Ranks and Subdegrees of the Dihedral Groups, $D_n$, Acting on Unordered r-Element Subsets

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Abstract

In this paper we determine the property of transitivity of the dihedral groups, $D_n$, acting on $X^{(1)}$, $X^{(2)}$, and $X^{(3)}$. We also compute the ranks and subdegrees of the respective actions.

Keywords: Transitive Action, Suborbits, Rank, Subdegrees

1. Introduction

Some properties of permutation groups, $S_n$, acting on r-element subsets have been investigated. The ranks and subdegrees of $S_n$ acting on 2-element subsets from the set $X = \{1, 2, \ldots, n\}$ was shown to be 3 and $1, 2(n - 2), \left(\frac{n}{2}\right)$ respectively by Higman in 1970 [3]. Faradzev and Ivanov, in 1990 [1], worked on the subdegrees of primitive permutation representation of $PSL(2, q)$. In 1992, Kamuti [5] constructed some suborbital graphs of $PGL(2, q)$ acting on the cosets of their maximal dihedral subgroups. Rimberia (2012) [4] worked on suborbits and suborbitals of $S_n$, acting on unordered and ordered r-element subsets.
In this paper we study some properties of the dihedral group, $D_n$, acting on unordered $r$-element subsets from the set $X = \{1, 2 \ldots n\}$.

2. Notations and preliminary definitions

2.1. Definition (The Dihedral group, $D_n$)

The group, $D_n$, is the group of all symmetries of a regular polygon. The group is of order $2n$ and is generated by a rotation of order $n$ and a reflection of order 2. Thus, $D_n = \langle x, y : x^n = y^2 = 1 \rangle$.

2.2. Notations

Throughout this paper, the notation, $G$, refers to a group. The notation, $X^{(r)}$, refers to a set of all unordered $r$-element subsets from the set $X = \{1, 2 \ldots n\}$, and $\binom{n}{r}$ denotes all combinations of $r$ from $n$.

2.3. Definition (Group Action)

Let $X$ be a finite non-empty set

A group action of $G$ on a set $X$ is a relation on the pair $(G, X)$ which assigns each $g \in G$ and $x \in X$ an element $gx \in X$. The relation satisfies the property of identity in $G$ and the property of associativity. Namely,

i) $lx = x$, for all $x \in X$ and $l \in G$
ii) $g_1(g_2x) = (g_1g_2)x$, for all $g_1, g_2 \in G$ and $x \in X$

2.4. Definition

The group action of $S_n$ on $X = \{1, 2 \ldots n\}$ is an equivalence relation

2.5. Definition (Orbit of a group action)

An equivalence relation partitions the set, on which it is acting, into disjoint equivalence classes. The classes are called the orbits of the action. For each $x \in (G, X)$, the orbit containing $x$ is denoted by $\text{orb}_G x$. The orbit is given by the set $\text{orb}_G x = \{gx | g \in G\}$. 


2.6. Definition (Transitive group action)

The action of a group $G$ on a set $X$ is said to be transitive if for each pair $x_1, x_2 \in X$, there exists $g \in G$ such that $gx_1 = x_2$. Thus, the action has only one orbit.

2.7. Definition (Doubly Transitive Group Action)

Let a group $G$ act a set $X$, with at least two elements. The action is called doubly transitive if for any two ordered pairs of distinct elements $(x_1, x_2)$ and $(y_1, y_2)$ there exists a $g \in G$ such that $y_1 = gx_1$ and $y_2 = gx_2$.

2.8. Definition (Fixed point set)

Let a group, $G$, act on a set $X$. The set of all elements $x \in X$ fixed by $g \in G$ is called the fixed point set of $G$ and it is denoted by $Fix(g)$.

2.9. Definition (Stabilizer of an element, $stab_G x$)

Let $G$ act on a set $X$. For a fixed point $x \in X$, the stabilizer of $x$ is the set of all elements $g \in G$ such that $gx = x$. The set is denoted by $G_x = \{g \in G: gx = x, \text{for a fixed } x \in X\}$. If $G_x = \{I\}$, then $G$ is said to be regular on $X$.

