Finite Groups Having Exactly 28 Elements of Maximal Order$^1$

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Abstract

Let $G$ be a finite group, $M(G)$ denotes the number of elements of maximal order of $G$. In this note a finite group $G$ with $M(G) = 28$ is determined.

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1 Introduction

All groups considered are finite. In this paper, $S_p$ denotes Sylow p-subgroup of $G$, $k$ denotes the maximal order of elements in $G$ and $A \rtimes B$ denotes the semidirect product of $A$ and $B$. For some natural number $m$ and $n$, $C_n^m$ always denotes the direct product of $m$ cyclic groups of order $n$.

For convenience, in the whole paper we always set:

$$G_1 = \langle a, b, c | a^4 = b^4 = 1, a^2 = c^2, [a, c] = a^2, [b, c] = a^2, [a, b] = 1 \rangle;$$

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\[ G_2 = \langle a, b, c | a^4 = b^4 = 1, c^2 = a^2b^2, [a, c] = a^2, [b, c] = c^2, [a, b] = 1 \rangle; \]

\[ G_3 = \langle a_1, a_2, a_3, a_4, a_5, a_6 | a_1^2 = a_2^2 = a_3^2 = a_4^2 = a_5^2 = a_6^2 = 1, [a_1, a_2] = a_5, [a_1, a_3] = a_6, [a_1, a_4] = [a_1, a_5] = [a_1, a_6] = 1, [a_2, a_3] = 1, [a_2, a_4] = a_5, [a_2, a_5] = [a_2, a_6] = 1, [a_3, a_4] = [a_3, a_5] = [a_3, a_6] = 1, [a_4, a_5] = [a_4, a_6] = 1, [a_5, a_6] = 1 \rangle. \]

For a finite group \( G \), we denote by \( M(G) \) the number of elements of maximal order of \( G \), and the maximal element order in \( G \) by \( k = k(G) \). There is a topic related to one of Thompson’s Conjectures:

**Thompson’s Conjecture** Let \( G \) be a finite group. For a positive integer \( d \), define \( G(d) = |\{ x \in G | \text{the order of } x \text{ is } d \}|. \) If \( S \) is a solvable group, \( G(d) = S(d) \) for \( d = 1, 2, \ldots \), then \( G \) is solvable.

Recently, some authors have investigated this topic in several articles (see [2], [5], [6], [8]). In particular, in [1] the authors gave a complete classification of the finite group with \( M(G) = 30 \), and the finite group with \( M(G) = 24 \) are classified in [4]. In this paper, we consider a finite group \( G \) satisfying \( M(G) = 28 \). Our main result of this paper is:

**Main Theorem** Suppose \( G \) is a finite group having exactly 28 elements of maximal order. Then \( G \) is solvable and one of the following holds:

1. If \( k = 4 \), then \( G \cong Q_8 \times C_4, G_1, G_2 \) or \( G_3 \);
2. If \( k = 6 \), then \( |G| = 2^a \cdot 3^b \), where \( 2 \leq a \leq 6 \) and \( 1 \leq b \leq 4 \);
3. If \( k = 10 \), then \( S_5 = C_5 \trianglelefteq G \), \( |C_G(S_5)| = C_5 \times C_2^2 \) and \( |G/C_G(S_5)| = 4 \);
4. If \( k \in \{29, 58\} \), then \( C_G(x) = \langle x \rangle \trianglelefteq G \). Therefore, \( G/C_G(x) \trianglelefteq \text{Aut}(C_k) \), where \( o(x) = k \).

By the above theorem, we have:

**Corollary** Thompson’s Conjecture holds if \( G \) has exactly 28 elements of maximal order.

## 2 Preliminaries

The following lemma reveals the relationship of \( M(G) \) and \( k \).

**Lemma 2.1** [8, Lemma 1] Suppose \( G \) has exactly \( n \) cyclic subgroups of order \( l \), then the number of elements of order \( l \) (denoted by \( n_l(G) \)) is \( n_l(G) = n\phi(l) \), where \( \phi(l) \) is the Euler function of \( l \). In particular, if \( n \) denotes the number of cyclic subgroups of \( G \) of maximal order \( k \), then \( M(G) = n\phi(k) \).

