Local Cohomology and Non Commutative Gorenstein Algebras

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Abstract

In this paper we continue the study of non connected graded Gorenstein algebras initiated in [13], the main result is the proof of a version of the Local Cohomology formula.

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1 Introduction

Non commutative versions of regular and Gorenstein algebras have been studied by several authors, [2], [3],[6],[7],[8],[9], [19], they usually deal with graded connected algebras. In [11],[15] we studied non connected Artin Schelter regular algebras and proved, that for such algebras the Local Cohomology formula and Serre duality hold. A natural example of such algebras is the preprojective $\mathbb{k}$-algebra of an Euclidean Diagram $Q$ [10].

In [13] we investigated non connected graded Artin Schelter Gorenstein algebras, an easy example is the following: let $\Gamma$ be a preprojective $\mathbb{k}$-algebra of an Euclidean Diagram $Q$ with only sinks and sources and $\Lambda=\mathbb{k}Q \triangleright D(\mathbb{k}Q)$ the trivial extension. Then the algebra $\Lambda \otimes_{\mathbb{k}} \Gamma$ is non connected and Artin Schelter Gorenstein.
The aim of the paper is to continue the study of these algebras and to provide a non connected version of the Local Cohomology formula. The article consists of two sections; in the first one we fix the notation and recall some basic results on Artin Schelter regular and Gorenstein algebras from [11], [12], [13], then we study the structure of a generalization of non connected AS Gorenstein algebras. In the second section we give an elementary proof of the Local Cohomology formula for such algebras.

2 Definitions and basic results

We recall first some basic definitions and results on graded Gorenstein algebras, then we concentrate in the study of the structure of graded Artin Schelter Gorenstein algebras. For further properties on graded and Gorenstein algebras we refer the reader to [1], [9],[12],[14].

Definition 2.1. Let $k$ be a field, a locally finite positively graded $k$-algebra is a positively graded $k$-algebra $\Lambda = \bigoplus_{i \geq 0} \Lambda_i$ such that:

i) $\Lambda_0 = k \times k \times ... k$

ii) $\dim_k \Lambda_i < \infty$.

Example 2.2. Given a finite quiver $Q$ and a field $k$, the quiver algebra is graded by path length, given an homogeneous ideal $I$ the quotient $kQ/I$ is a locally finite positively graded $k$-algebra.

The graded algebras we will consider here will be always locally finite positively graded $k$-algebras.

Definition 2.3. Given a $\mathbb{Z}$-graded module $M = \{M_i\}_{i \in \mathbb{Z}}$ over a locally finite positively graded $k$-algebra $\Lambda$, we say $M$ is locally finite if $\dim_k M_i < \infty$ for all $i$.

Given a $\mathbb{Z}$-graded module $M = \{M_i\}_{i \in \mathbb{Z}}$ we denote by $M[n]$ the n-th shift defined by $M[n]_i = M_{n+i}$ and we denote the n-truncation of $M$ by $M_{\geq n}$, where $M_{\geq n}$ is defined by $(M_{\geq n})_j = \begin{cases} 0 & \text{if } j < n \\ M_j & \text{if } j \geq n \end{cases}$

Definition 2.4. Given a positively graded $k$-algebra $\Lambda = \bigoplus_{i \geq 0} \Lambda_i$, the graded Jacobson radical is $m = \Lambda_{\geq 1}$.

We denote by $\text{Gr}_\Lambda$ the category of graded $\Lambda$-modules and degree zero maps. Given graded $\Lambda$-modules $M$ and $N$, $\text{Hom}_\Lambda(M,N)_k$ is the set of all maps $f : M \rightarrow N$ such that $f(M_j) \subseteq N_{j+k}$ and we call them, maps in degree $k$. By $\text{Hom}_\Lambda(M,N)$ we mean $\text{Hom}_\Lambda(M,N) = \bigoplus_{k \in \mathbb{Z}} \text{Hom}_\Lambda(M,N)_k$. The maps in
all degrees $\text{Hom}_\Lambda(M,N)$ is a graded $k$-vector space. We have isomorphisms: $\text{Hom}_\Lambda(M,N)_k \cong \text{Hom}_\Lambda(M,N[k])_0 \cong \text{Hom}_\Lambda(M[-k],N)_0$. If $M$ is finitely generated and $N$ locally finite, then $\text{Hom}_\Lambda(M,N)$ is locally finite.

In a similar way the $k$ extensions in degree zero, $\text{Ext}_\Lambda^k(\cdot,?)_0$ are the derived functors of $\text{Hom}_\Lambda(\cdot,?)_0$. We define $\text{Ext}_\Lambda^k(M,N)=\bigoplus_{n \in \mathbb{Z}} \text{Ext}_\Lambda^k(M,N[n])$ and $\text{Ext}_\Lambda^k(M,N)_0=\text{Hom}_\Lambda(M,N[k])_0$.

Denote by $\text{l.f.} \text{Gr}_\Lambda$ the full subcategory of $\text{Gr}_\Lambda$ consisting of all locally finite modules. Then $\text{l.f.} \text{Gr}_\Lambda$ is abelian and there is a duality $D: \text{l.f.} \text{Gr}_\Lambda \rightarrow \text{l.f.} \text{Gr}_\Lambda^{\text{op}}$ given by $D(M)_j=\text{Hom}_k(M_{-j},k)$ and $D(M)=\{D(M)_j\}_{j \in \mathbb{Z}}$.

A notion of regular for non commutative connected positively graded algebras was introduced by Artin and Schelter [1], a slight generalization is the following: (See [11],[12])

**Definition 2.5.** Let $\Lambda$ be a locally finite positively graded $k$-algebra. Then $\Lambda$ is called Artin-Schelter regular if the following conditions are satisfied:

i) There is an integer $n$ such that all graded simple have projective dimension $n$.

ii) For any graded simple $S$ and an integer $0 \leq i < n$, $\text{Ext}_\Lambda^i(S,\Lambda)=0$.

iii) The assignment $S \rightarrow \text{Ext}_\Lambda^n(S,\Lambda)$ gives a bijection between the graded simple $\Lambda$-modules and the graded simple $\Lambda^{\text{op}}$-modules.

The above definition was extended to graded categories in [15].

**Definition 2.6.** A ring $R$ is called Gorenstein if $R$ has finite injective dimension both as a left and as a right $R$-module. We denote the left (right) injective dimension by $\text{injdim}_R R$ ($\text{injdim}_R R$).

It was proved in [19] that Gorenstein implies $\text{injdim}_R R = \text{injdim}_R R$, but it is not known whether a one side condition implies the condition on both sides.

The notion of Artin-Schelter regular inspired a definition of Gorenstein for connected graded algebras that has been used by several authors like: [8],[9], [18], [20]. It has been called Artin Schelter Gorenstein (AS Gorenstein, for short). We will use here the following variation of that definition for non connected graded algebras:

**Definition 2.7.** Let $k$ be a field and $\Lambda$ a locally finite positively graded $k$-algebra. Then we say that $\Lambda$ is graded Artin Schelter Gorenstein if the following conditions are satisfied:

There exists a non negative integer $n$, called the graded injective dimension of $\Lambda$, such that:

i) For all graded simple $S_i$ concentrated in degree zero and non negative integers $j \neq n$, $\text{Ext}_\Lambda^j(S_i,\Lambda)=0$.

ii) We have an isomorphism $\text{Ext}_\Lambda^n(S_i,\Lambda)=S_i[-n]$, with $S_i$ a graded $\Lambda^{\text{op}}$-simple.
iii) For a non negative integer $k \neq n$, $\text{Ext}^k_{\Lambda \text{op}}(\text{Ext}^n_{\Lambda}(S_i, \Lambda), \Lambda) = 0$ and $\text{Ext}^n_{\Lambda \text{op}}(\text{Ext}^n_{\Lambda}(S_i, \Lambda), \Lambda) = S_i$.

Observe that in this definition we are not assuming the algebra is Gorenstein, and it is not clear that condition i) implies $\Lambda$ of finite injective dimension. However, our definition of graded AS Gorenstein is a two sided condition. Would be interesting to know if graded AS Gorenstein implies Gorenstein. We will some times assume that an algebra is both Gorenstein and graded AS Gorenstein.

We next recall a result from [13] that has an interesting corollary related to the above remark.

A proof using spectral sequences and results from [16] was suggested by Miyachi, being more elementary, we reproduce here the proof given in [13].

**Theorem 2.8.** Let $R$ be an arbitrary Gorenstein ring and $M$ a left $R$-module with a projective resolution consisting of finitely generated projective modules. Assume there is a non negative integer $n$ such that $\text{Ext}^j_R(M, R) = 0$ for $j \neq n$.

Then $\text{Ext}^n_R(M, R)$ satisfies the following conditions:

a) $\text{Ext}^j_{R \text{op}}(\text{Ext}^n_R(M, R), R) = 0$ for $j \neq n$.

b) $\text{Ext}^n_{R \text{op}}(\text{Ext}^n_R(M, R), R) \cong M$.

