A Short Proof of an Interpolation Result in Near-Rings

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Abstract

We give a short and elementary proof of an interpolation result for primitive near-rings which are not rings. It then turns out that this re-proves the well known interpolation theorem for 0-primitive near-rings. Hence, we can offer a very simple proof for this key result in the structure theory of near-rings.

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1 Introduction

We consider right near-rings, this means the right distributive law holds, but not necessarily the left distributive law. The notation is that of [2]. We will be concerned with primitive near-rings which play the same role in the structure theory of near-rings as primitive rings in ring theory do. In fact, any ring is a near-ring and whenever a primitive near-ring happens to be a ring, then primitivity in the near-ring sense coincides with primitivity in the ring sense. However, whenever we have primitive near-rings that are not rings, then the theory and proofs cannot be simply borrowed from ring theory (see [2] for an overview).

The following theorem is the near-ring counterpart to Jacobson’s density theorem in ring theory:
Theorem 1.1. ([2], Theorem 4.30) Let $N$ be a zero symmetric near-ring which is not a ring, acting 0-primitively on the $N$-group $\Gamma$. Let $n \in N$ and $\gamma_1, \ldots, \gamma_n$ be generators of the $N$-group $\Gamma$ such that for $i \neq j$, $(0 : \gamma_i) \neq (0 : \gamma_j)$. Let $\delta_i \in \Gamma$ for $i \in \{1, \ldots, n\}$. Then there is an element $k \in N$ such that $k\gamma_i = \delta_i$ for all $i \in \{1, \ldots, n\}$.

Theorem 1.1 is central for the structure theory of near-rings. Its standard proof given in the literature is not short and easy. In this paper we will also prove an interpolation result for 0-primitive near-rings whose proof is very elementary and short and differs from the standard proof of Theorem 1.1. The language and notation of our main theorem Theorem 2.2 differs from that of Theorem 1.1. In order to prove Theorem 2.2 we will only need very elementary considerations and hardly any previous knowledge in near-ring theory. We then will show that Theorem 1.1 and Theorem 2.2 are equivalent statements.

For reason of self-containment we briefly recall the most important notation. Let $\Gamma$ be an $N$-group of the near-ring $N$. A normal subgroup $\triangle \subseteq \Gamma$ is an $N$-ideal of $\Gamma$ if $\forall n \in N \forall \gamma \in \Gamma \forall \delta \in \triangle : n(\gamma + \delta) - n\gamma \in \triangle$. $N$-ideals of the natural $N$-group $N$ are called left ideals of the near-ring $N$. An $N$-group $\Gamma$ of the near-ring $N$ is of type 0 if it is non-zero, if there are no non-trivial $N$-ideals in $\Gamma$ and if there is an element $\gamma \in \Gamma$ such that $N\gamma = \Gamma$. Such an element $\gamma$ will be called a generator of the $N$-group $\Gamma$. The $N$-group is of type 1 if it is of type 0 and moreover we have $N\gamma = \Gamma$ or $N\gamma = \{0\}$ for any $\gamma \in \Gamma$. It is easy to see ([2], Proposition 3.4) that given a left ideal $L$ of the near-ring $N$ and given a generator $\gamma$ of $\Gamma$, then $L\gamma$ is an $N$-ideal in $\Gamma$.

Given an $N$-group $\Gamma$ and $\triangle \subseteq \Gamma$ then $(0 : \triangle) = \{n \in N|\forall \gamma \in \triangle : n\gamma = 0\}$ will be called the annihilator of $\triangle$. $\Gamma$ will be called faithful if $(0 : \Gamma) = \{0\}$.

A near-ring is called 0-primitive, 1-primitive respectively, if it acts on a faithful $N$-group $\Gamma$ of type 0, type 1 respectively. In such a situation we will say that the near-ring acts 0-primitively, 1-primitively respectively, on the $N$-group $\Gamma$.

2 When a near-ring is a ring and the main result

The following lemma is a nice tool for our considerations. It basically is the so called “Betsch-Wielandt” Lemma (see [1] or [2], Proposition 2.23, in particular [2], Proposition 3.4). Similar results can be found in [3] or [2], Proposition 2.22. The proof given in the following is a very elementary one.