By above lemma, we have:
Lemma 2.2 If \( M(G) = 28 \) and \( k \) is maximal element order of \( G \), then possible values of \( n, k \) and \( \phi(k) \) are given in the following table:

<table>
<thead>
<tr>
<th></th>
<th>( \phi(k) )</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>28</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>14</td>
<td>2</td>
<td>3,4,6</td>
</tr>
<tr>
<td>7</td>
<td>4</td>
<td>5,10,12</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>null</td>
</tr>
<tr>
<td>2</td>
<td>14</td>
<td>null</td>
</tr>
<tr>
<td>1</td>
<td>28</td>
<td>29,58</td>
</tr>
</tbody>
</table>

In proving our main theorem, the following two results will be frequently used.

Lemma 2.3 [1, Lemma 8] If the number of elements of maximal order \( k \) is \( m \), then there exists a positive integer \( \alpha \) such that \( |G| \) divides \( mk^\alpha \).

Lemma 2.4 [6, Lemma 2.5] Let \( P \) be a \( p \)-group with order \( p^t \) where \( p \) is a prime, and \( t \) is a positive integer. Suppose \( b \in Z(P) \), where \( o(b) = p^u = k \) with \( u \) a positive integer. Then \( P \) has at least \( (p-1)p^{t-1} \) elements of order \( k \).

Lemma 2.5 Let \( G \) be a finite 2-group. If \( \exp(G) = 4 \) and \( M(G) = 28 \). Then \( G \) is isomorphic to the following groups: \( Q_8 \times C_4, G_1, G_2 \) or \( G_3 \).

Proof If \( G \) is a nonabelian 2-group with \( \exp(G) = 4 \) and every \( x \) in \( G \) of order 2 is contained in \( Z(G) \). We prove that \( |G| \leq 64 \). Suppose that \( |G| > 64 \). Then \( G \) has a proper subgroup \( H \cong C_2 \times C_2 \times C_2 \times C_2 \times C_4 \). Since every element of order 2 is contained in \( Z(G) \) and \( \exp(G) = 4 \). Obviously, \( n_4(H) = 32 \), a contradiction. If \( G \) is nonabelian and there exists an element of order 2 which is not contained in \( Z(G) \), then \( |G| \leq 64 \) by [4, lemma4]. If \( G \) is abelian, let \( |G| = 2^t \). Then \( 2^{t-1} \leq 28 \) by Lemma 2.4. Hence \( t \leq 5 \) and \( |G| \leq 32 \). Therefore \( |G| = 32 \) or \( |G| = 64 \). If \( |G| = 64 \), then \( G \cong G_3 \) by [3]. If \( |G| = 32 \), then \( G \cong Q_8 \times C_4, G_1 \) or \( G_2 \) by [7, Part 3].

3 Proof of Main Theorem

By the hypothesis \( M(G) = 28 \), then \( k \neq 2,3 \) and 5 by [1, Lemma 6]. In the following we prove our theorem case by case for the remaining possible values of \( k \).

Case 1 \( k = 4 \). By Lemma 2.3, in this case \( G \) is a 2-group. By Lemma 2.5, \( G \) is isomorphic to one of the following groups: \( Q_8 \times C_4, G_1, G_2 \) or \( G_3 \). Thus (1) holds.
Case 2 $k = 6$. In this case $|G| = 2^\alpha 3^\beta$, where $\alpha > 0$ and $\beta > 0$ by Lemma 2.3. Let $x$ be an element of order 6. Then $|C_G(\langle x \rangle)| = 2^\alpha \cdot 3^\beta$. Since there exists no element of order 9 or 4 in $G$, we have $u \leq 3$ and $v \leq 3$ by $M(G) = 28$. Since $G$ has exactly 14 cyclic subgroups of order 6, we have $|G : N_G(\langle x \rangle)| = 1, 2, 3, 4, 6, 8, 9$ or 12. If there is an element $y$ of order 6 in $G$ such that $|G : N_G(\langle y \rangle)| = 6, 8$ or 9, then there exists another element $z$ of order 6 in $G$ such that $|G : N_G(\langle z \rangle)| = 1, 2, 3, 4, 6$ or 12. That is to say, $G$ always has an element $x$ of order 6 such that $|G : N_G(\langle x \rangle)| = 1, 2, 3, 4, 6$ or 12. Therefore $|G|2^\alpha 3^4$ since $|G| = |G : N_G(\langle x \rangle)| \cdot |N_G(\langle x \rangle) : C_G(\langle x \rangle)| \cdot |C_G(\langle x \rangle)|$. Thus (2) follows.

