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Construction of Units in $\mathbb{Z}C_{24}$

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Abstract

Torsion-free part of the unit group of $\mathbb{Z}C_{24}$ is generated by 5 units. In this paper, we firstly will give a parametrization of torsion-free units in $\mathbb{Z}C_{24}$ and later we shall characterize the unit group of $\mathbb{Z}C_{24}$ by using some ring extensions.

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1 Introduction

Let $C_n =< a : a^n = 1 >$ be a cyclic group of order n. One can construct the integral group ring $\mathbb{Z}C_n$. Let $U_1(\mathbb{Z}C_n)$ denote the group of units of augmentation 1 in $\mathbb{Z}C_n$. At first, let us modify the results of Higman on the unit group of integral group ring of a finite abelian group[1].

Theorem 1.1 *Let $C_n =< a : a^n = 1 >$ be a cyclic group of order n. Then, $U_1(\mathbb{Z}C_n) = C_n \times F$ where F is a torsion-free abelian group of finite rank $\rho = \frac{1}{2}(n + n_2 + 1 - 2l)$ where n is order of C_n, n_2 is the number of elements of order 2 in C_n and l is the number of all cyclic subgroups of C_n.*

Definition 1.2 *The group of symmetric units in $\mathbb{Z}C_n$ is defined by the set*

$$U(\mathbb{Z}C_n^+) = \{\gamma \in U(\mathbb{Z}C_n) : \gamma = \sum_{i=0}^{n-1} \gamma_i a^i, \gamma_i = \gamma_{n-i}\}$$

Theorem 1.3 *$U_1(\mathbb{Z}C_n^+) = C_n \times F^+$ where F^+ is a free abelian group of rank $\rho^+ = \frac{1}{2}\varphi(n) - 1$*

Corollary 1.4 *The torsion-free part of unit group of $\mathbb{Z}C_{24}$ is generated by 3 symmetric and 2 non-symmetric units.*

Proof. $n = 24$, $n_2 = 1$ and $l = 8$. Then, $\rho = 5$. Also, since $\phi(24) = 8$, $\rho^+ = \frac{1}{2}\phi(24) - 1 = 3.\square$

Theorem 1.1 and 1.3 reduces characterization of $U_1(\mathbb{Z}C_n)$ to construction of generators of F and F^+ respectively. We can easily conclude that $U_1(\mathbb{Z}C_n)$ is trivial if $n = 1, 2, 3, 4$ or 6. The torsion-free part of $U_1(\mathbb{Z}C_n)$ has only one generator for $n = 5, 8, 12$. For $n = 7, 9, 10$, the torsion-free part of $U_1(\mathbb{Z}C_n)$ is generated by two generators. Karpilovsky[3] was constructed the unit group of $U_1(\mathbb{Z}C_n)$ for $n = 8$. The torsion-free part of $U_1(\mathbb{Z}C_n)$ was characterized by Aleev for $n = 7$ and 9. Sehgal gave another description of $U_1(\mathbb{Z}C_n)$ for $n = 8$ by using fibre product diagram. Ari constructed the unit group of $U_1(\mathbb{Z}C_n)$ for $n = 8$ by using some classical methods of ring theory. Bilgin[4] gave a structure of the unit group $U_1(\mathbb{Z}C_n)$ for $n = 12$. Also, Kokluce and Kelebek gave a description of $U_1(\mathbb{Z}C_n)$ for $n = 7$ and 9 by using PARI software.

In this paper, we will start with giving a parametrization of torsion-free units in $U_1(\mathbb{Z}C_n)$ for $n = 24$ and later we shall construct the generators of torsion-free part of $U_1(\mathbb{Z}C_n)$ by using some ring extensions.

2 Parametrization of Torsion-Free Units

Theorem 2.1 *Let γ be a unit in $U_1(\mathbb{Z}C_{24})$. Then, all parameters of γ can be expressed by 8 free parameters.*

Proof. Let $C_{24} =< a >$ be a cyclic group of order 24, $H_1 =< a^{12} >$ and $H_2 =< a^8 >$ be its subgroups of prime order. Consider the following natural group homomorphisms:

$$\left.\begin{array}{rcl} \varphi_1 : C_{24} & \longrightarrow & C_{24}/H_1 \\ a^i & \mapsto & a^i H_1 \end{array}\right\} \quad \text{and} \quad \left.\begin{array}{rcl} \varphi_2 : C_{24} & \longrightarrow & C_{24}/H_2 \\ a^i & \mapsto & a^i H_2 \end{array}\right\}$$