Lemma 2.1. Let $N$ be a zero symmetric near-ring and $\Gamma = N\gamma$ an $N$-group with generator $\gamma$. Suppose there are two left ideals $L_1$ and $L_2$ in $N$ such that $L_1\gamma = L_2\gamma = \Gamma$ and $L_1 \cap L_2 \subseteq (0 : \gamma)$. Then $(\Gamma, +)$ is an abelian group and
for all \( n \in N \) and all \( \gamma_1, \gamma_2 \in \Gamma \) we have \( n(\gamma_1 + \gamma_2) = n\gamma_1 + n\gamma_2 \). In case \( \Gamma \) is a faithful \( N \)-group, \( N \) is a ring.

**Proof.** Let \( \gamma_1 \) and \( \gamma_2 \) be two elements in \( \Gamma \). Then there are \( l_1 \in L_1 \) and \( l_2 \in L_2 \) such that \( l_1 \gamma = \gamma_1 \) and \( l_2 \gamma = \gamma_2 \).

\((L_1, +)\) and \((L_2, +)\) are normal subgroups of \((N, +)\) and hence, \( l_1 + l_2 - l_1 \in L_2 \) so \( l_1 + l_2 - l_1 \in L_2 \). Similary, \( l_2 - l_1 - l_2 \in L_1 \) so \( l_1 + l_2 - l_1 - l_2 \in L_1 \). Consequently, \( l_1 + l_2 - l_1 - l_2 \in L_2 \cap L_1 \). By assumption \( L_2 \cap L_1 \subseteq (0 : \gamma) \). Therefore, \((l_1 + l_2 - l_1 - l_2) \gamma = l_1 \gamma + l_2 \gamma - l_1 \gamma - l_2 \gamma = \gamma_1 + \gamma_2 = \gamma_1 - \gamma_1 - \gamma_2 = 0 \). So, we have \( \gamma_1 + \gamma_2 = \gamma_2 + \gamma_1 \) and the group \((\Gamma, +)\) is abelian.

Now we proceed in a similar way. Let \( n \in N \) and \( l_1 \gamma = \gamma_1 \in \Gamma \) and \( l_2 \gamma = \gamma_2 \in \Gamma \). Consider the element \( m = n(\gamma_1 + \gamma_2) - n\gamma_2 - n\gamma_1 \in \Gamma \). We want to show, that \( m = 0 \). We have \( m = n(l_1 \gamma + l_2 \gamma) - nl_2 \gamma - nl_1 \gamma = (n(l_1 + l_2) - nl_2 - nl_1) \gamma \). Since \( L_1 \) is a left ideal we have \( n(l_1 + l_2) - nl_2 \in L_1 \) (see [2], Remark 1.28) and \( n(l_1 + l_2) - nl_2 - nl_1 \in L_1 \). Since \( L_2 \) is a normal subgroup of \((N, +)\) we have \( nl_1 - nl_2 - nl_1 = l \in L_2 \). Therefore, \(-nl_1 + l = -nl_2 - nl_1 \). So, using the fact that \( L_2 \) is a left ideal, we see that \( n(l_1 + l_2) - nl_2 - nl_1 = n(l_1 + l_2) - nl_1 + l \in L_2 \). It follows that \( n(l_1 + l_2) - nl_2 - nl_1 \in L_1 \cap L_2 \subseteq (0 : \gamma) \). Therefore, \( m = 0 \). This shows that for all \( n \in N \) and all \( \gamma_1, \gamma_2 \in \Gamma \) we have \( n(\gamma_1 + \gamma_2) = n\gamma_1 + n\gamma_2 \), then \( N \) is a ring. This is easy to see (see [2], Proposition 1.49).

We are already in a position to prove our main result.

