Ore Extensions over Weak \((\sigma, \delta)\)-Rigid Rings

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Abstract

Let \(R\) be a ring, \(\sigma\) an endomorphism of \(R\) and \(\delta\) a \(\sigma\)-derivation of \(R\). Recall that \(R\) is said to be a weak \((\sigma, \delta)\)-rigid ring if \(a(\sigma(a) + \delta(a)) \in N(R)\) implies and is implied by \(a \in N(R)\) for \(a \in R\) (where \(N(R)\) is the set of nilpotent elements of \(R\)). In this paper we give a necessary and sufficient condition for a commutative Noetherian ring to be a weak \((\sigma, \delta)\)-rigid ring.

Let \(\sigma\) be an endomorphism of a ring \(R\) and \(\delta\) a \(\sigma\)-derivation of \(R\) such that \(\sigma(\delta(a)) = \delta(\sigma(a))\), for all \(a \in R\). Then \(\sigma\) can be extended to an endomorphism (say \(\sigma\)) of \(R[x; \sigma, \delta]\) and \(\delta\) can be extended to a \(\sigma\)-derivation (say \(\delta\)) of \(R[x; \sigma, \delta]\).

With this we show that if \(R\) is a commutative Noetherian integral domain which is also an algebra over \(\mathbb{Q}\) (where \(\mathbb{Q}\) is the field of rational numbers), \(\sigma\) an automorphism of \(R\) and \(\delta\) a \(\sigma\)-derivation of \(R\) such that \(\sigma(\delta(a)) = \delta(\sigma(a))\), for all \(a \in R\). Then \(R\) is a weak \((\sigma, \delta)\)-rigid ring if and only if \(O(R) = R[x; \sigma, \delta]\) is a weak \((\sigma, \delta)\)-rigid ring.

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1. INTRODUCTION

All rings are associative with identity \(1 \neq 0\), unless otherwise stated. The prime radical and the set of nilpotent elements of \(R\) are denoted by \(P(R)\) and
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The set of natural numbers is denoted by \( \mathbb{N} \), the ring of integers is denoted by \( \mathbb{Z} \), the field of rational numbers is denoted by \( \mathbb{Q} \), the field of real numbers is denoted by \( \mathbb{R} \) and the field of complex numbers is denoted by \( \mathbb{C} \), unless otherwise stated. The set of minimal prime ideals of \( R \) is denoted by \( \text{Min.Spec}(R) \).

Now let \( R \) be a ring, \( \sigma \) an endomorphism of \( R \). Recall that \( \delta : R \to R \) an additive map such that

\[
\delta(ab) = \delta(a)\sigma(b) + a\delta(b), \text{ for all } a, b \in R
\]

is called a \( \sigma \)-derivation of \( R \).

**Example 1.1.** Let \( S \) be a ring and \( R = S \times S \). Then \( \sigma : R \to R \) is an endomorphism defined by

\[
\sigma((a, b)) = (b, a), \text{ for all } a, b \in S.
\]

Also \( \delta : R \to R \) defined by

\[
\delta((a, b)) = (a, b) - \sigma((a, b)) \text{ for } (a, b) \in R
\]

is a \( \sigma \)-derivation of \( R \).

Recall that the skew polynomial ring \( R[x; \sigma, \delta] \) is the set of polynomials

\[
\left\{ \sum_{i=1}^{n} x^i a_i, a_i \in R, n \in \mathbb{N} \right\}
\]

with usual addition of polynomials and multiplication subject to the relation \( ax = x\sigma(a) + \delta(a) \), for all \( a \in R \). We take any \( f(x) \in R[x; \sigma, \delta] \) to be of the form \( f(x) = \sum_{i=0}^{n} x^i a_i \) as in McConnell and Robson [8]. We denote \( R[x; \sigma, \delta] \) by \( O(R) \).