Case 3 $k = 10$. By Lemma 2.3, we may assume that $|G| = 2^\alpha 5^\beta 7^\gamma$, where $\alpha, \beta > 0$ and $\gamma = 0$, or 1.

If $\gamma = 0$, then $G$ is a $\{2, 3, 7\}$-group and $|G| = 2^\alpha 5^\beta$. Since the number of cyclic subgroups of order 10 in $G$ is 7, it follows that $|G : N_G(\langle x \rangle)| = 1, 2, 4$ or 5 for some element $x$ of order 10. If $|G : N_G(\langle x \rangle)| = 4$ or 5, then there must be another element $y$ of order 10 such that $|G : N_G(\langle y \rangle)| = 2$ or 1. Let $|C_G(\langle x \rangle)| = 2^u 5^v$. Then $u \leq 3$ and $v \leq 2$ since $C_G(\langle x \rangle)$ contains at most 28 elements of order 10. And it always follows that $|N_G(\langle x \rangle) / C_G(\langle x \rangle)| \leq 4$ and $C_G(\langle x \rangle)$ is a $\{2, 5\}$-group. So we get $|G|2^6 5^2$ since $|G| = |G : N_G(\langle x \rangle)| \cdot |N_G(\langle x \rangle) / C_G(\langle x \rangle)|$. Let $S_5 \in \text{Syl}_5(G)$. Then $S_5 \leq C_G(\langle x \rangle)$. If $v = 1$ and $S_5$ is not normal in $G$, then $|G : N_G(S_5)| \neq 1$. Assume $u = 2$ or 3. Then $C_G(\langle x \rangle)$ contains at least 8 elements of order 10. Since all elements of order 5 in $G$ are conjugate and $|G : N_G(S_5)| \geq 6$, there are at least 48 elements of order 10 in $G$, a contradiction. If $u = 1$, then $C_G(\langle x \rangle)$ contains 4 elements of order 10. Therefore $|G : N_G(S_5)| = 7$, which contradicts $\gamma = 0$. If $v = 1$ and $S_5$ is normal in $G$, then $|G / C_G(S_5)| \leq 4$ and $|C_G(S_5)| = C_5 \times C_2^2$. Thus (3) follows.

If $v = 2$, then $C_G(\langle x \rangle)$ contains at least 24 elements of order 10 since $S_5 \leq C_G(\langle x \rangle)$. Thus $M(G) \neq 28$, a contradiction.

If $\gamma = 1$, there exists an element $y$ of order 10 such that $|G : N_G(\langle y \rangle)| = 7$. Otherwise, as $7 \nmid |\text{Aut}(\langle y \rangle)|$, $G$ must have an element of order 70, a contradiction. Since $|N_G(\langle y \rangle) / C_G(\langle y \rangle)| \leq 4$, we have $S_5 \leq C_G(\langle y \rangle)$, for some $S_5 \in \text{Syl}_5(G)$. By Sylow’s Theorem, $|G : N_G(S_5)| = 5k^\prime + 1 \geq 56$ since $7 \mid |G|$, which implies that there are at least 224 elements of order 10, a contradiction.

Case 4 $k = 12$. By Lemma 2.3, we may assume that $|G| = 2^\alpha 3^\beta 7^\gamma$, where $\alpha, \beta > 0$ and $\gamma = 0$, or 1.