**Theorem 2.2.** Let \( N \) be a zero symmetric 0-primitive near-ring which is not a ring, acting 0-primitively on the \( N \)-group \( \Gamma \). Let \( 2 \leq n \in \mathbb{N} \) and \( L_1, \ldots, L_n \) be left ideals of \( N \). Let \( \gamma_1, \ldots, \gamma_n \) be generators of the \( N \)-group \( \Gamma \) such that for any \( i \in \{1, \ldots, n\} \), \( L_i \gamma_i = \Gamma \). Suppose that \( L_i \cap L_j \subseteq (0 : \{\gamma_i, \gamma_j\}) \) whenever \( i \neq j \). Let \( \delta_i \in \Gamma \) for \( i \in \{1, \ldots, n\} \). Then there is an element \( k \in N \) such that \( k\gamma_i = \delta_i \) for all \( i \in \{1, \ldots, n\} \).

**Proof.** For all \( i \in \{1, \ldots, n\} \) we have \( L_i \gamma_i = \Gamma \) by assumption. Hence, for any \( i \in \{1, \ldots, n\} \), there is an element \( l_i \in L_i \) such that \( l_i \gamma_i = \delta_i \). Let \( i \neq j \), \( i, j \in \{1, \ldots, n\} \). By assumption we have \( L_i \cap L_j \subseteq (0 : \{\gamma_i, \gamma_j\}) \). Hence, \( L_i \cap L_j \subseteq (0 : \gamma_j) \). Also we have \( L_j \gamma_j = \Gamma \). Suppose \( L_i \gamma_j = L_j \gamma_j = \Gamma \). Then Lemma 2.1 gives us that \( N \) is a ring, so we must have \( L_i \gamma_j \neq \Gamma \). Since \( \Gamma = L_j \gamma_j \), \( \gamma_j \) is a generator of the \( N \)-group \( \Gamma \). So, \( L_i \gamma_j \) is an \( N \)-ideal in \( \Gamma \). Since \( \Gamma \) is an \( N \)-group of type 0 it follows that \( L_i \gamma_j = \{0\} \).

So, for any \( i \in \{1, \ldots, n\} \) there is an element \( l_i \in L_i \) such that \( l_i \gamma_i = \delta_i \) and \( l_j \gamma_j = 0 \) if \( j \neq i \), \( j \in \{1, \ldots, n\} \). Let \( k = (l_1 + \ldots + l_n) \). Then, \( k\gamma_i = \delta_i \) for all \( i \in \{1, \ldots, n\} \).
Note that the proof of Theorem 2.2 only requires Lemma 2.1 with its straightforward proof and is especially simple. It will be shown that the statement of Theorem 2.2 is indeed equivalent to Theorem 1.1, whose standard proof given in the literature requires much more effort. Before, we will point out a corollary which simplifies the statement of Theorem 2.2 when considering minimal left ideals in a 0-primitive near-ring.

**Corollary 2.3.** Let $N$ be a zero symmetric and 0-primitive near-ring which is not a ring, acting 0-primitively on the $N$-group $\Gamma$. Let $2 \leq n \in \mathbb{N}$ and let $L_1, \ldots, L_n$ be different minimal left ideals of $N$ and $\gamma_1, \ldots, \gamma_n$ be generators of $\Gamma$ such that for $i \in \{1, \ldots, n\}$, $L_i\gamma_i = \Gamma$. Let $\delta_i \in \Gamma$ for $i \in \{1, \ldots, n\}$. Then there is an element $k \in N$ such that $k\gamma_i = \delta_i$ for all $i \in \{1, \ldots, n\}$.

**Proof.** The proof follows from Theorem 2.2 by observing that $L_i \cap L_j = \{0\}$ if $i \neq j$, $i, j \in \{1, \ldots, n\}$ because $L_i$ and $L_j$ are different minimal left ideals. $\square$

In Theorem 2.2 we supposed that for any $i \in \{1, \ldots, n\}$ there is an element $\gamma_i \in \Gamma$ such that $L_i\gamma_i = \Gamma$. If we have a 1-primitive near-ring $N$ which acts on the $N$-group $\Gamma$ of type 1, then this assumption will be naturally fulfilled by any non-zero left ideal in $N$. To see this, let $L$ be a left ideal of the 1-primitive near-ring $N$. Then, by faithfulness of $\Gamma$, $L\Gamma \neq \{0\}$. So, there is an element $\gamma \in \Gamma$ such that $\{0\} \neq L\gamma \subseteq N\gamma$. Since $\Gamma$ is of type 1 we must have $N\gamma = \Gamma$. So, $\gamma$ is a generator of $\Gamma$ and consequently $L\gamma$ is an $N$-ideal in $\Gamma$. Since $\Gamma$ has no non-trivial $N$-ideals, it follows that $L\gamma = \Gamma$.

