a^9 - a^{13} + a^{21})
+ r(a^2 - a^{10} - a^{14} + a^{22}) + s(a^3 - a^{11} - a^{15} + a^{23})
+ t(a^4 + a^8 - a^{16} - a^{20}) + x(a^5 + a^9 - a^{17} - a^{21})
+ y(a^6 + a^{10} - a^{18} - a^{22}) + z(a^7 + a^{11} - a^{19} - a^{23})
\]

Let us write once more regular the terms. Then,
\[
\gamma - 1 = -p(1 - a^{12})(1 - a^8) + qa(1 - a^{12})(1 - a^8) + ra^2(1 - a^{12})(1 - a^8)
+ sa^2(1 - a^{12})(1 - a^8) + ta^4(1 - a^{12})(1 + a^4) + xa^5(1 - a^{12})(1 + a^4)
+ ya^6(1 - a^{12})(1 + a^4) + za^7(1 - a^{12})(1 + a^4)
\]

Thus, if we denote \( \alpha_j = a^j(1 - a^{12})(1 - a^8) \) and \( \beta_j = a^{j+4}(1 - a^{12})(1 + a^4) \) for \( j = 0, 1, 2, 3 \) we conclude that
\[
\triangle(C_{24}, H_1) \cap \triangle(C_{24}, H_2) = \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2, \beta_3 \rangle
\]
as required. □

### 3 Characterization of Units

Now, we are ready to state and prove our main result in this paper. This result gives us an explicit characterization of units in \( \mathbb{Z}C_{24} \). Let \( u^+ \) and \( u^- \) denote symmetric and non-symmetric units respectively.

**Theorem 3.1**

\[
U_1(\mathbb{Z}C_{24}) = C_{24} \times <u^+_1, u^+_2, u^+_3, u^-_4, u^-_5>
\]

where
\[
\begin{align*}
\alpha^+_1 &= -1 - a^3 - a^6 + a^{12} + 2a^{15} + a^{18} \\
\alpha^+_2 &= 3 + 2a^2 + a^4 - a^8 - 2a^{10} - 2a^{12} - 2a^{14} + a^2 + a^{20} + 2a^{22} \\
\beta^-_4 &= 1 + 2a_0 + 2a_2 + \beta_0 \\
\beta^-_5 &= 1 - 2a_0 + \alpha_1 + \beta_0 + \beta_3
\end{align*}
\]

such that \( \alpha_j = a^j(1 - a^{12})(1 - a^8) \) and \( \beta_j = a^{j+4}(1 - a^{12})(1 + a^4), \quad (j = 0, 1, 2, 3) \).

**Proof.** First, let us consider \( \varepsilon = e^{\frac{2\pi i}{24}} = \cos\left(\frac{2\pi}{24}\right) + isin\left(\frac{2\pi}{24}\right) = \frac{\sqrt{6} + \sqrt{2}}{4} + i\frac{\sqrt{6} - \sqrt{2}}{4} \) and the following group isomorphism.
\[
\psi : <a> \quad \longrightarrow \quad <\varepsilon> \quad \left\{\begin{array}{l}
a^i \quad \mapsto \quad \varepsilon^i
\end{array}\right\}
\]
If we extend linearly ψ over \( \mathbb{Z} \), we get the following ring homomorphism:

\[
\bar{\psi} : \mathbb{Z} < a > \quad \longrightarrow \quad \mathbb{Z}[\varepsilon] \quad \quad \sum \gamma_k a^k \quad \longmapsto \quad \sum \gamma_k \varepsilon^k
\]

Now, let us take the images of \( \alpha_j \)'s and \( \beta_j \)'s under \( \bar{\psi} \). Then, we get the following image table:

| \( \bar{\psi}(\alpha_0) \) | \( \bar{\psi}(1 - a^8)\bar{\psi}(1 - a^{12}) = [1 - (\frac{-1+\sqrt{3}i}{2})][1 - (-1)] = 3 - \sqrt{3}i \) |
| \( \bar{\psi}(\alpha_1) \) | \( \bar{\psi}(a)\bar{\psi}(\alpha_0) = \frac{\sqrt{6} + 3\sqrt{2}}{2} + i(\frac{\sqrt{6} - \sqrt{2}}{2}) \) |
| \( \bar{\psi}(\alpha_2) \) | \( \bar{\psi}(a^2)\bar{\psi}(\alpha_0) = [\frac{\sqrt{4} + 1}{2}](3 - \sqrt{3}i) = 2\sqrt{3} \) |
| \( \bar{\psi}(\alpha_3) \) | \( \bar{\psi}(a^3)\bar{\psi}(\alpha_0) = [\frac{\sqrt{2} + i\sqrt{2}}{2}](3 - \sqrt{3}i) = \frac{3\sqrt{2} + \sqrt{6}}{2} + i(\frac{3\sqrt{2} - \sqrt{6}}{2}) \) |
| \( \bar{\psi}(\beta_0) \) | \( \bar{\psi}(a^4)\bar{\psi}(1 + a^4)\bar{\psi}(1 - a^{12}) = (\frac{1+\sqrt{3}i}{2})(\frac{3+\sqrt{3}i}{2}).2 = 2\sqrt{3}i \) |
| \( \bar{\psi}(\beta_1) \) | \( \bar{\psi}(a^5)\bar{\psi}(1 + a^4)\bar{\psi}(1 - a^{12}) = \frac{-3\sqrt{2} + \sqrt{6}}{2} + i(\frac{3\sqrt{2} + \sqrt{6}}{2}) \) |
| \( \bar{\psi}(\beta_2) \) | \( \bar{\psi}(a^6)\bar{\psi}(1 + a^4)\bar{\psi}(1 - a^{12}) = i(\frac{3+\sqrt{3}i}{2}).2 = -\sqrt{3} + 3i \) |
| \( \bar{\psi}(\beta_3) \) | \( \bar{\psi}(a^7)\bar{\psi}(1 + a^4)\bar{\psi}(1 - a^{12}) = -\sqrt{6} + i\sqrt{6} \) |

