On the Paper ”Generalized ideal elements in \textit{le-Γ}-semigroups” by K. Hila and E. Pisha
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Abstract

Comments on the paper in the title published in Communications Korean Mathematical Society. We give our results and make the main corrections.

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Keywords: Γ-semigroup, \textit{ve-Γ}-semigroup, \((m, n, β, γ)\)-ideal element, \((m, n, β)\)-ideal element, \((m, 0, β)\)-ideal element, \((0, m, β)\)-ideal element, \((m, n, β, γ)\)-regular element, \((m, n, β, γ)\)-regular \textit{ve-Γ}-semigroup, β-subidempotent element, β-subidempotent po-Γ-semigroup
1 Introduction

Concerning the paper in [1], we give our results and make the main corrections. The Theorem 2.4, Theorem 2.5, Theorem 2.6, Theorem 3.4, Theorem 3.9 of the paper are based on Lemma 2.3, but Lemma 2.3 is wrong. As Theorem 3.4 is wrong, Theorem 3.5 is wrong as well. The \((m,0)\)-ideal elements and the \((0,n)\)-ideal elements mentioned in Theorem 9.6 have not defined in the paper (look at Definition 2.1). Lemma 3.7 is also wrong as the expression \(a^n\) is used in it. In Lemma 2.3(3), the authors use the Definition 2.2 which clearly is not true: Let \(M\) be a \(\vee e\)-\(\Gamma\)-semigroup and \(m, n \in Z^+\). According to the paper, \(M\) is called \((m,n)\)-regular if for all \(a \in M\) and all \(\lambda, \mu \in \Gamma\) there exist \(\gamma_1, \gamma_2, \ldots, \gamma_m-1, \rho_1, \rho_2, \ldots, \rho_{n-1} \in \Gamma\) such that

\[ a \leq (a^{\gamma_1}a^{\gamma_2}a\ldots a^{\gamma_{m-1}}a)^{\lambda e \mu}(a^{\rho_1}, a^{\rho_2}, \ldots, a^{\rho_{n-1}}a). \]

Suppose \(m = 1\) (or \(n = 1\)). What is the \(\gamma_{m-1}\) (that is, the \(\gamma_0\)) in the expression \(a^{\gamma_1}a^{\gamma_2}a\ldots a^{\gamma_{m-1}}a\)? What is the \(\rho_{m-1} = \rho_0\) in the expression \(a^{\rho_1}a^{\rho_2}a\ldots a^{\rho_{m-1}}a\)? The Definition 2.2 of the paper should be corrected and then to check if its corrected form coincides with the definition of regular (that is, \((1,1)\)-regular) \(\vee e\)-\(\Gamma\)-semigroups. Lemma 2.3(3) is based on the definition of \((m,n)\)-ideal elements as well given in Definition 2.1 of the paper, which is also wrong. According to Definition 2.1, an element \(a\) of a \(\vee e\)-\(\Gamma\)-semigroup is an \((m,n)\)-ideal element \((m,n \in Z^+\) if there exist \(\gamma_1, \gamma_2, \ldots, \gamma_m-1, \rho_1, \rho_2, \ldots, \rho_{n-1} \in \Gamma\) such that

\[ (a^{\gamma_1}a^{\gamma_2}a\ldots a^{\gamma_{m-1}}a)^{\lambda e \mu}(a^{\rho_1}, a^{\rho_2}, \ldots, a^{\rho_{n-1}}a) \leq a \]

for all \(\lambda, \mu \in \Gamma\). The same question arises: Suppose \(m = 1\) (or \(n = 1\)). What is the \(\gamma_{m-1}\) (that is, the \(\gamma_0\)) in the expression \(a^{\gamma_1}a^{\gamma_2}a\ldots a^{\gamma_{m-1}}a\)? What is the \(\rho_{m-1} = \rho_0\) in the expression \(a^{\rho_1}a^{\rho_2}a\ldots a^{\rho_{m-1}}a\)? The Definition 2.1 should be corrected and then one has to check if its corrected form coincides with the definition of a bi-ideal element (that is, \((1,1)\)-ideal element). As far as the definition of a quasi-ideal element and the definition of bi-ideal element is concerned the authors gave the following definitions: An element \(a\) of a \(\vee e\)-\(\Gamma\)-semigroup is called a quasi-ideal element if, for for every \(\lambda \in \Gamma\), \(a \alpha e \land e \lambda a\) exists and \(a \alpha e \land e \lambda a \leq a\). The element \(a\) is called a bi-ideal element if \(a \alpha e \mu a \leq a\) for all \(\lambda, \mu \in \Gamma\). But the quasi-ideal elements should be bi-ideal elements as well. There is no such a proof in the paper, and it does not seem to be true.