We need another Lemma for the proof of our last theorem, Theorem 2.5.

**Lemma 2.4.** Let $N$ be a zero symmetric near-ring which is not a ring which acts 0-primitively on the $N$-group $\Gamma = N\gamma$. Let $L_1, \ldots, L_n$ be a finite collection of left ideals of the near-ring $N$. Suppose $\cap_{i=1}^n L_i \subseteq (0 : \gamma)$. Then there is a $j \in \{1, \ldots, n\}$ such that $L_j \subseteq (0 : \gamma)$.

**Proof.** The proof is done by induction on the number of left ideals appearing in the intersection. Let $m$ be this number. If $m = 1$. Then $L_1 \subseteq (0 : \gamma)$ and the statement is clear.

By induction hypothesis we assume $\cap_{i=1}^m L_i \subseteq (0 : \gamma)$ implies that there is $j \in \{1, \ldots, m\}$ such that $L_j \subseteq (0 : \gamma)$.

Now we assume $\cap_{i=1}^m L_i \cap L_{m+1} = \cap_{i=1}^{m+1} L_i \subseteq (0 : \gamma)$, where $L_{m+1}$ is a left ideal in $N$. Since $\Gamma$ is an $N$-group with generator $\gamma$ we have that $(\cap_{i=1}^m L_i)\gamma$ is an $N$-ideal of $\Gamma$ and also $L_{m+1}\gamma$ is an $N$-ideal of $\Gamma$. Since $\Gamma$ is an $N$-group of type 0 it only can be that $(\cap_{i=1}^m L_i)\gamma = \{0\}$ or $(\cap_{i=1}^m L_i)\gamma = \Gamma$. Similary, $L_{m+1}\gamma = \{0\}$ or $L_{m+1}\gamma = \Gamma$.

Let $(\cap_{i=1}^m L_i)\gamma = \{0\}$. Then $\cap_{i=1}^m L_i \subseteq (0 : \gamma)$ and by induction hypothesis there is a $k \in \{1, \ldots, m\}$ such that $L_k \subseteq (0 : \gamma)$ and the proof is complete. The proof is also complete if $L_{m+1}\gamma = \{0\}$. 
So, suppose \( (\cap_{i=1}^m L_i)\gamma = \Gamma \) and \( L_{m+1}\gamma = \Gamma \). Since \( \cap_{i=1}^{m+1} L_i \subseteq (0: \gamma) \) by assumption, it follows from Lemma 2.1 that \( N \) is a ring. So this situation cannot happen and the proof finally is complete.

Lemma 2.4 tells us something about the behaviour of left ideals in 0-primitive near-rings. So it could prove interesting when studying the structure of 0-primitive near-rings. A complete structural classification of 0-primitive near-rings is still open (see [2]). We finally want to show that Theorem 2.2 and Theorem 1.1 are indeed equivalent statements. Note that if each generator of \( \Gamma \) has the same annihilator, then the result of Theorem 1.1 is trivial. Hence, we are interested in near-rings where we have at least two different annihilators of generators of the \( N \)-group \( \Gamma \).

**Theorem 2.5.** Let \( N \) be a zero symmetric near-ring, \( N \) not a ring, which acts 0-primitively on the \( N \)-group \( \Gamma \). Then the following hold:

1. Suppose there exist left ideals \( L_1, \ldots, L_n \) of \( N \), \( 2 \leq n \in \mathbb{N} \), such that for any \( i \in \{1, \ldots, n\} \) there is an element \( \gamma_i \in \Gamma \) such that \( L_i\gamma_i = \Gamma \) and \( L_i \cap L_j \subseteq (0 : \{\gamma_i, \gamma_j\}) \) whenever \( i \neq j \). Then for all \( i, j \in \{1, \ldots, n\}, i \neq j \), \( (0 : \gamma_i) \neq (0 : \gamma_j) \) and the result of Theorem 2.2 follows from Theorem 1.1.