Also a ring \( R \) is said to be 2-primal if and only if \( P(R) = N(R) \) i.e., the set of nilpotent elements of \( R \) coincides with the prime radical of \( R \).

**Example 1.2.** [4]

1. Let \( R = F[x] \) be the polynomial ring over the field \( F \). Then \( R \) is 2-primal with \( P(R) = \{0\} \).
2. Let \( R = M_2(\mathbb{Q}) \), the set of \( 2 \times 2 \) matrices over \( \mathbb{Q} \). Then \( R[x] \) is a prime ring with non-zero nilpotent elements and, so cannot be 2-primal.

2-primal rings have been studied in recent years and are being treated by authors for different structures. We know that a ring \( R \) is 2-primal if the prime radical is completely semi-prime. Note that a reduced ring is 2-primal and a commutative ring is also 2-primal.

Krempa in [5] introduced \( \sigma \)-rigid rings; Kwak in [7] introduced \( \sigma(*) \)-rings. Ouyang in [9] introduced weak \( \sigma \)-rigid rings, where \( \sigma \) is an endomorphism of ring \( R \). Bhat in [2] gave a necessary and sufficient condition for a Noetherian ring to be a weak \( \sigma \)-rigid ring.
We discuss skew polynomial rings over weak \((\sigma, \delta)\)-rigid rings. We begin with the following definitions:

**Definition 1.3.** [1] Let \(R\) be a ring. Let \(\sigma\) be an endomorphism of \(R\) and \(\delta\) a \(\sigma\)-derivation of \(R\). Then \(R\) is said to be a \((\sigma, \delta)\)-ring if \(a(\sigma(a) + \delta(a)) \in P(R)\) implies that \(a \in P(R)\) for \(a \in R\).

**Example 1.4.** Let \(F\) be a field. Let \(R = F[x]\) be the polynomial ring over \(F\). Here \(P(R) = \{0\}\). Define \(\sigma : R \to R\) by

\[
\sigma(f(x)) = f(-x).
\]

Then it can be seen that \(\sigma\) is an automorphism of \(R\). Also \(\delta : R \to R\) defined by

\[
\delta(f(x)) = f(x) - \sigma(f(x))
\]

is a \(\sigma\)-derivation of \(R\). Now \(f(x)[\sigma(f(x)) + \delta(f(x))] \in P(R)\) implies that \(f(x)[f(-x) + f(x) - f(-x)] \in P(R)\) or \(f(x)^2 \in P(R) = \{0\}\). Hence \(f(x) = 0 \in P(R)\). Thus \(R\) is a \((\sigma, \delta)\)-ring.

**Definition 1.5.** [1] Let \(R\) be a ring. Let \(\sigma\) be an endomorphism of \(R\) and \(\delta\) a \(\sigma\)-derivation of \(R\). Then \(R\) is said to be a \((\sigma, \delta)\)-rigid ring if \(a(\sigma(a) + \delta(a)) = 0\) implies that \(a = 0\) for \(a \in R\).

**Example 1.6.** Let \(R = \mathbb{C}\) and \(\sigma : R \to R\) be defined by

\[
\sigma(a + ib) = a - ib, \text{ for all } a, b \in \mathbb{R}.
\]

Then \(\sigma\) is an automorphism of \(R\). Define \(\delta : R \to R\), \(\sigma\)-derivation of \(R\), as

\[
\delta(A) = A - \sigma(A)
\]

i.e., \(\delta(a+ib) = a+ib - \sigma(a+ib) = a+ib - (a-ib) = 2ib\). Then \(\delta\) is a \(\sigma\)-derivation of \(R\). Now \(A[\sigma(A) + \delta(A)] = 0\) gives \((a + ib)[\sigma(a + ib) + \delta(a + ib)] = 0\) i.e. \((a + ib)((a - ib) + 2ib) = 0\) or \((a + ib)(a + ib) = 0\) which implies that \(a = 0, b = 0\). Therefore, \(A = a + ib = 0\). Hence \(R\) is a \((\sigma, \delta)\)-rigid ring.