In case $R$ is graded and $M$ a graded $R$-module, the isomorphism is as graded $R$-modules.

**Proof.** Assume $M$ is of finite projective dimension $n$.

Let $n=0$. Then $M$ and $M^*$ are projective, $M^{**} \cong M$ and $\text{Ext}^\ell_R(M, R) = 0$ for $\ell \neq 0$.

Assume $\text{pd}M = n > 0$ and let: $0 \to P_n \to P_{n-1} \to \ldots P_1 \to P_0 \to M \to 0$ be a projective resolution with $P_j$ finitely generated for all $j$.

Dualizing with respect to the ring we obtain an exact sequence:

$0 \to P_0^* \to P_1^* \to \ldots P_{n-1}^* \to P_n^* \to \text{Ext}^n_R(M, R) \to 0$.

Dualizing again the complex: $0 \to P_{**} \to P_{**-1} \to \ldots P_1^{**} \to P_0^{**} \to 0$ is isomorphic to the complex:

$0 \to P_n \to P_{n-1} \to \ldots P_1 \to P_0 \to 0$.

It follows $\text{Ext}^i_{R \text{op}}(\text{Ext}^n_R(M, R), R) = 0$ for $i \neq n$ and $\text{Ext}^n_{R \text{op}}(\text{Ext}^n_R(M, R), R) \cong M$.

Assume $M$ is of infinite projective dimension.

Let $*) \ldots \to P_n \to P_{n-1} \to \ldots P_1 \to P_0 \to M \to 0$ be a projective resolution with $P_j$ finitely generated for all $j$.

Consider first the case $n=0$.

Dualizing with respect to the ring we obtain an exact sequence:

$0 \to M^* \to P_0^* \to P_1^* \to \ldots P_t^* \to P_{t+1}^* \to Y \to 0$, where $t$ is the injective dimension of $R$.

Then $M^* = \Omega^{t+2} Y$ and for $i > 0$, $\text{Ext}^i_R(M^*, R) = \text{Ext}^i_R(\Omega^{t+2} Y, R) = \text{Ext}^{i+t+1}_R(Y, R) = 0$, also $\text{Ext}^1_R(\Omega Y, R) = \text{Ext}^1_R(Y, R) = 0$.
Hence the exact sequences:
$$0 \rightarrow M^* \rightarrow P_0^* \rightarrow \Omega^{i+1}Y \rightarrow 0$$ and $$0 \rightarrow \Omega^{i+1}Y \rightarrow P_1^* \rightarrow \Omega^iY \rightarrow 0$$ induce exact sequences:

$$0 \rightarrow (\Omega^{i+1}Y)^* \rightarrow P_0^{**} \rightarrow M^{**} \rightarrow 0$$ and $$0 \rightarrow (\Omega^iY)^* \rightarrow P_1^{**} \rightarrow (\Omega^{i+1}Y)^* \rightarrow 0.$$ 

We have proved the sequence: $$P_1^{**} \rightarrow P_0^{**} \rightarrow M^{**} \rightarrow 0$$ is exact.

It follows $$M^* \cong M^{**}$$ and $$\text{Ext}_R^i(M^*,R) = 0$$ for $$i \neq n.$$ 

Assume now $$n > 0.$$ Dualizing *) with respect to the ring we get the complex:

$$0 \rightarrow M^* \rightarrow P_0^* \rightarrow P_1^* \rightarrow \ldots P_{n-1}^* \rightarrow P_{n-1}^* \rightarrow P_n^* \rightarrow P_{n+1}^* \rightarrow \ldots$$

whose homology is zero except at degree $$n,$$ where $$\text{Ker} f_{n+1}^* / \text{Im} f_n^* = \text{Ext}_R^n(M,R).$$

Let $$C = P_n^* \text{Im} f_n^*$$ and $$X = \text{Im} f_{n+1}^* = \text{Ker} f_{n+2}^* = P_n^*/\text{Ker} f_{n-1}^*.$$ 

There is an exact sequence:

*) $$0 \rightarrow \text{Ext}_R^n(M,R) \rightarrow C \rightarrow X \rightarrow 0.$$ 

Consider the exact sequence:

$$0 \rightarrow X \rightarrow P_{n+1}^* \rightarrow P_{n+2}^* \rightarrow \ldots P_{n+t-1}^* \rightarrow P_{n+t}^* \rightarrow Y \rightarrow 0,$$

with $$t$$ the injective dimension of $$R.$$ 

Then for $$i > 0,$$ we have: $$\text{Ext}_R^i(X,R) = \text{Ext}_R^i(\Omega^iY,R) = \text{Ext}_R^{i+t}(Y,R) = 0.$$ 

By the sequence *) and the long homology sequence we have an exact sequence:

$$\text{Ext}_R^i(X,R) \rightarrow \text{Ext}_R^i(C,R) \rightarrow \text{Ext}_R^i(\text{Ext}_R^n(M,R),R) \rightarrow \text{Ext}_R^{i+1}(X,R).$$

It follows that for each $$i \geq 1$$ there is an isomorphism: $$\text{Ext}_R^i(C,R) \cong \text{Ext}_R^i(\text{Ext}_R^n(M,R),R)$$ and that the sequence

$$0 \rightarrow X^* \rightarrow C^* \rightarrow \text{Ext}_R^n(M,R)^* \rightarrow 0$$

is exact.

Since $$M^* = 0,$$ the sequence: $$0 \rightarrow P_0^* \rightarrow P_1^* \rightarrow \ldots P_{n-1}^* \rightarrow P_n^* \rightarrow P_{n+1}^* \rightarrow C \rightarrow 0$$ is exact and $$\text{pd} C \leq n.$$ 

Being the complexes:
0→P_n→P_{n−1}→⋯P_1→P_0→0 and 0→P^{**}_n→P^{**}_{n−1}→⋯P^{**}_1→P^{**}_0→0 isomorphic, it follows Ext^i_{R^{op}}(C,R)=0 for i≠n, i≠0 and Ext^n_{R^{op}}(C,R)=M.

By the above observations, Ext^i_{R^{op}}(Ext^n_R(M,R),R)=0 for i≠0, i≠n and Ext^n_{R^{op}}(Ext^n_R(M,R),R)≅M.

Consider the following diagram with exact column and first row:

\[
\begin{array}{cccccccc}
0 & & & & & & & \\
\downarrow & & & & & & & \\
P^{*}_{n−1} & \rightarrow & \text{Ker}^*_n & \rightarrow & \text{Ext}^n_R(M,R) & \rightarrow & 0 \\
\downarrow 1 & & & & & & & \\
P^*_n & \xrightarrow{f^*_n} & P^*_n & \xrightarrow{f^*_n} & P^*_n & \rightarrow & P^{*+1}_n \\
\downarrow & & & & & \nearrow & & \\
\text{Im}^*_n & \rightarrow & 0
\end{array}
\]

Using the fact Ext^1_R(X,R)=Ext^1_R(\text{Im}^*_n, R)=0 we obtain by dualizing an exact commutative diagram:

\[
\begin{array}{cccccccc}
0 & & & & & & & \\
\downarrow & & & & & & & \\
(\text{Im}^*_n)^* & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow \\
\downarrow & & & & & & & \\
P^{**}_{n+1} & \xrightarrow{f^{**}_{n+1}} & P^{**}_n & \xrightarrow{f^{**}_n} & P^{**}_n & \rightarrow & P^{**+1}_n \\
\downarrow t & & & & & p & & \\
0 \rightarrow \text{Ext}^n_R(M,R)^* & \xrightarrow{S} & (\text{Ker}^*_n)^* & \rightarrow & P^{**+1}_n \\
\downarrow & & & & & & & \\
0
\end{array}
\]

By Five’s lemma, t is an epimorphism and st=pf^{**}_{n+1}=0 implies Ext^n_R(M,R)^*=0, as claimed.

**Corollary 2.9.** Let \( \Lambda \) be a locally finite positively graded \( k \)-algebra that is Gorenstein of injective dimension \( n \), such that all graded left and all graded right simple have projective resolutions consisting of finitely generated projective modules.

Assume the following conditions hold:

a) For all graded simple \( S \) and non negative integers \( i≠n \), Ext^i_\Lambda(S,\Lambda)=0.

b) Each right module Ext^n_\Lambda(S,\Lambda) is graded simple.

Then \( \Lambda \) is graded AS Gorenstein.
We come back to the more general notion of AS Gorenstein graded algebra and prove that, like in the Artin-Schelter regular case (See [15]), the functor \( \text{Ext}^n_A(-, \Lambda) \) induces a duality between the categories of graded \( \Lambda \)-modules of finite length and the corresponding category of \( \Lambda^{op} \)-modules.