2. Suppose there exist \( \gamma_1, \ldots, \gamma_n \), \( 2 \leq n \in \mathbb{N} \), being generators of the \( N \)-group \( \Gamma \) such that for \( i \neq j \), \( (0 : \gamma_i) \neq (0 : \gamma_j) \). Then there are left ideals \( L_1, \ldots, L_n \) of \( N \) such that \( L_i\gamma_i = \Gamma \) for any \( i \in \{1, \ldots, n\} \) and \( L_i \cap L_j \subseteq (0 : \{\gamma_i, \gamma_j\}) \) whenever \( i \neq j \). So the result of Theorem 1.1 follows from Theorem 2.2.

**Proof.** We prove the first statement: Suppose we have two different generators \( \gamma_i \) and \( \gamma_j \) fulfilling the assumptions of (1). We have that \( L_i\gamma_i = \Gamma \). So, by Lemma 2.1 we must have \( L_i\gamma_j \neq \Gamma \). Since \( \gamma_j \) is a generator of \( \Gamma \), \( L_i\gamma_j \) is an \( N \)-ideal in \( \Gamma \). \( \Gamma \) is an \( N \)-group of type 0, so we have that \( L_i\gamma_j = \{0\} \). Thus, \( L_i \subseteq (0 : \gamma_j) \) but \( L_i\gamma_i = \Gamma \). So \( L_i \not\subseteq (0 : \gamma_i) \) and hence \( (0 : \gamma_i) \neq (0 : \gamma_j) \).

Now we prove the second statement: Let \( \gamma_1, \ldots, \gamma_n \) be generators of the \( N \)-group \( \Gamma \) such that for \( i \neq j \), \( (0 : \gamma_i) \neq (0 : \gamma_j) \). We need to find left ideals \( L_1, \ldots, L_n \) such that \( L_i\gamma_i = \Gamma \) and \( L_i \cap L_j \subseteq (0 : \{\gamma_i, \gamma_j\}) \) if \( i \neq j \) for \( i, j \in \{1, \ldots, n\} \).

Let \( L_i := \cap_{s=1, s \neq i}^n (0 : \gamma_s) \). We will see in the following that these left ideals \( L_i, i \in \{1, \ldots, n\} \) will do the job.

Suppose that \( L_i \subseteq (0 : \gamma_i) \). Then Lemma 2.4 shows that there is \( k \in \{1, \ldots, n\} \setminus \{i\} \) such that \( (0 : \gamma_k) \subseteq (0 : \gamma_i) \). Since \( \Gamma \) is an \( N \)-group of type 0 with generator \( \gamma_k \), \( (0 : \gamma_k) \) is a maximal left ideal of \( N \) (see [2], Proposition 3.4). Maximality of \( (0 : \gamma_k) \) implies that \( (0 : \gamma_k) = (0 : \gamma_i) \) which is a contradiction.
to our assumptions. Thus, \( L_i \not\subseteq (0 : \gamma_i) \). So, \( \{0\} \neq L_i\gamma_i \) is a non-zero \( N \)-ideal of \( \Gamma \) and by 0-primitivity \( L_i\gamma_i = \Gamma \).

It remains to show that \( L_i \cap L_j \subseteq (0 : \{\gamma_i, \gamma_j\}) \) for all \( j \in \{1, \ldots, n\} \setminus \{i\} \).

Let \( a \in L_i \cap L_j \). Then \( a \in L_i \), so \( a\gamma_s = 0 \) for all \( s \in \{1, \ldots n\} \setminus \{i\} \). Since \( j \neq i \) we get that \( a\gamma_j = 0 \). But we also have that \( a \in L_j \), so \( a\gamma_s = 0 \) for all \( s \in \{1, \ldots n\} \setminus \{j\} \). Since \( i \neq j \), we have \( a\gamma_i = 0 \) and so we have that \( L_i \cap L_j \subseteq (0 : \{\gamma_i, \gamma_j\}) \).

\[ \square \]

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**References**


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