If we extend linearly φ_1 and φ_2 over integers, we obtain the following ring homomorphisms:

$$\left.\begin{array}{rccc} \bar{\varphi_1} : & \mathbb{Z}C_{24} & \longrightarrow & \mathbb{Z}(C_{24}/H_1) \\ & \sum_{i=0}^{23} \gamma_i a^i & \mapsto & \sum_{i=1}^{23} \gamma_i (a^i H_1) \end{array}\right\}$$

$$\left.\begin{array}{rccc} \bar{\varphi_2} : & \mathbb{Z}C_{24} & \longrightarrow & \mathbb{Z}(C_{24}/H_2) \\ & \sum_{i=0}^{23} \gamma_i a^i & \mapsto & \sum_{i=1}^{23} \gamma_i (a^i H_2) \end{array}\right\}$$

Let us pick $\gamma \in U_1(\mathbb{Z}C_{24})$ such that $\bar{\varphi_1}(\gamma) = H_1$ and $\bar{\varphi_2}(\gamma) = H_2$. Then,

$$\bar{\varphi_1}(\gamma) = H_1 \Longrightarrow \begin{cases} \gamma_0 + \gamma_{12} = 1 \\ \gamma_i + \gamma_{i+12} = 0 \end{cases} \quad i = 1, 2, ..., 11 \text{ and}$$

$$\bar{\varphi_2}(\gamma) = H_2 \Longrightarrow \begin{cases} \gamma_0 + \gamma_8 + \gamma_{16} = 1 \\ \gamma_i + \gamma_{i+8} + \gamma_{i+16} = 0 \end{cases} \quad i = 1, 2, ..., 7$$

Hence,by choosing freely the parameters

$$\gamma_{12} = p, \gamma_1 = q, \gamma_2 = r, \gamma_3 = s, \gamma_4 = t, \gamma_5 = x, \gamma_6 = y, \gamma_7 = z$$

we can get all the parameters as in the following table:

$\gamma_0 = 1 - p$	$\gamma_6 = y$	$\gamma_{12} = p$	$\gamma_{18} = -y$
$\gamma_1 = q$	$\gamma_7 = z$	$\gamma_{13} = -q$	$\gamma_{19} = -z$
$\gamma_2 = r$	$\gamma_8 = p + t$	$\gamma_{14} = -r$	$\gamma_{20} = -p - t$
$\gamma_3 = s$	$\gamma_9 = x - q$	$\gamma_{15} = -s$	$\gamma_{21} = q - x$
$\gamma_4 = t$	$\gamma_{10} = y - r$	$\gamma_{16} = -t$	$\gamma_{22} = r - y$
$\gamma_5 = x$	$\gamma_{11} = z - s$	$\gamma_{17} = -x$	$\gamma_{23} = s - z$

That is, all paramaters of a torsion-free unit $\gamma = \sum_{i=0}^{23} \gamma_i a^i$ in $\mathbb{Z}C_{24}$ can be expressed in terms of free parameters: $\gamma_{12}, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7$.□

Corollary 2.2 *Let* $C_{24} =< a >$ *be a cyclic group of order 24,* $H_1 =< a^{12} >$ *and* $H_2 =< a^8 >$ *be its subgroups of prime order. Then,*

$$\triangle(C_{24}, H_1) \cap \triangle(C_{24}, H_2) =< \alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2, \beta_3 >$$

where $\alpha_j = a^j(1 - a^{12})(1 - a^8)$ *and* $\beta_j = a^{j+4}(1 - a^{12})(1 + a^4)$ *for* $j = 0, 1, 2, 3.$

Proof. Let $\gamma \in U_1(\mathbb{Z}C_{24})$ such that $\bar{\varphi_1}(\gamma) = H_1$ and $\bar{\varphi_2}(\gamma) = H_2$. Then, $\bar{\varphi_1}(\gamma - 1) = 0$ and $\bar{\varphi_2}(\gamma - 1) = 0$. That is,

$$\gamma - 1 \in Ker(\bar{\varphi_1}) \cap Ker(\bar{\varphi_2}) = \triangle(C_{24}, H_1) \cap \triangle(C_{24}, H_2)$$