**Example 1.7.** Let \(R = \mathbb{Z}_2 \oplus \mathbb{Z}_2\). Then \(R\) is a commutative reduced ring. Define \(\sigma : R \to R\) by \(\sigma((a, b)) = (b, a)\) for \(a, b \in \mathbb{Z}_2\). Then it can be seen that \(\sigma\) is an automorphism of \(R\). Also \(\delta : R \to R\) defined by \(\delta((a, b)) = (a - b, 0)\) for \(a, b \in \mathbb{Z}_2\) is a \(\sigma\)-derivation of \(R\). Here \(P(R) = \{0\}\). But \(R\) is not a \((\sigma, \delta)\)-ring. For take \((a, b) = (0, b)\) for \(0 \neq b \in \mathbb{Z}_2\). It is also not a \((\sigma, \delta)\)-rigid ring.

**Definition 1.8.** [1] Let \(R\) be a ring. Let \(\sigma\) be an endomorphism of \(R\) and \(\delta\) a \(\sigma\)-derivation of \(R\). Then \(R\) is said to be a weak \((\sigma, \delta)\)-rigid ring if \(a(\sigma(a) + \delta(a)) \in N(R)\) implies and is implied by \(a \in N(R)\) for \(a \in R\).

**Example 1.9.** Let \(R = \mathbb{Z}[\sqrt{2}]\). Then \(\sigma : R \to R\) defined as

\[
\sigma(a + b\sqrt{2}) = a - b\sqrt{2} \text{ for } a + b\sqrt{2} \in R
\]
is an endomorphism of $R$. For any $s \in R$. Define $\delta_s : R \to R$ by
\[
\delta_s(a + b\sqrt{2}) = (a + b\sqrt{2})s - s\sigma(a + b\sqrt{2}) \text{ for } a + b\sqrt{2} \in R.
\]
Then $\delta_s$ is a $\sigma$-derivation of $R$. Here $N(R) = \{0\}$.
Let $(a + b\sqrt{2})\{\sigma(a + b\sqrt{2}) + \delta_s(a + b\sqrt{2})\} \in N(R)$
which gives $(a + b\sqrt{2})\{(a - b\sqrt{2}) + (a + b\sqrt{2})s - s\sigma(a + b\sqrt{2})\} \in N(R)$
or $(a + b\sqrt{2})\{a - b\sqrt{2} + as + bs\sqrt{2} - sa + sb\sqrt{2}\} \in N(R)$.
Hence $(a + b\sqrt{2})\{a + (2s - 1)b\sqrt{2}\} \in N(R) = \{0\}$ which gives $a = 0, b = 0$ or $a + b\sqrt{2} = 0 + 0\sqrt{2} \in N(R)$. Thus $R$ is a weak $(\sigma, \delta)$-rigid ring.

Also we have the following:

Let $\sigma$ be an endomorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$ such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$. Then $\sigma$ can be extended to an endomorphism (say $\overline{\sigma}$) of $R[x; \sigma, \delta]$ by
\[
\overline{\sigma}(\sum_{i=0}^{m} x^i a_i) = \sum_{i=0}^{m} x^i \sigma(a_i).
\]
Also $\delta$ can be extended to a $\overline{\sigma}$-derivation (say $\overline{\delta}$) of $R[x; \sigma, \delta]$ by
\[
\overline{\delta}(\sum_{i=0}^{m} x^i a_i) = \sum_{i=0}^{m} x^i \delta(a_i).
\]
We note that if $\sigma(\delta(a)) \neq \delta(\sigma(a))$, for all $a \in R$, then above does not hold.

For example take $f(x) = xa$, $g(x) = xb$ for $a, b \in R$.