**Proposition 2.10.** Let \( \Lambda \) be a graded AS Gorenstein algebra of graded injective dimension \( n \), and assume all graded simple left modules have projective resolutions consisting of finitely generated projective modules. Then for any graded left \( \Lambda \)-module \( M \) of finite length, the following is true:

1. For any non negative integers \( i \neq n \), \( \text{Ext}^i_A(M, \Lambda) = 0 \).
2. The right \( \Lambda \)-module \( \text{Ext}_A^n(M, \Lambda) \) has finite length and \( \ell(\text{Ext}_A^n(M, \Lambda)) = \ell(M) \).
3. \( \text{Ext}^n_{\Lambda^{op}}(\text{Ext}_A^n(M, \Lambda), \Lambda) \cong M \).

**Proof.** We consider first the case \( n=0 \).

This means that for any graded simple \( S \) the dual with respect to the ring \( S^* \) is simple, \( S \cong S^{**} \) and \( \text{Ext}^i_A(S, \Lambda) = 0 \) for \( i \neq 0 \).

We prove the claim by induction on \( \ell(M) \).

Let \( M' \) be a maximal graded submodule. Then \( M/M' = S \) is simple and the exact sequence: \( 0 \rightarrow M' \rightarrow M \rightarrow S \rightarrow 0 \) induces, by hypothesis, the exact sequence:

\[
\begin{array}{cccccc}
0 & \rightarrow & M' & \rightarrow & M & \rightarrow & S & \rightarrow & 0 \\
& & \downarrow \theta_{M'} & & \downarrow \theta_M & & \downarrow \theta_S \\
0 & \rightarrow & (M')^{**} & \rightarrow & M^{**} & \rightarrow & S^{**}
\end{array}
\]

By hypothesis \( \theta_{M'} \) and \( \theta_S \) are isomorphisms, by the short Five’s lemma \( \theta_M \) is an isomorphism. By induction hypothesis \( \ell(M') = \ell((M')^*) \) and the exactness of the sequence \( * \) implies \( \ell(M) = \ell(M^*) \).

Assume \( n > 0 \) and apply again induction on \( \ell(M) \). As before, \( M' \) is a maximal graded submodule of \( M \) and \( M/M' = S \).

From the long homology sequence we have an exact sequence:

\[
\text{Ext}^i_A(S, \Lambda) \rightarrow \text{Ext}^i_A(M, \Lambda) \rightarrow \text{Ext}^i_A(M', \Lambda)
\]

\[\text{Ext}^i_A(S, \Lambda) = \text{Ext}^i_A(M', \Lambda) = 0 \text{ for } i \neq n \text{ implies } \text{Ext}^i_A(M, \Lambda) = 0 \text{ for } i \neq n, \]

and \( \ell(\text{Ext}^n_A(M', \Lambda)) = \ell(M') \) implies \( \ell(\text{Ext}^n_A(M, \Lambda)) = \ell(M) \).

We only need to prove \( \text{Ext}^n_{\Lambda^{op}}(\text{Ext}^n_A(M, \Lambda), \Lambda) \cong M \).

Let

\[
\cdots \rightarrow Q_i \rightarrow Q_{i-1} \rightarrow \cdots Q_1 \rightarrow Q_0 \rightarrow M' \rightarrow 0
\]

and \( \cdots \rightarrow P_i \rightarrow P_{i-1} \rightarrow \cdots P_1 \rightarrow P_0 \rightarrow S \rightarrow 0 \)
be graded projective resolutions. By Horseshoe’s lemma, there is an exact commutative diagram:

**)  

\[
\begin{array}{cccccc}
0 & \to & Q_{i+1} & \to & Q_{i+1} \oplus P_{i+1} & \to & P_{i+1} & \to & 0 \\
f_{i+1} & \downarrow & h_{i+1} & \downarrow & g_{i+1} & \downarrow & \\
0 & \to & Q_i & \to & Q_i \oplus P_i & \to & P_i & \to & 0 \\
0 & \to & Q_1 & \to & Q_1 \oplus P_1 & \to & P_1 & \to & 0 \\
f_1 & \downarrow & h_1 & \downarrow & g_1 & \downarrow & \\
0 & \to & Q_0 & \to & Q_0 \oplus P_0 & \to & P_0 & \to & 0 \\
f_0 & \downarrow & h_0 & \downarrow & g_0 & \downarrow & \\
0 & \to & M' & \to & M & \to & S & \to & 0 \\
\end{array}
\]

Dualizing the diagram **) with respect to the ring we obtain an exact sequence of complexes:

\[
\begin{array}{cccccc}
0 & \to & P_0^* & \to & P_0^* \oplus Q_0^* & \to & Q_0^* & \to & 0 \\
g_1^* & \downarrow & h_1^* & \downarrow & f_1^* & \downarrow & \\
0 & \to & P_1^* & \to & P_1^* \oplus Q_1^* & \to & Q_1^* & \to & 0 \\
0 & \to & P_n^* & \to & P_n^* \oplus Q_n^* & \to & Q_n^* & \to & 0 \\
g_{n+1}^* & \downarrow & h_{n+1}^* & \downarrow & f_{n+1}^* & \downarrow & \\
0 & \to & P_{n+1}^* & \to & P_{n+1}^* \oplus Q_{n+1}^* & \to & Q_{n+1}^* & \to & 0 \\
\end{array}
\]

whose homology is zero except at degree n. Then we have an exact sequence of projective resolutions:
Local cohomology

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & P_0' & P_0^* \oplus Q_0^* & Q_0^* \\
\downarrow g_1^* & \downarrow h_1^* & \downarrow f_1^* & \\
0 & P_1^* & P_1^* \oplus Q_1^* & Q_1^* \\
\vdots & \vdots & \vdots & \\
\downarrow & \downarrow & \downarrow & \\
0 & P_n^* & P_n^* \oplus Q_n^* & Q_n^* \\
\downarrow & \downarrow & \downarrow & \\
0 & C_S & C_M & C_{M'} \\
\downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0 \\
\end{array}
\]

Setting as above \( C_S = P_n^*/\text{Img}_{n+1} \), \( C_M = P_n^* \oplus Q_n^*/\text{Imh}_{n+1} \), and \( C_{M'} = Q_n^*/\text{Imf}_{n+1} \), and \( X_S = P_n^*/\text{Ker}_{n+1} \), \( X_M = P_n^* \oplus Q_n^*/\text{Ker}_{n+1} \), and \( X_{M'} = Q_n^*/\text{Imf}_{n+1} \).

Then there is an exact commutative diagram:

\[
\begin{array}{ccc}
0 & \to & \text{Ext}_n^\Lambda(S, \Lambda) \\
\downarrow & & \downarrow \\
0 & \to & \text{Ext}_n^\Lambda(M, \Lambda) \\
\downarrow & & \downarrow \\
0 & \to & \text{Ext}_n^\Lambda(M', \Lambda) \\
\downarrow & & \downarrow \\
0 & \to & X_S \\
\downarrow & & \downarrow \\
0 & \to & X_M \\
\downarrow & & \downarrow \\
0 & \to & X_{M'} \\
\end{array}
\]

Applying the functor \( \text{Ext}_{\Lambda}^n(-, \Lambda) \) to the diagram ***) we have a commutative exact diagram:

\[
\begin{array}{ccc}
M' & \to & M \\
\downarrow \cong & & \downarrow \cong \\
0 & \to & \text{Ext}_\Lambda^n(C_{M'}, \Lambda) \\
\downarrow \psi_{M'} & & \downarrow \psi_M \\
0 & \to & \text{Ext}_\Lambda^n(C_M, \Lambda) \\
\downarrow \psi_S & & \downarrow \psi_S \\
0 & \to & \text{Ext}_\Lambda^n(C_{S}, \Lambda) \\
\end{array}
\]

where \( \psi_{M'} \) and \( \psi_{S} \) are isomorphisms. Therefore \( \psi_{M} \) is an isomorphism.

****

Lemma 2.11. Let \( \Lambda \) be an algebra over a field \( \mathbb{k} \) and \( M \) a finitely presented left \( \Lambda \)-module. Then for any left \( \Lambda \)-module \( X \) there is a natural isomorphism of \( \mathbb{k} \)-vector spaces: \( \text{Hom}_\mathbb{k}(\text{Hom}_\Lambda(M,X), \mathbb{k}) \cong \text{Hom}_\mathbb{k}(X, \mathbb{k}) \otimes_\Lambda M \).
Proof. Let \( P_1 \to P_0 \to M \to 0 \) be a finite projective presentation of \( M \). Tensoring with \( D(X) = \text{Hom}_k(X,k) \), we obtain an exact sequence of \( k \)-vector spaces:
\[
D(X) \otimes P_1 \to D(X) \otimes P_0 \to D(X) \otimes M \to 0,
\]
which is isomorphic to the exact sequence:
\[
\text{**}) \quad \text{Hom}_\Lambda(P_1^*,D(X)) \to \text{Hom}_\Lambda(P_0^*,D(X)) \to D(X) \otimes M \to 0.
\]
\( P_i^* \) is the dual with respect to the ring.