For each of the results of the papers in [2],[3], the authors tried to get its analogous in case of a \(\vee e\)-\(\Gamma\)-semigroup just, casually, putting \(\alpha, \beta\) (the elements of \(\Gamma\)) in some places. For shortly, they wrote \(a^m\) as the element \(a^{\gamma_1}a^{\gamma_2}a\ldots a^{\gamma_{m-1}}a\) for some \(\gamma_1, \gamma_2, \ldots, \gamma_{m-1} \in \Gamma\) \((m \in Z^+)\) which leads to the mistakes throughout the paper. Look, for example, at Lemma 2.3(1),(2). Besides, in the proof of Lemma 2.3(1) the authors wrote, \(a \gamma e \land a^m \lambda e \mu a^{k+1} \gamma e = a^2 \gamma e\) which certainly is not true. Except of the fact
that \( a^m, a^n \) has been used in Lemma 2.3(3), this part of the lemma has an additional mistake. According to Lemma 2.3(3), \(<a >_{(m,n)} = a \lor a^m \lambda e \mu a^n \) for every \( \lambda, \mu \in \Gamma \). \(<a >_{(m,n)} \) is uniquely defined, while they consider it equal to \( a \lor a^m \lambda e \mu a^n \) for every \( \lambda, \mu \in \Gamma \). If this is the case, the authors should prove that for every \( \lambda, \mu \in \Gamma \), \( a \lor a^m \lambda e \mu a^n \) is uniquely defined. Is it possible?\ That is, if \( \lambda, \mu \in \Gamma \) and \( \gamma, \delta \in \Gamma \) then is \( a \lor a^m \lambda e \mu a^n = a \lor a^m \gamma e \delta a^n \)?

Let us get \( m = 2, n = 2 \), for example. According to the paper by Hila and Pisha, an element \( a \) of \( M \) is called a \((2, 2)\)-ideal element if there exist \( \gamma, \delta \in M \) such that \( (a \gamma a) \xi e \zeta (a \delta a) \leq a \) for all \( \xi, \zeta \in \Gamma \). Then they write \( a^2 \xi e \zeta a^2 \leq a \), which actually means that \( a \gamma a = a \delta a \). In that case they should prove that \( a \gamma a = a \delta a \). Is it so, and why? So they cannot write \(<a >_{(2,2)} = a \lor a^2 \xi e \zeta a^2 \) for all \( \xi, \zeta \in \Gamma \). Shortly, Lemma 2.3 is without any sense, and so is the rest of the paper. Finally, the definition of a \( \vee e \Gamma \)-semigroup given in Definition 1.7 is also not correct. The authors say: Let ”\( M \) be a semilattice under \( \lor \ldots \)” which means that there exists an order relation \( \leq \) on \( M \) according to which \( M \) is a semilattice, that is, for any two elements \( a, b \in M \) there exists an element \( t \in M \) (denoted by \( a \lor b \) and called the supremum of \( a \) and \( b \)) such that \( t \geq a \), \( t \geq b \) and if \( h \in M \) such that \( h \geq a \) and \( h \geq b \), then \( t \leq h \) \((y \geq x \text{ means } x \leq y \text{ that is } (x, y) \in \leq \)\). Then, they say