With this we prove the following:

**Theorem A**: Let $R$ be a commutative Noetherian integral domain which is also an algebra over $\mathbb{Q}$. Let $\sigma$ an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$ such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$. Then $R$ is a weak $(\sigma, \delta)$-rigid ring if and only if $O(R) = R[x; \sigma, \delta]$ is a weak $(\overline{\sigma}, \overline{\delta})$-rigid ring. (This has been proved in Theorem (3.1)).

2. **Preliminaries**

To prove the main result of this paper we need the following Propositions and Theorems:

**Proposition 2.1**. Let $R$ be a ring, $\sigma$ an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$. Then for $u \neq 0$, $\sigma(u) + \delta(u) \neq 0$.

*Proof*. See Proposition (3.1) of [1].

**Theorem 2.2**. Let $R$ be a Noetherian integral domain which is also an algebra over $\mathbb{Q}$. Let $\sigma$ be an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$. If $R$ is a $(\sigma, \delta)$-ring, then $R$ is 2-primal.
Proof. See Theorem (3.2) of [1].

The converse of the above is not true.

Example 2.3. Let $R = F(x)$, the field of rational polynomials in one variable $x$ over the field $F$. Then $R$ is 2-primal with $P(R) = \{0\}$. Let $\sigma : R \to R$ be an endomorphism defined by

$$\sigma(f(x)) = f(0).$$

For $r \in R$, $\delta_r : R \to R$ is a $\sigma$-derivation defined as

$$\delta_r(a) = ar - r\sigma(a).$$

Then $R$ is not a $(\sigma, \delta)$-ring. For take $f(x) = xa + b, r = \frac{-b}{xa}$.

Towards the proof of the next Theorem, we require the following:

J. Krempa [5] has investigated the relation between minimal prime ideals and completely prime ideals of a ring $R$. With this he proved the following:

Theorem 2.4. For a ring $R$ the following conditions are equivalent:

1. $R$ is reduced.
2. $R$ is semiprime and all minimal prime ideals of $R$ are completely prime.
3. $R$ is a subdirect product of domains.

Theorem 2.5. Let $R$ be a Noetherian integral domain which is also an algebra over $\mathbb{Q}$. Let $\sigma$ be an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$. If $R$ is a $(\sigma, \delta)$-ring, then $P(R)$ is completely semi-prime.

Proof. As in proof of Theorem (2.2), $R$ is a reduced ring and by using Theorem (2.4), the result follows.

The converse of the above is not true.

Example 2.6. Let $F$ be a field, $R = F \times F$. Let $\sigma : R \to R$ be an automorphism defined as

$$\sigma((a, b)) = (b, a) \text{ for } a, b \in F.$$

Here $P(R)$ is a completely semi-prime ring, as $R$ is a reduced ring. For $r \in F$.

Define $\delta_r : R \to R$ by

$$\delta_r((a, b)) = (a, b)r - r\sigma((a, b)) \text{ for } a, b \in F.$$

Then $\delta_r$ is a $\sigma$-derivation of $R$. But $R$ is not a $(\sigma, \delta)$-ring. For take $A = (1, -1), r = \frac{1}{2}$.

Theorem 2.7. Let $R$ be a Noetherian integral domain which is also an algebra over $\mathbb{Q}$. Let $\sigma$ be an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$ such that $R$ is a $(\sigma, \delta)$-ring. Then $R$ is a weak $(\sigma, \delta)$-rigid ring. Conversely a 2-primal weak $(\sigma, \delta)$-rigid ring is a $(\sigma, \delta)$-ring.
Proof. See Theorem (3.8) of [1].

**Theorem 2.8.** Let $R$ be a Noetherian integral domain which is also an algebra over $\mathbb{Q}$. Let $\sigma$ be an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$ such that $\sigma(U) = U$ and $\delta(U) \subseteq U$ where $U \in \text{Min.Spec}(R)$. Then $R$ is a $(\sigma, \delta)$-ring if and only if for each $U \in \text{Min.Spec}(R)$, $\sigma(U) + \delta(U) = U$ and $U$ is a completely prime ideal of $R$.