Let us take a unit by using Corollary 2.2 as follows:

\[ u = 1 - p\alpha_0 + q\alpha_1 + r\alpha_2 + s\alpha_3 + t\beta_0 + x\beta_1 + y\beta_2 + z\beta_3 \]

Then, we get the image of \( u \) under \( \bar{\psi} \) as follows:

\[ \bar{\psi}(u) = 1 - p\bar{\psi}(\alpha_0) + q\bar{\psi}(\alpha_1) + r\bar{\psi}(\alpha_2) + s\bar{\psi}(\alpha_3) + t\bar{\psi}(\beta_0) + x\bar{\psi}(\beta_1) + y\bar{\psi}(\beta_2) + z\bar{\psi}(\beta_3) \]

By considering the above equalities, we clearly get the image:

\[
\bar{\psi}(u) = (1 - 3p) + \frac{3}{2}(q + s - x)\sqrt{2} + (2r - y)\sqrt{3} + \frac{1}{2}(q + s + x - 2z)\sqrt{6} + 3yi + \frac{3}{2}(-q + s + x)i\sqrt{2} + (p + 2t)i\sqrt{3} + \frac{1}{2}(q - s + x + 2z)i\sqrt{6}
\]

Since the parameters of imaginary terms must be zero, the followings are obtained:

\[
\begin{align*}
\begin{cases}
y = 0 \\
-q + s + x = 0 \\
p + 2t = 0 \\
q - s + x + 2z = 0
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
y = 0 \\
p = -2t \\
q = s + x \\
x = -z
\end{cases}
\end{align*}
\]
Therefore, we conclude that \( \tilde{\psi}(u) \) is contained in unit group of the ring extensions \( Z[\sqrt{2}], Z[\sqrt{3}] \) or \( Z[\sqrt{6}] \). Now, let us investigate these cases respectively.

**Case 1:**
\[ \tilde{\psi}(u) \in U(Z[\sqrt{2}]) = (1 + \sqrt{2})^k \implies 2r = 0, s + 2x = 0 \]
\[ \implies \tilde{\psi}(u) = (1 + 6t) + 3s\sqrt{2} = -17 + 12\sqrt{2} \]
\[ \implies r = 0, s = 4, t = -3, x = -2 \]

So, \( u_1 = 1 - 6\alpha_0 + 2\alpha_1 + 4\alpha_3 - 3\beta_0 - 2\beta_1 + 2\beta_7 \).

**Case 2:**
\[ \tilde{\psi}(u) \in U(Z[\sqrt{3}]) = (2 + \sqrt{3})^k \implies 3s = 0, s + 2x = 0 \]
\[ \implies \tilde{\psi}(u) = (1 + 6t) + 2r\sqrt{3} = 7 + 4\sqrt{3} \]
\[ \implies r = 2, s = 0, t = 1, x = 0 \]

Hence, \( u_2 = 1 + 2\alpha_0 + 2\alpha_2 + \beta_0 \).

**Case 3:**
\[ \tilde{\psi}(u) \in U(Z[\sqrt{6}]) = (5 + 2\sqrt{6})^k \implies 3s = 0, 2r = 0 \]
\[ \implies \tilde{\psi}(u) = (1 + 6t) + 2x\sqrt{6} = -5 + 2\sqrt{6} \]
\[ \implies r = 0, s = 0, t = -1, x = 1 \]

Thus, \( u_3 = 1 - 2\alpha_0 + \alpha_1 - \beta_0 + \beta_1 - \beta_3 \). If we pay attention, we can easily see that the unit is obtained in the case 1 is symmetric. Let us denote this unit by \( u_3^+ = 1 - 6\alpha_0 + 2\alpha_1 + 4\alpha_3 - 3\beta_0 - 2\beta_1 + 2\beta_7 \).

Whereas, we can easily see that the units are obtained in the cases 2 and 3 are non-symmetric. Let us denote these units by \( u_4^- \) and \( u_5^- \) respectively. Now, we give some important remarks to obtain the other symmetric units.

**Lemma 3.2** [3] Since \( C_8 = \langle x : x^8 = 1 \rangle \),

\[ U_1(ZC_8) = C_8 \times < -1 - x - x^2 + x^4 + 2x^5 + x^6 > \]

**Lemma 3.3** [4] Since \( C_{12} = \langle x : x^{12} = 1 \rangle \),

\[ U_1(ZC_{12}) = C_{12} \times < 3 + 2x + x^2 - x^4 - 2x^5 - 2x^6 - 2x^7 - x^8 + x^{10} + 2x^{11} > \]
if we define an embedding \( < a^3 > \hookrightarrow < a > \) by \( a^i \mapsto a^{3i} \) we can obtain a symmetric generator of \( U_1(ZC_{24}) \) as follows:

\[ u_1^+ = -1 - a^3 - a^6 + a^{12} + 2a^{15} + a^{18} \]

In a similar way, if we define \( < a^2 > \hookrightarrow < a > \) by \( a^i \mapsto a^{2i} \) we get the second symmetric generator of \( U_1(ZC_{24}) \) as below:

\[ u_2^+ = 3 + 2a^2 + a^4 - a^8 - 2a^{10} - 2a^{12} - 2a^{14} - a^{16} + a^{20} + 2a^{22} \]

Hence, the required is obtained. □
References


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