”The usual order relation ” \( \leq \)” on \( M \) is defined in the following way

\[
a \leq b \iff a \lor b = b.
\]

and they add that \( a \leq b \) implies \( a \gamma c \leq b \gamma c \) and \( c \gamma a \leq c \gamma b \) for all \( c \in M \) and all \( \gamma \in \Gamma \). This has no sense, it is wrong, because the order defines the semilattice. Besides, they should mention that \( a \leq b \) implies \( a \gamma c \leq b \gamma c \) and \( c \gamma a \leq c \gamma b \) for all \( c \in M \) and all \( \gamma \in \Gamma \) immediately after the Definition 1.7 in line 20 of page 375 and not on lines 22-23 as they did. Its proof is as follows: Let \( M \) be a \( \vee e \Gamma \)-semigroup, \( a \leq b, \gamma \in \Gamma \) and \( c \in M \). Since \( a \leq b \), we have \( a \lor b = b \). Then, by Definition 1.1(3) of the paper, we have \((a \lor b) \gamma c = b \gamma c \). Since \( M \) is a \( \vee e \Gamma \)-semigroup, \((a \lor b) \gamma c = a \gamma c \lor b \gamma c \). Then \( a \gamma c \leq a \gamma c \lor b \gamma c = b \gamma c \), and \( a \gamma c \leq b \gamma c \). Similarly \( a \leq b \) implies \( c \gamma a \leq c \gamma b \) for all \( c \in M \) and all \( \gamma \in \Gamma \). This means that every \( \vee e \Gamma \)-semigroup is a \( poe \Gamma \)-semigroup. In addition, instead of ”for all \( a,b,c \in M \)” written in the paper, is much better to write for all \( c \in M \) since \( a \leq b \) that is \((a,b) \in \leq (\subseteq M \times M)\) implies \( a,b \in M \).

The authors tried to extend the results given in [2],[3] from \( \vee e \)-semigroups to ordered \( \Gamma \)-semigroups, but there is nothing correct in this paper, as the expression \( a^n \) has been used throughout the paper.

## 2 Main Results

In the following we correct the results given by the authors in Lemma 2.3, Theorem 2.4 and Theorem 2.5 which correspond to the result in [2] (Lemma 1,
Theorem 1, Theorem 2 in [2]). Based on our results, the authors might correct the rest of their paper which corresponds to the results given in [3].

The first two properties of Lemma 2.3 might be corrected as follows:

**Definition 1.** Let $M$ be a $\Gamma$-semigroup, $a \in M$, $\gamma \in \Gamma$ and $m \in N$ ($N = \{1, 2, \ldots, n\}$ is the set of natural numbers). Then

1. if $m = 1$, we define $a^1_\gamma := a$
2. if $m \geq 2$, we define $a^m_\gamma := a\gamma a\gamma \ldots \gamma a$

$m - 1$-times the $\gamma$, $m$-times the $a$).

**Remark 2.** If $a \in M$, $\gamma \in \Gamma$ and $m, n \in N$, then we have

$$a^m_\gamma a^n_\gamma = a^{m+n}_\gamma.$$

The first two properties of Lemma 1 in [1] can be formulated as follows:

**Lemma 3.** Let $M$ be a $\vee e$-$\Gamma$-semigroup, $a, b \in M$, $\gamma \in \Gamma$ and $m, n \in N$. Then we have

1. $(a \vee a^m_\gamma b)^m_\gamma e = a^m_\gamma e$.
2. $e\gamma(a \vee b)^n_\gamma a = e\gamma a^n_\gamma$.

**Proof.** (1) For $m = 1$, condition (1) is satisfied. Indeed: Let $a, b \in M$ and $\gamma \in \Gamma$. Then we have

$$a \vee a^m_\gamma b)^m_\gamma e = (a \vee a^m_\gamma b)^1_\gamma e = (a \vee a\gamma b)^1_\gamma e$$

$$= a\gamma e \vee a\gamma b\gamma e.$$

Since $b\gamma e \leq e$, we have $a\gamma b\gamma e \leq a\gamma e$, then $a\gamma e \vee a\gamma b\gamma e = a\gamma e$, and

$$(a \vee a^m_\gamma b)^m_\gamma e = a\gamma e = a^1_\gamma e = a^m_\gamma e.$$

Suppose condition (1) is satisfied for $m = k \geq 1$. That is, suppose that for every $c, d \in M$ and every $\delta \in \Gamma$, we have

$$(c \vee c^k_\delta d)^k_\delta e = c^k_\delta e.$$

Then it is satisfied for $m = k + 1$ as well. That is, for every $a, b \in M$ and every $\gamma \in \Gamma$, we have