**Proof.** See Theorem (3.9) of [1].

**Theorem 2.9.** Let $R$ be a commutative Noetherian integral domain which is also an algebra over $\mathbb{Q}$. Let $\sigma$ be an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$ such that $R$ is a $(\sigma, \delta)$-ring. Then $O(N(R)) = N(O(R))$.

**Proof.** By Theorem (2.2), we know that $R$ a $(\sigma, \delta)$-ring is 2-primal. Also $O(N(R)) \subseteq N(O(R))$. We will show that $N(O(R)) \subseteq O(N(R))$. Let $f = \sum_{i=0}^{m} x^i a_i \in N(O(R))$. Then $(f)(O(R)) \subseteq N(O(R))$, and $(f)(R) \subseteq N(O(R))$. Let $((f)(R))^k = 0$, $k > 0$. Then equating leading term to zero, we get
\[(x^m a_m R)^k = 0.\]

After simplification equating leading term to zero, we get
\[x^{km} \sigma^{(k-1)m}(a_m R) \sigma^{(k-2)m}(a_m R) \sigma^{(k-3)m}(a_m R) \ldots (a_m R) = 0.\]

Therefore,
\[\sigma^{(k-1)m}(a_m R) \sigma^{(k-2)m}(a_m R) \sigma^{(k-3)m}(a_m R) \ldots (a_m R) = 0 \subseteq P,\]

for all $P \in \text{Min. Spec}(R)$. This implies that
\[\sigma^{(k-j)m}(a_m R) \subseteq P, \text{ for some } j, 1 \leq j \leq k.\]

Therefore, by using Theorem (2.8)
\[a_m R \subseteq \sigma^{-(k-j)m}(P) \subseteq \sigma(P) + \delta(P) = P.\]

So we have $a_m R \subseteq P$, for all $P \in \text{Min.Spec}(R)$. Therefore, $a_m \in P(R)$ and $R$ being 2-primal implies that $a_m \in N(R)$. Now $x^m a_m \in O(N(R)) \subseteq N(O(R))$ implies that $\sum_{i=0}^{m-1} x^i a_i \in N(O(R))$ and with the same process in a finite number of steps, it can be seen that
\[a_i \in P(R) = N(R), \text{ } 0 \leq i \leq m - 1.\]

Therefore, $f \in O(N(R))$. Hence $N(O(R)) \subseteq O(N(R))$ and result follows. \qed
3. Proof of Main Result

**Theorem 3.1.** Let $R$ be a commutative Noetherian integral domain which is also an algebra over $\mathbb{Q}$. Let $\sigma$ an automorphism of $R$ and $\delta$ a $\sigma$-derivation of $R$ such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$. Then $R$ is a weak $(\sigma, \delta)$-rigid ring if and only if $O(R) = R[x; \sigma, \delta]$ is a weak $(\overline{\sigma}, \overline{\delta})$-rigid ring.

**Proof.** Let $R$ be a weak $(\sigma, \delta)$-rigid ring. Then by Theorem (2.7), $R$ is a $(\sigma, \delta)$-ring. Also by Theorem (2.9), $O(N(R)) = N(O(R))$. We show that $R[x; \sigma, \delta]$ is a weak $(\overline{\sigma}, \overline{\delta})$-rigid ring. Let $f \in O(R)$ say $f = \sum_{i=0}^m x^i a_i$ be such that $f[\overline{\sigma}(f) + \overline{\delta}(f)] \in N(O(R))$. We use induction on $m$ to prove the result. For $m = 1$, $f = xa_1 + a_0$.