If $\gamma = 0$, then $G$ is a $\{2, 3, 7\}$-group and $|G| = 2^\alpha 3^\beta$. Since the number of cyclic subgroups of order 12 in $G$ is 7, it follows that $|G : N_G(\langle x \rangle)| = 1, 2, 3, 4$ or 6 for some element $x$ of order 12. If $|G : N_G(\langle x \rangle)| = 3$ or 6, then there must be another element $y$ of order 12 such that $|G : N_G(\langle y \rangle)| = 1, 2$ or 4. Hence there is an element $x$ of order 12 such that $|G : N_G(\langle x \rangle)| = 1, 2$ or 4. Let
$|C_G(\langle x \rangle)| = 2^n \cdot 3^r$. By Lemma 2.4, $C_G(\langle x \rangle)$ has at least $2^{n-1}$ elements of order 4. On the other hand, all 3-elements of $C_G(\langle x \rangle)$ is of order 3 since $C_G(\langle x \rangle)$ has no element of order 9. Hence we have $2 \cdot 2^{n-1} + 2(3^r - 1) - 4 \leq 28$ by Lemma 2.4 and our assumption. Therefore we get $1 \leq v \leq 2$ and $2 \leq u \leq 4$. So we get $|G| \geq 2^8 \cdot 3^2$ since $|G| = |G : N_G(\langle x \rangle)| \cdot |N_G(\langle x \rangle) : C_G(\langle x \rangle)| \cdot |C_G(\langle x \rangle)|$. Let $S_3 \in Syl_3(G)$. Then $S_3 \leq C_G(x)$. If $v = 1$ and $S_3$ is normal in $G$, then $n_3(G) = 2$ and hence, $28 = M(G) = n_3(G)n_2(C_G(S_3)) = 2n_2(C_G(S_3))$, so $n_2(C_G(S_3)) = 14$, which is a contradiction, because the number of the elements of order 2 is always an odd number. a contradiction. If $v = 1$ and $S_3$ is not normal in $G$, then $|G : N_G(S_3)| \neq 1$. Since $|G : N_G(S_3)| = 2s > 2, n_3(G) = 2s + 1 > 4$ and hence, $28 = M(G) = n_1(G) = n_3(G)n_2(C(G(S_3))) = 2^{s+1}n_4(C_G(S_3))$, which is impossible. Suppose now that $v = 2$. Then $C = C_G(\langle x \rangle) = C_4 \times C_2^2$ contains 16 elements of order 12. Choose $y \in G \setminus C$ be an element of order 12, then $C_G(y)$ also contains 16 elements of order 12. We prove that for every $t \in G \setminus C_G(x)$ with $o(t) = 12$, $C_G(t) \cap C$ contains no element of order 12. Otherwise, there is $z \in C \cap C_G(t)$ with $o(z) = 12$. Since $C$ and $C_G(t)$ are abelian, we have $C_G(t) \leq C_G(z)$ and $C \leq C_G(z)$. Noting that all the centralizers of cyclic subgroup of order 12 are conjugate in $G$, we know that $C, C_G(t)$ and $C_G(z)$ are also conjugate. Hence $C = C_G(t) = C_G(z)$, a contradiction. Hence $C \cup C_G(t)$ contains 32 elements of order 12, a contradiction.

If $\gamma = 1$, there exists an element $x$ of order 12 such that $|G : N_G(\langle x \rangle)| = 7$ and hence all the cyclic subgroups of order 12 are conjugate in $G$. Since $|N_G(\langle x \rangle)/C_G(x)| \leq 4$, we have $S_3 \leq C_G(x)$, for some $S_3 \in Syl_3(G)$. Let $C = C_G(x), S_2 \in Syl_3(G)$ and $S_3 \in Syl_3(G)$. By Lemma 2.4 and our assumption, we have $1 \leq v \leq 2$ and $2 \leq u \leq 4$. Obviously $\langle x \rangle \leq Z(C)$, the center of $C$. Suppose that $3^2$ divides $|C|$. Then $C > \langle x \rangle$. If $C$ is abelian, then $|C| = 2^2 \cdot 3^2$ and $C$ contains exactly 16 elements of order 12. By the same argument as above, we can get a contradiction. If $|C| = 36$, then we can get $C$ is abelian since $\langle x \rangle \leq Z(C)$. Therefore we may assume that $|C| > 36$ and $C$ is not abelian. Hence $|C| \geq 72$. Obviously $C_G(S_3) = S_3 \times S_2$, where $S_2 \in Syl_2(C_G(S_3))$. Since $4 \in \pi_e(Z(C))$ and $S_3 \leq C, 4 \in \pi_e(S_2)$. So for every $y \in S_3$, $n_4(C_G(y)) \geq 2$.

We continue the proof in the following cases:

**Subcase 1** $S_3 \leq G$, then since $9 \notin \pi_e(G)$, $S_3$ is 3-elementary abelian. So $G/C_G(S_3) \leq GL_2(3)$. Thus $7||C_G(S_3)||$ and hence, $21 \in \pi_e(G)$, which is a contradiction.

**Subcase 2** $S_3 \nsubseteq G$, then $n_3(G) \geq 24$, and $28 = n_{12}(G) \geq 24 \cdot 2$, using (**), which is impossible.

**Case 5** $k \in \{29, 58\}$. Let $x$ be an element of order $k$. Then $C_G(x) = \langle x \rangle \leq G$. Therefore, $G/C_G(x) \leq Aut(C_k)$ and $C_G(x) \cong C_k$. Thus (5) holds.
References


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