$$(a \vee a^{k+1}_\gamma b)^{k+1}_\gamma e = a^{k+1}_\gamma e.$$

Indeed: Let $a, b \in M$ and $\gamma \in \Gamma$. Then we have

$$(a \vee a^{k+1}_\gamma b)^{k+1}_\gamma e = (a \vee a^{k+1}_\gamma b)^1_\gamma (a \vee a^{k+1}_\gamma b)^k_\gamma e) (by \text{Remark } 2)$$

$$= (a \vee a^{k+1}_\gamma b)^1_\gamma ((a \vee a^{k+1}_\gamma b)^k_\gamma e) (since M is associative)$$

$$= (a \vee a^{k+1}_\gamma b)^\gamma ((a \vee a^{k+1}_\gamma b)^k_\gamma e) (by \text{Definition } 1(1))$$

$$= (a \vee a^{k+1}_\gamma b)^\gamma (a \vee (a^k_\gamma a^1_\gamma b)^k_\gamma e) (by \text{Remark } 2)$$
\[ (a \lor a_\gamma^{k+1} \gamma b)^\gamma \left( (a \lor (a_\gamma^k \gamma a) \gamma b)^k \gamma e \right) \] (by Definition 1(1))

\[ = (a \lor a_\gamma^{k+1} \gamma b)^\gamma \left( a \lor a_\gamma^k \gamma (a \gamma b)^k \gamma e \right) \] (since \( M \) is associative)

\[ = (a \lor a_\gamma^{k+1} \gamma b)^\gamma (a_\gamma^k \gamma e) \] (by the assumption)

\[ = (a_\gamma^{k+1} \gamma e) \lor (a_\gamma^{k+1} \gamma b \gamma a_\gamma^k \gamma e) \] (since \( M \) is a \( \lor e \)-\( \Gamma \)-semigroup)

\[ = (a_\gamma^{k+1} \gamma e) \lor (a_\gamma^{k+1} \gamma b \gamma a_\gamma^k \gamma e) \] (by Remark 2).

Since \( b_\gamma a_\gamma^k \gamma e \leq e \), we have \( a_\gamma^{k+1} \gamma b_\gamma a_\gamma^k \gamma e \leq a_\gamma^{k+1} \gamma e \), and so

\[ a_\gamma^{k+1} \gamma e \lor a_\gamma^{k+1} \gamma b_\gamma a_\gamma^k \gamma e = a_\gamma^{k+1} \gamma e. \]

Thus we have \((a \lor a_\gamma^{k+1} \gamma b)^k \gamma e = a_\gamma^{k+1} \gamma e\). Condition (2) can be proved in a similar way.

**Corollary 4.** Let \( M \) be a \( \lor e \)-\( \Gamma \)-semigroup, \( a, b \in M \), \( \beta, \gamma \in \Gamma \) and \( m, n \in N \).

Then we have

1. \((a \lor a_\gamma^m \beta e_\gamma a_\gamma^n)^m \beta e = a_\beta^m \beta e\).
2. \(e_\gamma(a \lor a_\beta^m \beta e_\gamma a_\gamma^n)^n \gamma = e_\gamma a_\gamma^n\).

**Proof.** (1) Since \( M \) is associative, we have

\[ (a \lor a_\beta^m \beta e_\gamma a_\gamma^n)^m \beta e = (a \lor a_\beta^m \beta (e_\gamma a_\gamma^n))^m \beta e. \]

We put \( e_\gamma a_\gamma^n = c \) and, by Lemma 3(1), we have

\[ (a \lor a_\beta^m \beta e_\gamma a_\gamma^n)^m \beta e = (a \lor a_\beta^m \beta c)^m \beta e = a_\beta^m \beta e. \]

(2) We have \( e_\gamma(a \lor a_\beta^m \beta e_\gamma a_\gamma^n)^n \gamma = e_\gamma(a \lor (a_\beta^m \beta c) \gamma a_\gamma^n)^n \gamma \). We set \( a_\beta^m \beta c = d \) and, by Lemma 3(2), we get \( e_\gamma(a \lor a_\beta^m \beta e_\gamma a_\gamma^n)^n \gamma = e_\gamma(a \lor d_\gamma a_\gamma^n)^n \gamma = e_\gamma a_\gamma^n\).