Now $f[\overline{\sigma}(f) + \overline{\delta}(f)] \in N(O(R))$ implies that

$$(xa_1 + a_0)[\overline{\sigma}(xa_1 + a_0) + \overline{\delta}(xa_1 + a_0)] \in N(O(R)) = O(N(R))$$

i.e.,

$$x^2 \sigma^2(a_1) + x \delta(a_1) \sigma(a_1) + x \sigma(a_0) \sigma(a_1) + \delta(a_0) \sigma(a_1) + xa_1 \sigma(a_0) + a_0 \sigma(a_0) + xa_1 \delta(a_1) + xa_1 \delta(a_0) + a_0 x \delta(a_1) + a_0 \delta(a_0) \in O(N(R))$$

or

$$x^2 \sigma^2(a_1) + x \delta(a_1) \sigma(a_1) + x \sigma(a_0) \sigma(a_1) + \delta(a_0) \sigma(a_1) + xa_1 \sigma(a_0) + a_0 \sigma(a_0) + x(x \sigma(a_1) + \delta(a_1)) \delta(a_1) + xa_1 \delta(a_0) + (x \sigma(a_0) + \delta(a_0)) \delta(a_1) + a_0 \delta(a_0) \in O(N(R)).$$

Now coefficient of leading term $\sigma(a_1)(\sigma(a_1) + \delta(a_1)) \in N(R)$, which implies $a_1 \in N(R)$, by Proposition (3) of [6]. Also coefficient of constant term

$$a_0(\sigma(a_0) + \delta(a_0)) + \delta(a_0) (\sigma(a_1) + \delta(a_1)) \in N(R).$$

Now $a_1 \in N(R)$, $\sigma(N(R)) = N(R)$, by Proposition (3) of [6]. Therefore, $\sigma(a_1) \in N(R)$. Also $R$ is commutative and hence 2-primal. Therefore, by Proposition (1.1) of [3], $\delta(a_1) \in N(R)$. Since $N(R)$ is an ideal, so $\delta(a_0)(\sigma(a_1) + \delta(a_1)) \in N(R)$. Therefore, (3.1) implies that $a_0(\sigma(a_0) + \delta(a_0)) \in N(R)$ which gives $a_0 \in N(R)$, as $R$ is a weak $(\sigma, \delta)$-rigid ring. Hence $f \in O(N(R)) = N(O(R))$.

Suppose the result is true for $m = k$. We prove for $m = k + 1$. Now $f[\overline{\sigma}(f) + \overline{\delta}(f)] \in N(O(R))$ implies that

$$(x^{k+1}a_{k+1} + ... + a_0)[x^{k+1} \sigma(a_{k+1}) + ... + \sigma(a_0) + x^{k+1} \delta(a_{k+1}) + ... + \delta(a_0)] \in N(O(R)) = O(N(R))$$

or

$$x^{2k+2} \sigma^{k+2}(a_{k+1}) + x^{2k+1} \sigma^k(a_{k+1}) \sigma(a_k) + x^{2k+1} \sigma^{k+1}(a_k) \sigma(a_{k+1}) + ... + x^{2k+2} \sigma^{k+1}(a_{k+1}) \delta(a_{k+1}) + ...$$
which gives on rearranging
\[
x^{2k+2}(\sigma^{k+2}(a_{k+1})+\sigma^{k+1}(a_{k+1})\delta(a_{k+1})) + \text{term of } x^{2k+1} + g[\sigma(g)+\delta(g)] \in O(N(R))
\]
where \( g = \sum_{i=0}^{k} x^ia_i \). Hence \( \sigma^{k+2}(a_{k+1})+\sigma^{k+1}(a_{k+1})\delta(a_{k+1}) \in N(R) \) which implies that \( a_{k+1} \in N(R) \). Also coefficient of \( x^{2k+1} \in N(R) \) and so \( g[\sigma(g)+\delta(g)] \in N(R) \). But degree of \( g \) is \( k \), therefore, by induction hypothesis, the result is true for all \( m \).
Converse is obvious.

\[\square\]

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