By adjunction, the sequence \( \text{**} \) is isomorphic to the exact sequence:
\[
\text{Hom}_{/DG}(P_1^* \otimes X,k) \to \text{Hom}_{/DG}(P_0^* \otimes X,k) \to D(X) \otimes M \to 0.
\]
In addition, \( P_i^* \otimes X \cong \text{Hom}_\Lambda(P_i,X) \) and \( \text{**} \) is isomorphic to:
\[
D(\text{Hom}_\Lambda(P_1,X)) \to D(\text{Hom}_\Lambda(P_0,X)) \to D(X) \otimes M \to 0.
\]
In the other hand, the presentation \( \text{*)} \) induces an exact sequence:
\[
0 \to \text{Hom}_\Lambda(M,X) \to \text{Hom}_\Lambda(P_0,X) \to \text{Hom}_\Lambda(P_1,X).
\]
Dualizing it we obtain the exact sequence:
\[
D(\text{Hom}_\Lambda(P_1,X)) \to D(\text{Hom}_\Lambda(P_0,X)) \to D(\text{Hom}_\Lambda(M,X)) \to 0.
\]
It follows: \( D(X) \otimes M \cong D(\text{Hom}_\Lambda(M,X)) \).

As a corollary we get:

**Proposition 2.12.** Let \( \Lambda \) be a left coherent algebra over a field \( k \). Then for any injective \( I \) its dual \( \text{Hom}_k(I,k) \) is flat.

Proof. Let \( a \) be a finitely generated left ideal of \( \Lambda \), by definition of coherent, \( a \) is finitely presented, hence by Lemma 2.11, there is a natural isomorphism:
\[
D(I) \otimes_\Lambda a \cong D(\text{Hom}_\Lambda(a,I)).
\]
The exact sequence \( 0 \to a \xrightarrow{j} \Lambda \to \Lambda/a \to 0 \) induces an exact sequence:
\[
0 \to \text{Hom}_\Lambda(\Lambda/a,I) \to \text{Hom}_\Lambda(\Lambda,I) \to \text{Hom}_\Lambda(a,I) \to 0.
\]
We obtain by dualizing it the exact sequence:
\[
0 \to D(\text{Hom}_\Lambda(a,I)) \to D(\text{Hom}_\Lambda(\Lambda,I)) \to D(\text{Hom}_\Lambda(\Lambda/a,I)) \to 0,
\]
which is isomorphic to the exact sequence: \( D(I) \otimes_\Lambda a \xrightarrow{1 \otimes j} D(I) \otimes_\Lambda A \to D(I) \otimes_\Lambda \Lambda/a \to 0 \).
Therefore: \( 1 \otimes j \) is a monomorphism.
It follows by [17] Proposition 3.58, that \( D(I) \) is flat.

The Gorenstein property, or more generally the Cohen Macaulay property, is related with the existence of a dualizing object, in the commutative case the ring or another bimodule [4], in other instances a dualizing complex, for example the injective resolution of the ring [18], [20], in the case of Artin Schelter regular algebras \( \Lambda \) considered in [11], the dualizing object was a shift of \( \Lambda \), in the case we are considering the dualizing object is also a bimodule, but its description is more subtle, we dedicate the remain of this section to find its structure.

**Proposition 2.13.** [13] Let \( \Lambda \) be a graded AS Gorenstein \( k \)-algebra of graded injective dimension \( n \). Then for any graded simple \( S_j \) concentrated in degree zero, there exists a unique indecomposable projective \( Q_{\sigma(j)} \) and a non negative
integer \( n_j \) such that \( \text{Ext}^n_{\Lambda}(S_j, Q_{\sigma(j)})[-n_j]_0 \neq 0 \) and the assignment \( S_j \to Q_{\sigma(j)} \) is a bijection.

**Proof.** We have the following isomorphisms:

\[
\text{Ext}^n_{\Lambda}(S_j, \Lambda) = \bigoplus_{k \in \mathbb{Z}} \text{Ext}^n_{\Lambda}(S_j, \Lambda)_k \cong S'_j[-n_j],
\]

where \( S'_j \) has dimension one as \( k \)-vector space. Since the isomorphism is as \( k \)-graded vector spaces, \( \text{Ext}^n_{\Lambda}(S_j, \Lambda)_k \neq 0 \) implies \( k = -n_j \). The algebra \( \Lambda \) decomposes in sum of indecomposables \( \Lambda \cong \bigoplus Q_i \). Then \( \text{Ext}^n_{\Lambda}(S_j, Q_i[-n_j])_0 = S'_j[-n_j] \) and there exists a unique integer \( \sigma(j) \) such that \( \text{Ext}^n_{\Lambda}(S_j, Q_{\sigma(j)})[-n_j]_0 \neq 0 \).

We will prove that the function \( j \to \sigma(j) \) is injective, hence bijective.

Assume for some simple \( S_k \) there is an isomorphism \( Q_{\sigma(j)} \cong Q_{\sigma(k)} \). But both \( \text{Ext}^n_{\Lambda}(S_j, Q_{\sigma(j)}) \neq 0 \) and \( \text{Ext}^n_{\Lambda}(S_k, Q_{\sigma(k)}) \neq 0 \).

There are isomorphisms:

\[
\text{Ext}^n_{\Lambda}(S_j, Q_{\sigma(j)}) = \text{Ext}^n_{\Lambda}(S_j, \Lambda) \otimes_{\Lambda} Q_{\sigma(j)} \cong S'_j \otimes_{\Lambda} Q_{\sigma(j)}[-n_j] \quad \text{and} \quad \text{Ext}^n_{\Lambda}(S_k, Q_{\sigma(k)}) = \text{Ext}^n_{\Lambda}(S_k, \Lambda) \otimes_{\Lambda} Q_{\sigma(k)} \cong S'_k \otimes_{\Lambda} Q_{\sigma(k)}[-n_k].
\]

After dualizing we obtain the following isomorphisms:

\[
D(S'_j \otimes_{\Lambda} Q_{\sigma(j)}[-n_j]) = \text{Hom}_{\Lambda^o}(S'_j, D(Q_{\sigma(j)})[n_j]) \neq 0 \quad \text{and} \quad D(S'_k \otimes_{\Lambda} Q_{\sigma(k)}[-n_k]) = \text{Hom}_{\Lambda^o}(S'_k, D(Q_{\sigma(k)})[n_k]) \neq 0.
\]

Since \( Q_{\sigma(j)} \cong Q_{\sigma(k)} \) and \( D(Q_{\sigma(j)}) \cong D(Q_{\sigma(k)}) \) has simple socle, it follows \( S'_j \cong S'_k \). In fact \( D(S'_j) \cong Q_{\sigma(j)} / JQ_{\sigma(j)} \cong S_{\sigma(j)} \).

We have proved \( \text{Ext}^n_{\Lambda}(S_j, \Lambda) \cong D(S_{\sigma(j)})[-n_j] \).

What we want to do next is to prove that \( \bigoplus_{j=1}^{m} Q_j[n_j] \) has a structure of \( \Lambda^\Lambda \)-bimodule and it is projective both as left and as right \( \Lambda \)-module.

Let \( Q_k \) be an indecomposable projective \( \Lambda \)-module with injective resolution:

\[
0 \to Q_k \to I^{(1)}_k \to I^{(2)}_k \to \cdots \to I^{(n_k)}_k \to I^{(n_k+1)}_k \to \cdots.
\]

Here \( I^{(n_k)}_k \) decomposes as \( I^{(n_k)}_k = I^{I(n_k)}_k \oplus I^{I(n_k)}_k \), where \( I^{I(n_k)}_k \) is torsion and \( I^{I(n_k)}_k \) is torsion free in the sense of [M].

Then \( 0 \neq \text{Ext}^n_{\Lambda}(S_{\sigma^{-1}(k)[n_{\sigma^{-1}(k)}]}, Q_k)_0 = \text{Hom}_{\Lambda}(S_{\sigma^{-1}(k)[n_{\sigma^{-1}(k)}]}, I^{(n_k)}_k)_0 = \text{Hom}_{\Lambda}(S_{\sigma^{-1}(k)[n_{\sigma^{-1}(k)}]}, I^{I(n_k)}_k)_0 = \text{Hom}_{\Lambda}(S_{\sigma^{-1}(k)[n_{\sigma^{-1}(k)}]}, I^{I(n_k)}_k)_0 = \text{Hom}_{\Lambda}(S_{\sigma^{-1}(k)[n_{\sigma^{-1}(k)}]}, I^{I(n_k)}_k)_0 = D(S_k)[-n_{\sigma^{-1}(k)}].\]

Therefore: \( I^{I(n_k)}_k = D(Q^*_\sigma^{-1}(k))[-n_{\sigma^{-1}(k)}].\)

Since \( \Lambda \) has a decomposition \( \Lambda \cong \bigoplus_{i=1}^{m} Q_i \), it has an injective resolution:

\[
0 \to \Lambda \to I_0 \to I_1 \to \cdots \to I_{n-1} \to I_n \to I_{n+1} \to \cdots \text{ with } I_j = \bigoplus_{k=1}^{m} I^{(k)}_j \text{ and } I_n = I'_n \oplus I''_n,
\]

with \( I'_n = \bigoplus_{k=1}^{m} I^{I(n_k)}_k \) and \( I''_n = \bigoplus_{k=1}^{m} I^{I(n_k)}_k \).