By the table in Theorem 2.1, we can form a unit in $\mathbb{Z}C_{24}$ as follows

$$\begin{aligned} \gamma \;=\; & 1 - p + qa + ra^2 + sa^3 + ta^4 + xa^5 + ya^6 + za^7 + (p + t)a^8 + (x - q)a^9 \\ & +(y - r)a^{10} + (z - s)a^{11} + pa^{12} - qa^{13} - ra^{14} - sa^{15} - ta^{16} - xa^{17} \\ & -ya^{18} - za^{19} - (p + t)a^{20} - (x - q)a^{21} - (y - r)a^{22} - (z - s)a^{23} \end{aligned}$$

Then,

$$\begin{aligned} \gamma - 1 \;=\; & -p + qa + ra^2 + sa^3 + ta^4 + xa^5 + ya^6 + za^7 + (p + t)a^8 \\ & +(x - q)a^9 + (y - r)a^{10} + (z - s)a^{11} + pa^{12} - qa^{13} - ra^{14} \\ & -sa^{15} - ta^{16} - xa^{17} - ya^{18} - za^{19} - (p + t)a^{20} \\ & -(x - q)a^{21} - (y - r)a^{22} - (z - s)a^{23} \end{aligned}$$

If we arrange all the parameters with respect to p, q, r, s, t, x, y, z, we get the form

$$\begin{aligned}\gamma - 1 &= -p(1-a^8-a^{12}+a^{20}) + q(a-a^9-a^{13}+a^{21}) \\ &+r(a^2-a^{10}-a^{14}+a^{22}) + s(a^3-a^{11}-a^{15}+a^{23}) \\ &+t(a^4+a^8-a^{16}-a^{20}) + x(a^5+a^9-a^{17}-a^{21}) \\ &+y(a^6+a^{10}-a^{18}-a^{22}) + z(a^7+a^{11}-a^{19}-a^{23})\end{aligned}$$

Let us write once more regular the terms. Then,

$$\begin{aligned}\gamma - 1 &= -p(1-a^{12})(1-a^8) + qa(1-a^{12})(1-a^8) + ra^2(1-a^{12})(1-a^8) \\ &+sa^3(1-a^{12})(1-a^8) + ta^4(1-a^{12})(1+a^4) + xa^5(1-a^{12})(1+a^4) \\ &+ya^6(1-a^{12})(1+a^4) + za^7(1-a^{12})(1+a^4)\end{aligned}$$

Thus, if we denote $\alpha_j = a^j(1-a^{12})(1-a^8)$ and $\beta_j = a^{j+4}(1-a^{12})(1+a^4)$ for $j = 0, 1, 2, 3$ we conclude that

$$\triangle(C_{24}, H_1) \cap \triangle(C_{24}, H_2) =< \alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2, \beta_3 >$$

as required.□

3 Characterization of Units

Now, we are ready to state and prove our main result in this paper. This result gives us an explicit characterization of units in $\mathbb{Z}C_{24}$. Let u^+ and u^- denote symmetric and non-symmetric units respectively.

Theorem 3.1

$$U_1(\mathbb{Z}C_{24}) = C_{24} \times < u_1^+, u_2^+, u_3^+, u_4^-, u_5^- >$$

where

$$\begin{aligned}u_1^+ &= -1 - a^3 - a^6 + a^{12} + 2a^{15} + a^{18} \\ u_2^+ &= 3 + 2a^2 + a^4 - a^8 - 2a^{10} - 2a^{12} - 2a^{14} - a^{16} + a^{20} + 2a^{22} \\ u_3^+ &= 1 - 6\alpha_0 + 2\alpha_1 + 4\alpha_3 - 3\beta_0 - 2\beta_1 + 2\beta_3 \\ u_4^- &= 1 + 2\alpha_0 + 2\alpha_2 + \beta_0 \\ u_5^- &= 1 - 2\alpha_0 + \alpha_1 - \beta_0 + \beta_1 - \beta_3\end{aligned}$$

such that $\alpha_j = a^j(1-a^{12})(1-a^8)$ *and* $\beta_j = a^{j+4}(1-a^{12})(1+a^4)$, $(j = 0, 1, 2, 3)$.