As far as the third property of Lemma 1 in [1] is concerned, we first have to introduce the following definition:

**Definition 5.** Let \( M \) be a \( \lor e \)-\( \Gamma \)-semigroup, \( m, n \in N \) and \( \beta, \gamma \in \Gamma \). An element \( a \) of \( M \) is called an \((m, n, \beta, \gamma)\)-ideal element if

\[ a_\beta^m \beta e_\gamma a_\gamma^n \leq a. \]

For \( \beta = \gamma \), the element \( a \) is called an \((m, n, \beta)\)-ideal element. An element \( a \) of \( M \) is called an \((m, 0, \beta)\)-ideal element if

\[ a_\beta^m \beta e \leq a. \]

It is called an \((0, m, \beta)\)-ideal element if \( e_\beta a_\beta^m \leq a \). For \( a \in M \) denote by \( < a >_{(m, n, \beta, \gamma)} \) the \((m, n, \beta, \gamma)\)-ideal element of \( M \) generated by \( a \); denote by
< a >_{(m,0,\beta)} the \((m,0,\beta)\)-ideal element of \(M\) generated by \(a\) and < \(a\) >_{(0,m,\beta)} the \((0,m,\beta)\)-ideal element of \(M\) generated by \(a\). We denote by \(I_{(m,n,\beta,\gamma)}\) the set of \((m,n,\beta,\gamma)\)-ideal elements of \(M\) and by \(I_{(m,0,\beta)}\) (resp. \(I_{(0,m,\beta)}\)) the set of \((m,0,\beta)\) (resp. \((0,m,\beta)\))-ideal elements of \(M\).

The Lemma 2.3(3) in [1] should be formulated as follows:

**Lemma 6.** Let \(M\) be a \(\forall e-\Gamma\)-semigroup, \(a \in M\), \(m,n \in N\) and \(\beta, \gamma \in \Gamma\). Then we have

\[
< a >_{(m,n,\beta,\gamma)} = a \vee a^m_\beta b e \gamma a^n_\gamma.
\]

**Proof.** The element \(a \vee a^m_\beta b e \gamma a^n_\gamma\) is an \((m,n,\beta,\gamma)\)-ideal element of \(M\). That is,

\[
(a \vee a^m_\beta b e \gamma a^n_\gamma)^m_\beta b e \gamma (a \vee a^m_\beta b e \gamma a^n_\gamma)^n_\gamma \leq a \vee a^m_\beta b e \gamma a^n_\gamma.
\]

Indeed,

\[
(a \vee a^m_\beta b e \gamma a^n_\gamma)^m_\beta b e \gamma (a \vee a^m_\beta b e \gamma a^n_\gamma)^n_\gamma = (a \vee a^m_\beta b e \gamma a^n_\gamma)^m_\beta e \gamma (a \vee a^m_\beta b e \gamma a^n_\gamma)^n_\gamma = (a \vee a^m_\beta e \gamma a^n_\gamma)^m_\beta (e \gamma (a \vee a^m_\beta b e \gamma a^n_\gamma)^n_\gamma) = a^m_\beta b e \gamma a^n_\gamma (by \text{Corollary 4(1)})
\]

\[
\leq a \vee a^m_\beta b e \gamma a^n_\gamma.
\]

Clearly, \(a \leq a \vee a^m_\beta b e \gamma a^n_\gamma\). Let now \(t\) be an \((m,n,\beta,\gamma)\)-ideal element of \(M\) such that \(t \geq a\). Then \(a \vee a^m_\beta b e \gamma a^n_\gamma \leq t\). Indeed: Since \(t \geq a\), we have \(a \leq t^m_\beta\) and \(a^n_\gamma \leq t^n_\gamma\). Then we have \(a \vee a^m_\beta b e \gamma a^n_\gamma \leq a \vee t^m_\beta b e \gamma t^n_\gamma \leq t\).