The module \( I''_n \) is torsion free and \( I'_n = \bigoplus_{k=1}^{m} D(Q^*_\sigma^{-1}(k))[-n_{\sigma^{-1}(k)}] = \bigoplus_{k=1}^{m} D(Q^*_j)[-n_j] \), after shifting.

We give \( I'_n \) a structure of \( \Lambda \)-\( \Lambda \)-bimodule as follows:
Let \( \lambda \in \Lambda \) be a non zero homogeneous element of \( \Lambda \) and \( \phi_\lambda : \Lambda \rightarrow \Lambda \) given by right multiplication, \( \phi_\lambda(x) = x\lambda \).
This map extends to a map of resolutions:

\[
0 \rightarrow \Lambda \xrightarrow{\varepsilon} I_0 \xrightarrow{d_0} I_1 \rightarrow I_n \rightarrow I_{n+1} \rightarrow \cdots \\
\phi_\lambda \downarrow \quad \phi_0^\lambda \downarrow \quad \phi_1^\lambda \downarrow \quad \phi_n^\lambda \downarrow \quad \phi_{n+1}^\lambda \downarrow \\
0 \rightarrow \Lambda \xrightarrow{\varepsilon} I_0 \xrightarrow{d_0} I_1 \rightarrow I_n \rightarrow I_{n+1} \rightarrow \cdots 
\]

Since \( I_n'' \) is torsion free \( \text{Hom}_\Lambda(I_n',I_n'') = 0 \) and \( \phi_\lambda^n : I_n' \oplus I_n'' \rightarrow I_n' \oplus I_n'' \) has triangular form:

\[
\phi_\lambda^n = \begin{bmatrix}
\wedge \\
\phi_\lambda \\
\rho \\
0 \\
\sigma
\end{bmatrix}.
\]

We claim the map: \( \phi_\lambda : I_n' \rightarrow I_n'' \) is unique.
Assume \( (\psi_\lambda^0, \psi_\lambda^1, \psi_\lambda^2, \ldots, \psi_\lambda^n, \psi_\lambda^{n+1}, \ldots) \) is another lifting of \( \phi_\lambda \).

The map \( \psi_\lambda^n \) has triangular form

\[
\psi_\lambda^n = \begin{bmatrix}
\wedge \\
\psi_\lambda \\
\varphi \\
0 \\
\rho
\end{bmatrix}.
\]

Then there exists a homotopy \( \{s_k\}, s_k : I_k \rightarrow I_{k-1} \) such that

\[
\phi_\lambda^n = s_{n-1}s_n + s_{n+1}d_n + \psi_\lambda^n.
\]
There are decompositions:

\[
s_n = (s'_n, s''_n), d_n = (d'_n, d''_n), s_{n+1} = \begin{bmatrix} s'_{n+1} \\ s''_{n+1} \end{bmatrix}, d_{n-1} = \begin{bmatrix} d'_{n-1} \\ d''_{n-1} \end{bmatrix}.
\]

It follows: \( d_{n-1}s_n + s_{n+1}d_n = \begin{bmatrix} d'_{n-1} \\ d''_{n-1} \end{bmatrix} \begin{bmatrix} s'_{n} \\ s''_{n} \end{bmatrix} + \begin{bmatrix} s'_{n+1} \\ s''_{n+1} \end{bmatrix} \begin{bmatrix} d'_n \\ d''_n \end{bmatrix}.
\]
Since \( I_{n-1} \) and \( I_{n+1} \) are torsion free we have:

\[
s'_{n} = 0, d'_{n-1} = 0, d_{n-1}s_n + s_{n+1}d_n = \begin{bmatrix} 0 & d'_{n-1}s'_{n+1} + s''_{n+1}d''_n \\ 0 & d''_{n-1}s''_{n+1} + s'_{n+1}d'_n \end{bmatrix}.
\]

It follows:

\[
\phi_\lambda^n = \begin{bmatrix}
\wedge \\
\phi_\lambda \\
\rho \\
0 \\
\sigma
\end{bmatrix} = \begin{bmatrix}
\wedge \\
\psi_\lambda \\
\varphi + d'_{n-1}s'_{n+1} + s''_{n+1}d''_n \\
0 \\
\varphi + d''_{n-1}s''_{n+1} + s'_{n+1}d'_n
\end{bmatrix}.
\]

Therefore: \( \phi_\lambda = \psi_\lambda \).

It is clear that there is a ring homomorphism \( \Lambda \rightarrow \text{End}_\Lambda(I_n') \) given by \( \lambda \rightarrow \phi_\lambda \)
which gives \( I_n' \) the structure of \( \Lambda \)-\( \Lambda \) bimodule.

Similarly to the left sided situation, \( \Lambda \) has a right injective resolution:

\[
0 \rightarrow \Lambda \rightarrow J_0 \rightarrow J_1 \rightarrow \cdots \rightarrow J_{n-1} \rightarrow J_n \rightarrow J_{n+1} \rightarrow \cdots
\]

\( J_n = J_n' \oplus J_n'' \) with \( J_n' \) torsion and \( J_n'' \) torsion free and \( J_n' = \bigoplus_{j=1}^m D(Q_j)[-n_j] \) is a \( \Lambda \)-\( \Lambda \) bimodule. It follows \( D(J_n') = \bigoplus_{j=1}^m Q_j[n_j] \) is a \( \Lambda \)-\( \Lambda \) bimodule.

In order to continue the study of the bimodule structure of \( I_n' \) and \( J_n' \) we need the following:
Lemma 2.14. Let $\Lambda$ be a positively graded locally finite $k$-algebra, $E$ a finitely cogenerated injective left $\Lambda$-module and $\{M_\alpha, \pi_\alpha\}_{\alpha \in A}$ an inverse system of locally finite graded left $\Lambda$-modules such that $M=\lim M_\alpha$ is locally finite. Then we have a natural isomorphism: $\text{Hom}_\Lambda(\lim M_\alpha, E) \cong \lim \text{Hom}_\Lambda(M_\alpha, E)$.

Proof. Since $E$ is finitely cogenerated it is locally finite. It was proved in [12] that $D$ is a duality in the category of locally finite graded modules. Applying the duality and adjunction we have natural isomorphisms:

$$\text{Hom}_\Lambda(\text{Hom}_\Lambda(\lim M_\alpha, E), k) \cong D(E) \otimes \lim M_\alpha.$$

Using the fact $D(E)$ is a finitely generated projective, we have isomorphisms:

$$D(E) \otimes \lim M_\alpha \cong \text{Hom}_\Lambda(D(E)^*, \lim M_\alpha) \cong \lim \text{Hom}_\Lambda(D(E)^*, M_\alpha) \cong \lim (D(E) \otimes M_\alpha).$$

It follows from [7] that $\lim D(\text{Hom}_\Lambda(M_\alpha, E)) \cong D(\lim D(\text{Hom}_\Lambda(M_\alpha, E)))$.

Dualizing again we obtain the isomorphism:

$$\text{Hom}_\Lambda(\lim M_\alpha, E) \cong \lim \text{Hom}_\Lambda(M_\alpha, E).$$

As a corollary we have:

Lemma 2.15. Let $\Lambda$ be a positively graded locally finite $k$-algebra, $X, Y$ graded left $\Lambda$-modules with $Y$ locally finite. Then for any non negative integer $n$, there is a natural isomorphism: $\text{Ext}^n_\Lambda(X, Y) \cong D(\text{Tor}^\Lambda_n(D(Y), X))$.

Proof. Consider a projective resolution of $X$:

$$0 \rightarrow Q_{n+1} \rightarrow Q_n \rightarrow Q_{n-1} \rightarrow \cdots \rightarrow Q_1 \rightarrow Q_0 \rightarrow X \rightarrow 0.$$

Tensoring with $D(Y)$ we obtain a complex *):

$$D(Y) \otimes \Lambda Q_{n+1} \rightarrow D(Y) \otimes \Lambda Q_n \rightarrow D(Y) \otimes \Lambda Q_{n-1} \rightarrow \cdots \rightarrow D(Y) \otimes \Lambda Q_1 \rightarrow D(Y) \otimes \Lambda Q_0 \rightarrow 0$$

whose n-th homology is $\text{Tor}^\Lambda_n(D(Y), X)$.