Proof. First, let us consider $\varepsilon = e^{\frac{2\pi i}{24}} = cos(\frac{2\pi}{24}) + isin(\frac{2\pi}{24}) = \frac{\sqrt{6}+\sqrt{2}}{4} + i\frac{\sqrt{6}-\sqrt{2}}{4}$ and the following group isomorphism.

$$\left.\begin{array}{ccc}\psi :< a > & \longrightarrow & < \varepsilon > \\ a^i & \mapsto & \varepsilon^i\end{array}\right\}$$

If we extend linearly ψ over $\mathbb{Z}$, we get the following ring homomorphism:

$$\left.\begin{array}{ccc} \bar{\psi}:\mathbb{Z}<a> & \longrightarrow & \mathbb{Z}[\varepsilon] \\ \sum \gamma_i a^i & \mapsto & \sum \gamma_i \varepsilon^i \end{array}\right\}$$

Now, let us take the images of α_j's and β_j's under $\bar{\psi}$. Then, we get the following image table:

$\bar{\psi}(\alpha_0) = \bar{\psi}(1-a^8)\bar{\psi}(1-a^{12}) = [1-(\frac{-1+\sqrt{3}i}{2})][1-(-1)] = 3-\sqrt{3}i$
$\bar{\psi}(\alpha_1) = \bar{\psi}(a)\bar{\psi}(\alpha_0) = \frac{\sqrt{6}+3\sqrt{2}}{2} + i\frac{(\sqrt{6}-3\sqrt{2})}{2}$
$\bar{\psi}(\alpha_2) = \bar{\psi}(a^2)\bar{\psi}(\alpha_0) = [\frac{\sqrt{3}+i}{2}](3-\sqrt{3}i) = 2\sqrt{3}$
$\bar{\psi}(\alpha_3) = \bar{\psi}(a^3)\bar{\psi}(\alpha_0) = [\frac{\sqrt{2}+i\sqrt{2}}{2}](3-\sqrt{3}i) = \frac{3\sqrt{2}+\sqrt{6}}{2} + i\frac{(3\sqrt{2}-\sqrt{6})}{2}$
$\bar{\psi}(\beta_0) = \bar{\psi}(a^4)\bar{\psi}(1+a^4)\bar{\psi}(1-a^{12}) = (\frac{1+\sqrt{3}i}{2})(\frac{3+\sqrt{3}i}{2}).2 = 2\sqrt{3}i$
$\bar{\psi}(\beta_1) = \bar{\psi}(a^5)\bar{\psi}(1+a^4)\bar{\psi}(1-a^{12}) = \frac{-3\sqrt{2}+\sqrt{6}}{2} + i\frac{(3\sqrt{2}+\sqrt{6})}{2}$
$\bar{\psi}(\beta_2) = \bar{\psi}(a^6)\bar{\psi}(1+a^4)\bar{\psi}(1-a^{12}) = i.(\frac{3+\sqrt{3}i}{2}).2 = -\sqrt{3}+3i$
$\bar{\psi}(\beta_3) = \bar{\psi}(a^7)\bar{\psi}(1+a^4)\bar{\psi}(1-a^{12}) = -\sqrt{6}+i\sqrt{6}$

Let us take a unit by using Corollary 2.2 as follows:

$$u = 1 - p\alpha_0 + q\alpha_1 + r\alpha_2 + s\alpha_3 + t\beta_0 + x\beta_1 + y\beta_2 + z\beta_3$$

Then, we get the image of u under $\bar{\psi}$ as follows:

$$\bar{\psi}(u) = 1 - p\bar{\psi}(\alpha_0) + q\bar{\psi}(\alpha_1) + r\bar{\psi}(\alpha_2) + s\bar{\psi}(\alpha_3) + t\bar{\psi}(\beta_0) + x\bar{\psi}(\beta_1) + y\bar{\psi}(\beta_2) + z\bar{\psi}(\beta_3)$$

By considering the above equalities, we clearly get the image:

$$\begin{aligned} \bar{\psi}(u) \;=\; & (1-3p) + \tfrac{3}{2}(q+s-x)\sqrt{2} + (2r-y)\sqrt{3} + \tfrac{1}{2}(q+s+x-2z)\sqrt{6} \\ & +3y.i + \tfrac{3}{2}(-q+s+x)i\sqrt{2} + (p+2t)i\sqrt{3} + \tfrac{1}{2}(q-s+x+2z)i\sqrt{6} \end{aligned}$$