2.10. Orbit-stabilizer Theorem (Rose, 1978, P.72)[7]

Let $G$ act on a finite set, $X$. Then the size of $orb_G x$ is the index $|G: stab_G x|$. Thus, $|orb_G x| = |G: stab_G x|$, for some $x \in X$.

2.11. Cauchy-Frobenius Lemma (Harrary, 1969, P.98)[2]

Let $G$ act on a finite set $X$. The number of orbits of $G$ in $X$ is given by $\frac{1}{|G|} \sum |Fix(g)|$.

2.12. Definition (Cycle-type)

If a finite group $G$ acts on a set $X$ with $n$ elements, then each $g \in G$ corresponds to a permutation $\sigma$ of $X$ which can be written uniquely as a product of disjoint cycles. If $\sigma$ has $\alpha_1$ cycles of length 1, $\alpha_2$ cycles of length 2, ... $\alpha_n$ cycles of length $n$, then $\sigma$ and $g$ are said to have the cycle type $(\alpha_1, \alpha_2, ..., \alpha_n)$. 

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2.13. Definition (Rank of a group)
Let $G$ act transitively on a non-empty set $X$. The orbits $\Delta_0, \Delta_1, \Delta_2, \ldots, \Delta_{r-1}$ of $G_x$ on $X$ are called suborbits of $G$. The rank of $G$ is $r$ and the sizes

$$n_i = |\Delta_i|(i = 0, 1, 2 \ldots r - 1),$$

the lengths of the suborbits, are known as the subdegrees of $G$.

The rank of $G$ is $2$ iff $G$ is doubly transitive.

3. Transitivity, Ranks and Subdegrees of $D_n$ acting on $X^{(1)}$, $X^{(2)}$ and $X^{(3)}$

3.1. Action of $D_n$ on $X^{(1)}$
Let $G = D_n$ and $X = \{1, 2, 3 \ldots n\}$

Then $X^{(1)} = \{\{1\}, \{2\} \ldots \{n\}\}$

The action of $g \in G$ in $X^{(1)}$ is defined by, $g(\{x\}) = \{g(x)\}$

Theorem 3.1

The group $D_n$ acts transitively on $X^{(1)}$

Proof:
Suppose $g \in G$ has the cycle type $(\alpha_1, \alpha_2 \ldots \alpha_n)$ then $g \in G$ fixes $\{x\} \in X^{(1)}$ if $x$ comes from a $1$ - cycle or from $(n-1)$ – cycle. Hence $|Fix(g)| = 2\alpha_1$

Using Cauchy - Frobenius Lemma, the number of orbits of $G$ in $X^{(1)}$ is $1$. Hence the action of $D_n$ on $X^{(1)}$ is transitive.

Let $\{x\} \in X^{(1)}$, $|Stab_{G_x}| = 2 \Rightarrow |G_{\{x\}}| = 2$

Using the Orbit-stabilizer Theorem,$|Orb_{G_x}(\{x\}| = \alpha_1$ But the trivial suborbit corresponding to $\{x\}$ has length $1$.

The number of non-trivial $G_x$-Orbits is then $\frac{\alpha_1 - 1}{2}$

Hence,
For $\alpha_1$ odd, the rank of $G$ is $\frac{\alpha_1-1}{2} + 1 = \frac{\alpha_1+1}{2}$ and the subdegrees are $1, 2, 2, 2 ... \frac{(n-1)}{2}$’s.

When $\alpha_1$ is even, $|\text{Stab}_{G_x} x| = 2$ and hence $|\text{Orob}_{G_x} x| = \alpha_1$.

Since $g \in G_x$ fixes elements in $X^{(1)}$ pairwise, there are 2 suborbits of length 1.

Hence the rank of $G$ is $\frac{\alpha_1-2}{2} + 2 = \frac{\alpha_1+2}{2}$ and the subdegrees are $1, 1, 2, 2, 2 ... \frac{(n-2)}{2}$’s.

### 3.2. Action of $G = D_n$ on $X^{(2)}$

Let $\{x_1, x_2\} \in X^{(2)}$. The action of $g \in G$ on $\{x_1, x_2\}$ is defined by

$g \{x_1, x_2\} = \{g(x_1), g(x_2)\}$. An element $g \in G$ fixes $\{x_1, x_2\}$ if each of the elements in $\{x_1, x_2\}$ comes from a 1-cycle in $G$ or the set $\{x_1, x_2\}$ is a 2-cycle in $G$. The number of elements in $\text{fix}(g)$ is then $\left(\frac{\alpha_1}{2}\right) + \alpha_2$.