Dualizing *) we obtain a complex **):

$$0 \rightarrow D(D(Y) \otimes \Lambda Q_{n+1}) \rightarrow D(D(Y) \otimes \Lambda Q_n) \rightarrow \cdots \rightarrow D(D(Y) \otimes \Lambda Q_1) \rightarrow D(D(Y) \otimes \Lambda Q_0) \rightarrow 0$$

with n-th homology $D(\text{Tor}^\Lambda_n(D(Y), X))$.

Since we have natural isomorphisms:

$$D(D(Y) \otimes \Lambda Q_i) \cong \text{Hom}_\Lambda(Q_i, D^2(Y)) \cong \text{Hom}_\Lambda(Q_i, Y),$$

the complex **) is isomorphic to the complex:

$$0 \rightarrow \text{Hom}_\Lambda(Y, Q_0) \rightarrow \cdots \rightarrow \text{Hom}_\Lambda(Y, Q_{n-1}) \rightarrow \text{Hom}_\Lambda(Y, Q_n) \rightarrow \text{Hom}_\Lambda(Y, Q_{n+1}) \rightarrow \cdots$$

whose n-th homology $\text{Ext}^n_\Lambda(X, Y)$.

It follows: $\text{Ext}^n_\Lambda(X, Y) \cong D(\text{Tor}^\Lambda_n(D(Y), X))$. 

Corollary 2.16. Let $\Lambda$ be a graded AS Gorenstein algebra of graded injective dimension $n$. Then for any graded left $\Lambda$-module $M$, there is a natural isomorphism:

$$\text{Ext}^n_\Lambda(M, \oplus_{j=1}^m Q_j[n_j]) \cong D(\text{Tor}^\Lambda_n(J_n, M)).$$
Corollary 2.17. Let $\Lambda$ be a positively graded locally finite $k$-algebra, $\{X_\alpha, j_\alpha\}$ be a graded direct system of $\Lambda$-modules and $Y$ a locally finite $\Lambda$-module. Then for any non negative integer $i$, there is a natural isomorphism: $\text{Ext}^i_{\Lambda}(\lim X_\alpha, Y) \cong \lim \text{Ext}^i_{\Lambda}(X_\alpha, Y)$.

Proof. From the isomorphism: $\text{Tor}^i_\Lambda(D(Y), \lim X_\alpha) \cong \lim \text{Tor}^i_\Lambda(D(Y), X_\alpha)$, we obtain a chain of isomorphisms:

$$\text{Ext}^i_{\Lambda}(\lim X_\alpha, Y) \cong D(\text{Tor}^i_\Lambda(D(Y), \lim X_\alpha) \cong D(\lim \text{Tor}^i_\Lambda(D(Y), X_\alpha)) \cong \lim D(\text{Tor}^i_\Lambda(D(Y), X_\alpha)) \cong \lim \text{Ext}^i_{\Lambda}(X_\alpha, Y).$$

As an application of these observations we have the following:

Let $\Lambda$ be a graded Gorenstein algebra, such that all graded simple have projective resolutions consisting of finitely generated projective, $m$ be the graded radical of $\Lambda$ and $k$ a positive integer. Then there are isomorphisms:

$$\text{Ext}^i_{\Lambda}(\Lambda/\Lambda_{\geq k}, \Lambda) \cong \text{Hom}_{\Lambda}(\Lambda/\Lambda_{\geq k}, I'_n)$$

and

$$\text{D}(\text{Hom}_{\Lambda}(\Lambda/\Lambda_{\geq k}, I'_n)) \cong D(I'_n) \otimes \Lambda/\Lambda_{\geq k}.$$

Dualizing again and using adjunction:

$$\text{Hom}_{\Lambda}(\Lambda/\Lambda_{\geq k}, I'_n) \cong \text{Hom}_{\Lambda^{op}}(D(I'_n), D(\Lambda/\Lambda_{\geq k})).$$

Using the fact $D(I'_n)$ is a finitely generated projective, $\text{Hom}_{\Lambda}(\Lambda/\Lambda_{\geq k}, I'_n) \cong D(\Lambda/\Lambda_{\geq k}) \otimes D(I'_n)^*$. It follows

$$\lim \text{Ext}^i_{\Lambda}(\Lambda/\Lambda_{\geq k}, \Lambda) \cong \lim (D(\Lambda/\Lambda_{\geq k}) \otimes D(I'_n)^*) \cong (\lim D(\Lambda/\Lambda_{\geq k})) \otimes D(I'_n)^* \cong D(I'_n)^* \cong \text{Hom}_{\Lambda^{op}}(D(I'_n), D(\Lambda)) \cong \text{Hom}_{\Lambda}(\Lambda, I'_n) \cong I'_n.$$

Similarly, $\lim \text{Ext}^i_{\Lambda}(\Lambda/\Lambda_{\geq k}, \Lambda) \cong J'_n$.

Observe that the functor $\lim \text{Hom}_{\Lambda}(\Lambda/\Lambda_{\geq k}, \cdot)$ is left exact and it has left derived functors $\lim \text{Ext}^i_{\Lambda}(\Lambda/\Lambda_{\geq k}, \cdot)$.

Definition 2.18. Let $\Lambda$ be a positively graded locally finite $k$-algebra and $m$ the graded Jacobson radical. The $i$-th local cohomology of the module $M$ is $\lim \text{Ext}^i_{\Lambda}(\Lambda/\Lambda_{\geq k}, M)$.

We write $\Gamma^m(M) = \lim \text{Hom}_{\Lambda}(\Lambda/\Lambda_{\geq k}, M)$ and $\Gamma^i_i(M) = \lim \text{Ext}^i_{\Lambda}(\Lambda/\Lambda_{\geq k}, M)$.

We have proved above that for a graded AS Gorenstein algebra of graded injective dimension $n$, such that all graded simple have projective resolutions consisting of finitely generated projective, $\Gamma^m_m(\Lambda) = I'_n$ and $\Gamma^m_{m+1}(\Lambda) = J'_n$.

Lemma 2.19. Let $\Lambda$, $\Gamma$ be a positively graded $k$-algebras such that the graded simple have projective resolutions consisting of finitely generated projective modules, $m_\Lambda$, $m_\Gamma$, the graded Jacobson radicals of $\Lambda$ and $\Gamma$, respectively and $m_{\Lambda \otimes_k \Gamma}$ the graded Jacobson radical of $\Lambda \otimes_k \Gamma$. Then $\Gamma_{m_{\Lambda \otimes_k \Gamma}} = \Gamma_{m_\Lambda} \circ m_{\Lambda} = \Gamma_{m_\Gamma} \circ m_{\Gamma}$.

Proof. From inequalities: $(\Lambda \otimes_k \Gamma)_{\geq k} \supseteq \Lambda_0 \otimes_k \Gamma_{\geq k} + \Lambda_{\geq k} \otimes_k \Gamma \supseteq (\Lambda \otimes_k \Gamma)_{\geq 2k}$, it follows:

$$\lim \text{Hom}_{\Lambda \otimes \Gamma}(\Lambda \otimes \Gamma/((\Lambda \otimes \Gamma)_{\geq 2k}, \cdot) = \lim \text{Hom}_{\Lambda \otimes \Gamma}(\Lambda \otimes \Gamma/(\Lambda_0 \otimes_k \Gamma_{\geq k} + \Lambda_{\geq k} \otimes_k \Gamma_0), \cdot).$$
Local cohomology

\[ \lim_{\rightarrow} \text{Hom}(\Lambda \otimes \Gamma \rightarrow \Lambda/\Lambda \geq k, \text{Hom}(\Lambda/\Lambda \geq k, \rightarrow)) = \lim_{\rightarrow} \text{Hom}(\Gamma/\Gamma \geq k, \lim \text{Hom}(\Lambda \rightarrow)) = \Gamma_{m} \circ \Gamma_{m} \Lambda. \]

Also \[ \lim_{\rightarrow} \text{Hom}(\Lambda \otimes \Gamma \rightarrow \Lambda/\Lambda \geq k, \rightarrow) = \lim_{\rightarrow} \text{Hom}(\Lambda/\Lambda \geq k, \text{Hom}(\Lambda \rightarrow)) = \Gamma_{m} \circ \Gamma_{m} \Lambda. \]

The following result can be found in [9], for completeness we include it here.

**Proposition 2.20.** Let \( \Lambda \) be a positively graded \( k \)-algebras such that the graded simple have projective resolutions consisting of finitely generated projective, \( m \) the graded radical of \( \Lambda \) and \( m^{op} \) the graded radical of \( \Lambda^{op} \). Then for any integer \( k \), \( \Gamma_{m}^{k}(\Lambda) = \Gamma_{m}^{k}(\Lambda) \).