Since the parameters of imaginary terms must be zero, the followings are obtained:

$$\left.\begin{array}{rcl} y & = & 0 \\ -q+s+x & = & 0 \\ p+2t & = & 0 \\ q-s+x+2z & = & 0 \end{array}\right\} \Longrightarrow \begin{array}{rcl} y & = & 0 \\ p & = & -2t \\ q & = & s+x \\ x & = & -z \end{array}$$

So, $\bar{\psi}(u)$ can be shortly expressed as follows:

$$\bar{\psi}(u) = (1+6t) + 3s\sqrt{2} + 2r\sqrt{3} + (s+2x)\sqrt{6}$$

Therefore, we conclude that $\bar{\psi}(u)$ is contained in unit group of the ring extensions $\mathbb{Z}[\sqrt{2}], \mathbb{Z}[\sqrt{3}]$ or $\mathbb{Z}[\sqrt{6}]$. Now, let us investigate these cases respectively.

Case 1:

$$\begin{aligned} \bar{\psi}(u) \in U(\mathbb{Z}[\sqrt{2}]) = (1+\sqrt{2})^k &\implies 2r = 0, s + 2x = 0 \\ &\implies \bar{\psi}(u) = (1+6t) + 3s\sqrt{2} = -17 + 12\sqrt{2} \\ &\implies r = 0, s = 4, t = -3, x = -2 \end{aligned}$$

So, $u_1 = 1 - 6\alpha_0 + 2\alpha_1 + 4\alpha_3 - 3\beta_0 - 2\beta_1 + 2\beta_7$.

Case 2:

$$\begin{aligned} \bar{\psi}(u) \in U(\mathbb{Z}[\sqrt{3}]) = (2+\sqrt{3})^k &\implies 3s = 0, s + 2x = 0 \\ &\implies \bar{\psi}(u) = (1+6t) + 2r\sqrt{3} = 7 + 4\sqrt{3} \\ &\implies r = 2, s = 0, t = 1, x = 0 \end{aligned}$$

Hence, $u_2 = 1 + 2\alpha_0 + 2\alpha_2 + \beta_0$.

Case 3:

$$\begin{aligned} \bar{\psi}(u) \in U(\mathbb{Z}[\sqrt{6}]) = (5+2\sqrt{6})^k &\implies 3s = 0, 2r = 0 \\ &\implies \bar{\psi}(u) = (1+6t) + 2x\sqrt{6} = -5 + 2\sqrt{6} \\ &\implies r = 0, s = 0, t = -1, x = 1 \end{aligned}$$

Thus, $u_3 = 1 - 2\alpha_0 + \alpha_1 - \beta_0 + \beta_1 - \beta_3$. If we pay attention, we can easily see that the unit is obtained in the case 1 is symmetric. Let us denote this unit by

$$u_3^+ = 1 - 6\alpha_0 + 2\alpha_1 + 4\alpha_3 - 3\beta_0 - 2\beta_1 + 2\beta_7$$

Whereas, we can easily see that the units are obtained in the cases 2 and 3 are non-symmetric. Let us denote these units by u_4^- and u_5^- respectively. Now, we give some important remarks to obtain the other symmetric units.

Lemma 3.2 *[3] Since $C_8 =< x : x^8 = 1 >$,*

$$U_1(\mathbb{Z}C_8) = C_8 \times < -1 - x - x^2 + x^4 + 2x^5 + x^6 >$$

Lemma 3.3 *[4] Since $C_{12} =< x : x^{12} = 1 >$,*

$$U_1(\mathbb{Z}C_{12}) = C_{12} \times < 3 + 2x + x^2 - x^4 - 2x^5 - 2x^6 - 2x^7 - x^8 + x^{10} + 2x^{11} >$$

if we define an embedding $< a^3 > \hookrightarrow < a >$ by $a^i \mapsto a^{3i}$ we can obtain a symmetric generator of $U_1(\mathbb{Z}C_{24})$ as follows:

$$u_1^+ = -1 - a^3 - a^6 + a^{12} + 2a^{15} + a^{18}$$

In a similar way, if we define $< a^2 > \hookrightarrow < a >$ by $a^i \mapsto a^{2i}$ we get the second symmetric generator of $U_1(\mathbb{Z}C_{24})$ as below:

$$u_2^+ = 3 + 2a^2 + a^4 - a^8 - 2a^{10} - 2a^{12} - 2a^{14} - a^{16} + a^{20} + 2a^{22}$$

Hence, the required is obtained.□

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