When $\alpha_1$ is odd, $|\text{Fix}(g)|$ in $X^{(2)}$ is given by $\left(\frac{\alpha_1}{2}\right) + (\alpha_1 - 1) \frac{\alpha_1}{2}$.

When $\alpha_1$ is even, $|\text{Fix}(g)|$ in $X^{(2)}$ is given by $\left(\frac{\alpha_1}{2}\right) + \frac{\alpha_1^2}{2}$.

#### Theorem 3.2

$G = D_n$ acts transitively on $X^{(2)}$

**Proof:**

Using Cauchy - Frobenius Lemma, the number of $G$-Orbits in $X^{(2)}$ is given by

$$\frac{1}{2\alpha_1} \left\{ \left(\frac{\alpha_1}{2}\right) + (\alpha_1 - 1) \frac{\alpha_1}{2} \right\} = \frac{\alpha_1 - 1}{2}$$

For transitivity, $\frac{\alpha_1-1}{2} = 1 \quad \Rightarrow \quad \alpha_1 = 3$

Hence, $D_n$ acts transitively on $X^{(2)}$ when $n = 3$. 

3.3. Ranks and subdegrees of the action of $D_3$ on $X^{(2)}$

Let $\{x_1, x_2\} \in X^{(2)}$

$|\text{Stab}_G\{x_1, x_2\}| = 2$

Since $|X^{(2)}| = 3$, and there has to be a suborbit of length 1 corresponding to $\{x_1, x_2\}$ the suborbits of $G$ are 2

These are $\Delta_0 = \{\{1,2\}\}$, $\Delta_1 = \{\{2,3\}, \{1,3\}\}$

The subdegrees are 1,2 and the rank of $G$ is 2. Hence the action of $D_3$ on $X^{(2)}$ is doubly transitive.

3.4. Action of $G = D_n$ on $X^{(3)}$

Let $\{x_1, x_2, x_3\} \in X^{(3)}$. The action of $g \in G$ in $X^{(3)}$ is defined by

$g(\{x_1, x_2, x_3\}) = \{g(x_1), g(x_2), g(x_3)\}$. An element $g \in G$ fixes $\{x_1, x_2, x_3\}$ if each of $x_1$, $x_2$ and $x_3$ comes from a 1-cycle in $G$ or one of $x_1$, $x_2$, $x_3$ comes from a 1-cycle and the other two come from 2-cycle in $G$ or $\{x_1, x_2, x_3\}$ comes from a 3-cycle in $G$. But $g \in G$ is a 3-cycle if $\alpha_1$ is a multiple of 3.

If $x_1$, $x_2$ and $x_3$ come from 1-cycle in $G$, then $|\text{Fix}(g)| = \left(\begin{array}{c} \alpha_1 \\ 3 \end{array}\right)$

If $\{x_1, x_2, x_3\}$ come from 1-cycle and 2-cycle, then the number $\text{fix}(g)$ is $\frac{\alpha_1}{2}(\alpha_1 - 1)$.

If $\{x_1, x_2, x_3\}$ come from a 3-cycle, then the number $\text{fix}(g)$ is $\frac{2}{3}\alpha_1$. This exists when $\alpha_1$ is a multiple of 3.

Theorem 3.3

The action of $D_n$ on $X^{(3)}$ is transitive.