**Definition 7.** Let \(M\) be a \(\forall e-\Gamma\)-semigroup, \(m,n \in N\) and \(\beta, \gamma \in \Gamma\). An element \(a\) of \(M\) is called \((m,n,\beta,\gamma)\)-regular if \(a \leq a^m_\beta b e \gamma a^n_\gamma\). \(M\) is called \((m,n,\beta,\gamma)\)-regular if every element of \(M\) is so.

**Theorem 8.** Let \(M\) be a \(\forall e-\Gamma\)-semigroup, \(m,n \in N\) and \(\beta, \gamma \in \Gamma\). The following are equivalent:

1. \(M\) is \((m,n,\beta,\gamma)\)-regular.
2. \(a^m_\beta b e \gamma a^n_\gamma = a\) for every \(a \in I_{(m,n,\beta,\gamma)}\).

**Proof.** (1) \(\implies\) (2). Let \(a \in I_{(m,n,\beta,\gamma)}\). Then \(a^m_\beta b e \gamma a^n_\gamma \leq a\). Since \(a \in M\), by hypothesis, we have \(a \leq a^m_\beta b e \gamma a^n_\gamma\). Thus we have \(a^m_\beta b e \gamma a^n_\gamma = a\).

(2) \(\implies\) (1). Let \(a \in M\). Then \(a \leq a^m_\beta b e \gamma a^n_\gamma\). Indeed: Since < \(a >_{(m,n,\beta,\gamma)}\) is an \((m,n,\beta,\gamma)\)-ideal element of \(M\), by hypothesis, we have

\[
\left( < a >_{(m,n,\beta,\gamma)} \right)^m_\beta b e \gamma \left( < a >_{(m,n,\beta,\gamma)} \right)^n_\gamma = < a >_{(m,n,\beta,\gamma)}.
\]

Then, by Lemma 6, we get

\[
a \leq a \vee a^m_\beta b e \gamma a^n_\gamma = (a \vee a^m_\beta b e \gamma a^n_\gamma)^m_\beta b e \gamma (a \vee a^m_\beta b e \gamma a^n_\gamma)^n_\gamma.
\]
Then we have

\[ (1) = a \quad \text{(since } M \text{ is associative}) \]

\[ (2) = a \gamma b \quad \text{and property (2) is satisfied.} \]

By Lemma 3(1), we have

\[ (1) \]

Proof. (1) By Lemma 3(1), we have \( (a \lor a_{\beta}^{m}\beta e\alpha^{n}_{\gamma})_{\beta}^{m}\beta e = a_{\beta}^{m}\beta e \leq a \lor a_{\beta}^{m}\beta e \), so \( a \lor a_{\beta}^{m}\beta e \) is an \( (m, 0, \beta) \)-ideal element of \( M \) containing \( a \). If now \( t \) is an \( (m, 0, \beta) \)-ideal element of \( M \) such that \( a \leq t \), then we have \( a \lor a_{\beta}^{m}\beta e \leq t \lor t_{\beta}^{m}\beta e = t \). The proof of (2) is similar.

By Lemma 3, the following lemma holds:

Lemma 10. Let \( M \) be a \( \vee e \)-semigroup, \( a, b \in M \), \( m, n, \beta, \gamma \in \Gamma \). Then we have

\[ (1) < a > (m, 0, \beta) = a \lor a_{\beta}^{m}\beta e. \]

\[ (2) < a > (0, m, \beta) = a \lor e_{\beta}a^{n}_{\gamma}. \]

Theorem 12. Let \( M \) be an \( \ell e \)-semigroup, \( m, n \in \mathbb{N} \) and \( \beta, \gamma \in \Gamma \). Suppose \( M \) is \( \beta \)-subidempotent. The following are equivalent:

(1) \( M \) is \( (m, n, \beta, \gamma) \)-regular.

(2) \( a \land b = a_{\beta}^{m}\beta b \land a_{\gamma}^{m}\beta b_{\gamma} \) for every \( a \in I_{(m, 0, \beta)} \) and every \( b \in I_{(0, n, \gamma)} \).