**Proof.** Let \( 0 \rightarrow \Lambda \rightarrow E_{0} \rightarrow E_{1} \rightarrow E_{t} \rightarrow \rightarrow ... \) be an injective resolution of \( \Lambda \) as bimodule, it is easy to prove that each \( E_{t} \) is injective both as left and as right module and let \( E \) be the complex: \( 0 \rightarrow E_{0} \rightarrow E_{1} \rightarrow E_{t} \rightarrow E_{t+1} \rightarrow \rightarrow ... \)

Then we have:
\[
\Gamma_{m \otimes \Lambda^{op}}(E) = \Gamma_{m}^{\otimes \Lambda^{op}}(E) = \lim \text{Hom}(\Lambda/\Lambda \geq k, \text{Hom}(\Lambda/\Lambda \geq k, \rightarrow)) = \lim \text{Hom}(\Lambda/\Lambda \geq k, \text{Hom}(\Lambda/\Lambda \geq k, \rightarrow)) = \Gamma_{m}^{k}(E).
\]

Similarly, \( \Gamma_{m \otimes \Lambda^{op}}(E) = \Gamma_{m}^{k}(E) \).

Taking the \( k \)-th homology of the complex we obtain \( \Gamma_{m}^{k}(\Lambda) = \Gamma_{m}^{k}(\Lambda) \).

As an application of this equality we have the following:

**Corollary 2.21.** For a graded AS Gorenstein algebra \( \Lambda \) of graded injective dimension \( n \), such that all graded simple have projective resolutions consisting of finitely generated projective, \( I_{n}^{*} \cong I_{n}^{*} \) as a bimodule.

### 3 Local Cohomology

The aim of this section is to prove the Local Cohomology formula for a class of graded Gorenstein algebras. We already have all the ingredients to prove it for graded Gorenstein algebras of finite local cohomology dimension and such that all graded simple have graded projective resolutions consisting of finitely generated projective modules.

**Definition 3.1.** We say that a graded AS Gorenstein algebra \( \Lambda \) has finite left local cohomology dimension, if there is a non negative integer \( \ell_{0} \) such that for any left \( \Lambda \)-module \( M \) and any integer \( \ell > \ell_{0} \), \( \Gamma_{m}^{\ell}(M) = 0 \).
Let $\Lambda$ be an algebra satisfying these conditions, consider the projection maps $\pi_k : \Lambda/\Lambda_{\geq k} \to \Lambda/\Lambda_{\geq k-1}$, the collection $\{\pi_k, \Lambda/\Lambda_{\geq k}\}_{k \geq 0}$ forms a graded inverse system.

For each $k$, there is a minimal graded projective resolution: $\mathfrak{P}^{(k)} \to \Lambda/\Lambda_{\geq k}$.

$\mathfrak{P}^{(k)}$: $P_n^{(k)} \to P_{n-1}^{(k)} \to \ldots \to P_1^{(k)} \to P_0^{(k)} \to \Lambda/\Lambda_{\geq k} \to 0$.

The projections $\pi_{n+1} : \Lambda/\Lambda_{\geq k} \to \Lambda/\Lambda_{\geq k-1}$ induce maps of projective resolutions:

$$\ldots P_{n+1}^{(k)} \to P_n^{(k)} \to P_{n-1}^{(k)} \to \ldots P_1^{(k)} \to P_0^{(k)} \to \Lambda/\Lambda_{\geq k} \to 0$$

Dualizing with respect to the ring we get a map of complexes:

$\mathfrak{P}^{*(k-1)}$: $0 \to P_0^{*(k-1)} \to P_1^{*(k-1)} \to P_2^{*(k-1)} \to \ldots P_n^{*(k-1)} \to P_{n+1}^{*(k-1)} \to 0$

$\mathfrak{P}^{*(k)}$: $0 \to P_0^{*(k)} \to P_1^{*(k)} \to P_2^{*(k)} \to \ldots P_n^{*(k)} \to P_{n+1}^{*(k)} \to 0$

where $H^i(\mathfrak{P}^{*(k)}) = \text{Ext}^i_\Lambda(\Lambda/\Lambda_{\geq k}, \Lambda) = 0$ for $i \neq n$ and $H^n(\mathfrak{P}^{*(k)}) = \text{Ext}^n_\Lambda(\Lambda/\Lambda_{\geq k}, \Lambda)$ is different from zero and of finite length.

In addition, for any graded left $\Lambda$-module $M$, $\mathfrak{P}^{*(k)} \otimes M \cong \text{Hom}_\Lambda(\mathfrak{P}^{(k)}, M)$.

Hence, $H^i(\mathfrak{P}^{*(k)} \otimes M) \cong \text{Ext}^i_\Lambda(\Lambda/\Lambda_{\geq k}, M)$.

We have a direct system of complexes $\{\mathfrak{P}^{*(k)}, (\pi_k)^*\}$ and $\mathfrak{F} = \text{lim} \mathfrak{P}^{*(k)}$ is a complex of flat modules. Since $\text{lim}$ is an exact functor, $H^i(\mathfrak{F}) = H^i(\text{lim} \mathfrak{P}^{*(k)}) = \text{lim} H^i(\mathfrak{P}^{*(k)}) = \text{lim} \text{Ext}^i_\Lambda(\Lambda/\Lambda_{\geq k}, \Lambda)$, and $\text{lim} (\mathfrak{P}^{*(k)} \otimes M) \cong \text{lim} \text{Hom}_\Lambda(\mathfrak{P}^{(k)} \otimes M)$.

Therefore: $H^i(\text{lim} \mathfrak{P}^{*(k)} \otimes M) \cong H^i(\text{lim} \text{Hom}_\Lambda(\mathfrak{P}^{(k)} \otimes M)) \cong \text{lim} H^i(\text{Hom}_\Lambda(\mathfrak{P}^{(k)} \otimes M)) \cong \text{lim} \text{Ext}^i_\Lambda(\Lambda/\Lambda_{\geq k}, M) = \Gamma_m^{(k)}(M)$.

We see next that the assumption of finite cohomological dimension $\ell_0$ imposes strong restrictions on the complex $\mathfrak{F}$.

$\mathfrak{F}$: $0 \to F_0 \to F_1 \to F_2 \to \ldots F_j \to \ldots \to F_{\ell_0} \to F_{\ell_0+1} \to F_{\ell_0+2} \to \ldots$

The complex $\mathfrak{F}$ has homology $H^i(\mathfrak{F}) = \text{lim} \text{Ext}^i_\Lambda(\Lambda/\Lambda_{\geq k}, \Lambda) = 0$ for $i \neq n$.

Since $\Gamma_m^{(k)}(\Lambda) = \Gamma_m^{(n)} \neq 0$, $\ell_0 \geq n$ and $H^i(\mathfrak{F} \otimes M) = 0$ for $i > \ell_0$.

Consider the exact sequence: $F_{\ell_0} \to F_{\ell_0+1} \to F_{\ell_0+2} \to \ldots C \to 0$.

*) is part of a flat resolution of $C$, tensoring with a graded left $\Lambda$-module $M$ we obtain a complex: $F_{\ell_0} \otimes M \to F_{\ell_0+1} \otimes M \to F_{\ell_0+2} \otimes M \to 0$, where $\text{Ker} d_{\ell_0+1} \otimes 1/\text{Im} d_{\ell_0} \otimes 1 = \Gamma^{(n+1)}(\mathfrak{F} \otimes M) = \text{Tor}^{(1)}_\Lambda(C, M) = 0$ for all graded $\Lambda$-modules $M$.

This implies $C$ is flat.
Consider the exact sequence: \(0 \to C_1 \to F_{e_0+2} \overset{d_{e_0+2}}{\to} C \to 0\). By the long homology sequence there is an exact sequence:

\[
\text{Tor}_2^A(C,M) \to \text{Tor}_1^A(C_1,M) \to \text{Tor}_1^A(F_{e_0+2},M)
\]

Since \(C\) and \(F_{e_0+2}\) are flat, it follows \(\text{Tor}_1^A(C_1,M)=0\) for all \(M\), hence \(C_1\) is also flat.

We use induction to get an exact sequence: \(0 \to \text{Ker}d_n \to F_n \to \text{Im}d_n \to 0\) with \(F_n\) and \(\text{Im}d_n\) flat. It follows \(F_n' = \text{Ker}d_n\) is flat and we have a complex \(\mathfrak{F}'\) of flat modules:

\[
\mathfrak{F}': 0 \to F_0 \overset{d_0}{\to} F_1 \overset{d_1}{\to} \cdots \to F_{n-1} \overset{d_{n-1}}{\to} F_n' \to 0
\]

such that \(H^i(\mathfrak{F}') = H^i(\mathfrak{F}) = 0\) for \(i \neq n\) and \(H^n(\mathfrak{F}') = H^n(\mathfrak{F}) = \lim\ Ext^n_A(\Lambda/\Lambda_\geq k, \Lambda) = I'_n\).