Proof:

Let $\{x_1, x_2, x_3\} \in X^{(3)}$

Using Cauchy - Frobenius Lemma, the number of $G$-orbits in $X^{(3)}$ is given by
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\[ \frac{1}{2\alpha_1} \left\{ \left(\frac{\alpha_1}{3}\right) + \frac{\alpha_1}{2}(\alpha_1 - 1) \right\} \], for \( \alpha_1 \) a non-multiple of 3

\[ = \frac{1}{12}(\alpha^2 - 4) \]

For transitivity,

\[ \frac{1}{12}(\alpha_1^2 - 4) = 1 \]

\[ \alpha_1 = 4 \]

Hence, \( D_4 \) acts transitively on \( X^{(3)} \)

3.5. Ranks and Subdegrees of the action of \( D_4 \) on \( X^{(3)} \)

Let \( \{x_1, x_2, x_3\} \in X^{(3)} \)

\[ |\text{Stab}_G(x_1, x_2, x_3)| = 2 \text{ and } |X^{(3)}| = \left(\frac{4}{3}\right) = 4. \] Since there exists \( a \in G \) which fixes a pair of elements in \( X^{(3)} \), there are 2 suborbits of length 1. The remaining 2 elements belong to 1 suborbit as they are the pairwise from \( (\alpha_1 - 2) \) elements. Hence there is always a suborbit of length 2 in the actions of \( D_n \).

The rank of the action of \( D_4 \) on \( X^{(3)} \) is 3.

The suborbits of the action are \( \Delta_0 = \{1, 2, 3\} \), \( \Delta_1 = \{1, 3, 4\} \), \( \Delta_2 = \{1, 2, 4\}, \{2, 3, 4\} \)

Hence the subdegrees of \( G = D_4 \) are 1, 1, 2

Similarly, the number \( \text{fix}(g) \) when \( \alpha_1 \) is an odd multiple of 3 is

\[ \left(\frac{\alpha_1}{3}\right) + \frac{\alpha_1}{2}(\alpha_1 - 1)(\alpha_1 - 2) + \frac{2}{3}\alpha_1 \]

Hence, for transitivity

\[ \frac{1}{2\alpha_1} \{4\alpha_1^2 - 12\alpha_1 + 12\} = 1 \]

\[ 4\alpha_1^2 - 12\alpha_1 = 0 \]
\[ \alpha_1 = 3 \]

\(D_3\) acts transitively on \(X^{(3)}\). But this is a trivial action on one element.

**Table 1:** Table of ranks and subdegrees of \(D_n\) acting on \(X^{(r)}\), \(n \leq 8\)

| G  | \(|X^{(r)}|\) | r  | rank | subdeges   |
|----|--------------|----|------|------------|
| \(D_3\) | 3          | 2  | 2    | 1,2        |
| \(D_4\) | 4          | 3  | 3    | 1,1,2      |
| \(D_5\) | 5          | 4  | 3    | 1,2,2      |
| \(D_6\) | 6          | 5  | 4    | 1,1,2,2    |
| \(D_7\) | 7          | 6  | 4    | 1,2,2,2    |
| \(D_8\) | 8          | 7  | 5    | 1,1,2,2,2  |

**Theorem 3.4**

The action of \(D_n\) on \(X^{(r)}\) is transitive when \(r = n - 1\)

**Proof:**

Case 1: When \(n\) is odd

Let \(G = D_n\) act on the set \(X = \{1, 2 ... n\}\)

An element \(x \in X\) is fixed by \(g \in G\) if \(x\) comes from 1-cycle in \(G\) or \(x\) comes from \((n-1)\)-cycle.

If \(g \in G\) fixes 1 element in \(X\), it automatically fixes the remaining \(n - 1\) elements as a set.

Hence, the number \(fix(g)\) in \(X^{(1)} = fix(g)\) in \(X^{(n-1)}\). Transitivity of \(G\) on\(X^{(1)}\) implies transitivity of \(G\) on\(X^{(n-1)}\). Transitivity of \(G\) on \(X^{(1)}\) has been proved in Theorem 3.1, and hence the proof.

Case 2: When \(n\) is even

An element \(x \in X\) is fixed by \(g \in G\) if \(x\) comes from 1-cycle in\(G\) or \(x\) is fixed pairwise. The first alternative is case 1. If \(\{x_1, x_2\}\) is pairwise fixed by \(g\), then the remaining \(n - 2\) elements are automatically fixed by \(g\). We can choose the \(n-2\) elements and 1 element from the set \(\{x_1, x_2\}\) to have a set of \(n-1\) elements, still
fixed by $g$. But the action of $G$ on $X^{(n-1)}$ is transitive in case 1. Hence, the action of $G$ on $X^{(r)}$ is transitive when $r = n - 1$.

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