Proof. (1) \( \Rightarrow \) (2). Let \( a \in I_{(m, 0, \beta)} \) and \( b \in I_{(0, n, \gamma)} \). Since \( a \in I_{(m, 0, \beta)} \), we have \( a_{\beta}^{m}\beta e \leq a \), then \( a_{\beta}^{m}\beta b \leq a_{\beta}^{m}\beta e \leq a \). Since \( b \in I_{(0, n, \gamma)} \), we have \( e_{\gamma}b_{\gamma}^{n} \leq b \), then \( a_{\beta}^{m}\beta b \land a_{\gamma}^{m}\beta b_{\gamma} \leq a \land b \). Thus we have \( a_{\beta}^{m}\beta b \land a_{\gamma}^{m}\beta b_{\gamma} \leq a \land b \). Since \( a \land b \in M \) and \( M \) is \( (m, n, \beta, \gamma) \)-regular, we have \( a \land b \leq (a \land b)_{\beta}^{m}\beta e_{\gamma}(a \land b)_{\gamma}^{n} \). Since \( a \land b \leq a \), we have \( (a \land b)_{\beta}^{m}\beta e_{\gamma}(a \land b)_{\gamma}^{n} \leq a_{\beta}^{m}\beta e_{\gamma}b_{\gamma}^{n} \). Since \( a \land b \leq b \), we have \( (a \land b)_{\gamma}^{n} \leq b_{\gamma}^{n} \). Then

\[ a \land b \leq (a \land b)_{\beta}^{m}\beta e_{\gamma}(a \land b)_{\gamma}^{n} \leq a_{\beta}^{m}\beta e_{\gamma}b_{\gamma}^{n}. \]

Since \( a_{\beta}^{m}\beta e_{\gamma}b_{\gamma}^{n} \leq a_{\beta}^{m}\beta b \) and \( (a, m, \beta, \gamma)b_{\gamma}^{n} \leq a_{\beta}^{m}\beta b_{\gamma}^{n} \), we have

\[ a \land b \leq a_{\beta}^{m}\beta e_{\gamma}b_{\gamma}^{n} \leq a_{\beta}^{m}\beta b \land a_{\gamma}^{m}\beta b_{\gamma}^{n}, \]

and property (2) is satisfied.

(2) \( \Rightarrow \) (1). We remark first that \( M \) has the following property:

\[ c \land d \leq c \beta d \quad \text{for every } c \in I_{(m, 0, \beta)} \text{ and every } d \in I_{(0, n, \gamma)} \ldots \ldots \quad (\ast) \]
Indeed: Let $c \in I_{(m,0,\beta)}$ and $d \in I_{(0,n,\gamma)}$. By (2), we have
\[ c \wedge d \leq c^m \beta d \wedge c \gamma d^n \leq c^m \beta d. \]
Since $M$ is $\beta$-subidempotent, we have $c^m \beta \leq c$, so $c \wedge d \leq c \beta d$, and (*) holds.
Let now $a \in M$. Then $a \leq a^m \beta e \gamma a^n$. Indeed:
\[ a \leq < a >_{(m,0,\beta)} \wedge < a >_{(0,n,\gamma)} \leq < a >_{(m,0,\beta)} \beta \leq < a >_{(0,n,\gamma)} \beta < a >_{(0,n,\gamma)} \text{ (by (*))}. \]
Moreover,
\[< a >_{(m,0,\beta)} = < a >_{(m,0,\beta)} \wedge e = \left( < a >_{(m,0,\beta)} \right)^m \beta e \wedge < a >_{(m,0,\beta)} \gamma e^n \text{ (by (2))} \leq \left( < a >_{(m,0,\beta)} \right)^m \beta e = (a \vee a^m \beta e)^m \beta e \text{ (by Lemma 10)} = a^m \beta e \text{ (by Lemma 11)}. \]
On the other hand, by Lemma 10, we have $< a >_{(m,0,\beta)} \geq a^m \beta e$. Thus we have $< a >_{(m,0,\beta)} = a^m \beta e$. Similarly we have $< a >_{(0,n,\gamma)} = e \gamma a^n$. Therefore, we obtain
\[ a \leq (a^m \beta e) \beta (e \gamma a^n) = a^m \beta (e \beta e) \gamma a^n \leq a^m \beta e \gamma a^n, \]
and $M$ is $(m, n, \beta, \gamma)$-regular.

References


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