Also for any graded left \(\Lambda\)-module \(M\), the sequence:

\[
0 \to \text{Tor}_1^A(\text{Im} d_n,M) \to F'_n \otimes M \to F_n \otimes M \to \text{Im} d_n \otimes M \to 0
\]

is exact, and \(\text{Im} d_n\) flat implies \(0 \to F'_n \otimes M \to F_n \otimes M \to \text{Im} d_n \otimes M \to 0\) is exact. Similarly, the sequence:

\[
0 \to \text{Im} d_{n+1} \otimes M \to F_{n+1} \otimes M \to \text{Im} d_n \otimes M \to 0
\]

is exact and the sequence:

\[
0 \to F'_n \otimes M \to F_n \otimes M \to F_{n+1} \otimes M \to \text{Im} d_{n+1} \otimes M
\]

is exact.

It follows \(H^n(\mathfrak{F} \otimes M) = H^n(\mathfrak{F}' \otimes M) = \Gamma^n(M)\).

Since \(\mathfrak{F}' \to I'_n\) is a flat resolution of \(I'_n\). The complex \(\mathfrak{F}' \otimes M\) has \(i\)-th homology \(\text{Tor}_{n-i}^A(I'_n,M)\). It follows \(\text{Tor}_{n-i}^A(I'_n,M) = \lim\ Ext^n_A(\Lambda/\Lambda_\geq k, M) = \Gamma_m^n(M)\). Applying the duality we obtain the local cohomology formula:

\[
D(\lim\ Ext^n_A(\Lambda/\Lambda_\geq k, M)) = \text{Ext}_A^{n-i}(M, D(\Gamma_m^n(\Lambda))), \quad \text{for all } 0 \leq i \leq n.
\]

We have proved the following:

**Theorem 3.2.** Let \(\Lambda\) be a graded AS Gorenstein algebra of graded injective dimension \(n\) and such that all graded simple modules have projective resolutions consisting of finitely generated projective modules and assume \(\Lambda\) has finite local cohomology dimension. Then for any graded left module \(M\) there is a natural isomorphism: \(D(\lim\ Ext^n_A(\Lambda/\Lambda_\geq k, M)) = \text{Ext}_A^{n-i}(M, D(\Gamma_m^n(\Lambda))), \quad \text{for } 0 \leq i \leq n\).

The theorem generalizes results for Artin-Schelter regular algebras proved in [11].

We will finish the paper giving a family of examples of graded AS Gorenstein algebras of finite local cohomology dimension.

Let \(\Lambda = \mathbb{k}Q_1/I_1\) be a graded selfinjective non semisimple algebra, \(\Gamma = \mathbb{k}Q_2/I_2\) an Artin-Schelter regular algebra of global dimension \(n\), in the sense of [12],[15], denote by \(r\) and \(m\), the graded Jacobson radicals of \(\Lambda\) and \(\Gamma\), respectively and \(\Lambda_0 = \Lambda/r\), \(\Gamma_0 = \Gamma/m\).

The following formula was proved in [13]:

\[
\text{Ext}_{A \otimes \Gamma}^i(\Lambda_0 \otimes \Gamma_0, \Lambda \otimes \Gamma) \cong \bigoplus_{i+j=m} \text{Ext}^i_{A}(\Lambda_0, \Lambda) \otimes_{\mathbb{k}} \text{Ext}^j_{\Gamma}(\Gamma_0, \Gamma).
\]

Since \(\Lambda\) is selfinjective \(\text{Ext}^i_{A \otimes \Gamma}(\Lambda_0 \otimes \Gamma_0, \Lambda \otimes \Gamma) \cong \text{Hom}_A(\Lambda_0, \Lambda) \otimes_{\mathbb{k}} \text{Ext}^i_{\Gamma}(\Gamma_0, \Gamma)\), \(\Gamma\) Artin-Schelter regular implies \(\text{Ext}^t_{\Gamma}(\Gamma_0, \Gamma) = 0\) for \(t \neq 0\), \(\text{Ext}^0_{\Gamma}(\Gamma_0, \Gamma) = \Gamma_0^{op}[-n]\).
It follows:
\[ \text{Ext}_*^t(\Lambda \otimes_k \Gamma_0, \Lambda \otimes \Gamma) = 0 \text{ for } t \neq n \text{ and } \text{Ext}_*^n(\Lambda_0 \otimes_k \Gamma_0, \Lambda \otimes_k \Gamma) = \Lambda_0^{op} \otimes_k \Gamma^{op}[n]. \]
We have proved \( \Lambda \otimes \Gamma \) is graded Gorenstein of injective dimension \( n \).

The graded radical of \( \Lambda \otimes \Gamma \) is \( m = \Lambda \otimes m + r \otimes \Gamma \). Then \( m_k = \sum_{i+j=k} r_i \otimes m_j \).

Assume \( r_{t-1} \neq 0 \) and \( r' = 0 \). For \( k \geq t \),
\[ m_k = \left( \sum_{i+j=t-1} r_i \otimes m_j \right) \Lambda \otimes m^{k-t+1} \subseteq (\Lambda \otimes \Gamma) \Lambda \otimes m^{k-t+1} = \Lambda \otimes m^{k-t+1}. \]

We have inequalities:
\[ \Lambda \otimes m^k \subseteq m^k \subseteq \Lambda \otimes m^{k-t+1} \subseteq m^{k-t+1}. \]

Hence, we have surjective maps:
\[ \Lambda \otimes \Gamma / \Lambda \otimes m^k \rightarrow \Lambda \otimes \Gamma / m^k \rightarrow \Lambda \otimes \Gamma / m^{k-t+1} \rightarrow \Lambda \otimes \Gamma / m^{k-t+1}, \]
which induce maps:
\[ \text{Ext}_*^j(\Lambda \otimes \Gamma / m^{k-t+1}, M) \rightarrow \text{Ext}_*^j(\Lambda \otimes \Gamma / m^{k-t+1}, M) \rightarrow \text{Ext}_*^j(\Lambda \otimes \Gamma / m^k, M) \rightarrow \text{Ext}_*^j(\Lambda \otimes \Gamma / m^k, M). \]

Taking direct limits the sequence:
\[ \lim \text{Ext}_*^j(\Lambda \otimes \Gamma / m^{k-t+1}, M) \rightarrow \lim \text{Ext}_*^j(\Lambda \otimes (\Gamma / m^{k-t+1}), M) \rightarrow \lim \text{Ext}_*^j(\Lambda \otimes (\Gamma / m^k), M) \rightarrow \lim \text{Ext}_*^j(\Lambda \otimes (\Gamma / m^k), M) \]

Using the fact:
\[ \lim \text{Ext}_*^j(\Lambda \otimes (\Gamma / m^{k-t+1}), M) = \lim \text{Ext}_*^j(\Lambda \otimes (\Gamma / m^k), M) \]

and \( \lim \text{Ext}_*^j(\Lambda \otimes (\Gamma / m^{k-t+1}), M) = \lim \text{Ext}_*^j(\Lambda \otimes (\Gamma / m^k), M) \), it follows:
\[ \lim \text{Ext}_*^j(\Lambda \otimes (\Gamma / m^k), M) = \sum \text{Ext}_*^j(\Lambda \otimes (\Gamma / m^k), M). \]

Let \( 0 \rightarrow Q_n^{(k)} \rightarrow Q_{n-1}^{(k)} \rightarrow \cdots \rightarrow Q_1^{(k)} \rightarrow \Gamma \rightarrow \Gamma / m^k \rightarrow 0 \) be a graded projective resolution of \( \Gamma / m^k \).

Then \( 0 \rightarrow \Lambda \otimes Q_n^{(k)} \rightarrow \Lambda \otimes Q_{n-1}^{(k)} \rightarrow \cdots \rightarrow \Lambda \otimes Q_1^{(k)} \rightarrow \Lambda \otimes \Gamma \rightarrow \Lambda \otimes \Gamma / m^k \rightarrow 0 \) is a graded projective resolution of \( \Lambda \otimes \Gamma / m^k \) and \( \text{Ext}_*^j(\Lambda \otimes \Gamma / m^k, M) = 0 \) for \( j > n \). It follows \( H_m^j(M) = \lim \text{Ext}_*^j(\Lambda \otimes \Gamma / m^k, M) = 0 \) for \( j > n \).

We have proved \( \Lambda \otimes \Gamma \) has local cohomology dimension \( n \).

We give two concrete examples in which this situation natural arises:

1) Let \( \Lambda = k[x_1, x_2, \ldots, x_n] \) be the exterior algebra in \( n \)-variables and \( \Gamma = \Lambda[x_1, x_2, \ldots, x_n] \) the polynomial algebra. Then the ring of polynomial forms \( \Gamma \otimes \Lambda \) is Gorenstein of finite local cohomology dimension \( n \).

This example appears as the cohomology ring of the group algebra of an elementary abelian \( p \)-group, over a field of characteristic \( p > 2 \). [5]

2) Let \( Q \) be a non Dynkin quiver with only sinks and sources \( k \) a field, \( \Lambda \) the trivial extension \( kQ \) of \( \text{D}(kQ) \) and \( \Gamma \) the preprojective algebra [10] corresponding to the quiver \( Q \). Then \( \Gamma \otimes \Lambda \) is Gorenstein of finite local cohomology dimension 2.

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References


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