Finite Groups Having Exactly 22 Elements of Maximal Order\textsuperscript{1}

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Abstract

Let $G$ be a finite group, $M(G)$ denotes the number of elements of maximal order of $G$. In this note a finite group $G$ with $M(G) = 22$ is determined.

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1 Introduction

For a finite group $G$, we denote by $M(G)$ the number of elements of maximal order of $G$, and the maximal element order in $G$ by $k = k(G)$. There is a topic related to one of Thompson’s Conjectures:

Thompson’s Conjecture Let $G$ be a finite group. For a positive integer $d$, define $G(d) = \{ \{ x \in G \mid \text{the order of } x \text{ is } d \} \}$. If $S$ is a solvable group, $G(d) = S(d)$ for $d = 1, 2, \ldots$, then $G$ is solvable.

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Recently, some authors have investigated this topic in several articles (see [3], [4]). In particular, in [1] the authors gave a complete classification of the finite group with $M(G) = 30$, and the finite group with $M(G) = 24$ are classified in [2]. In this paper, we consider a finite group $G$ satisfying $M(G) = 22$. Our main result of this paper is:

**Main Theorem** Suppose $G$ is a finite group having exactly 22 elements of maximal order. Then $G$ is solvable and one of the following holds:

1. if $k = 6$, then $|G| = 2^\alpha \cdot 3^\beta$, where $2 \leq \alpha \leq 5$ and $1 \leq \beta \leq 3$;
2. if $k \in \{23, 46\}$, then $C_G(x) = \langle x \rangle \leq G$. Therefore, $G/C_G(x) \cong \text{Aut}(C_k)$, where $o(x) = k$.

By the above theorem, we have:

**Corollary** Thompson’s Conjecture holds if $G$ has exactly 22 elements of maximal order.

## 2 Preliminaries

The following lemma reveals the relationship of $M(G)$ and $k$.

**Lemma 2.1** [4, Lemma 1] Suppose $G$ has exactly $n$ cyclic subgroups of order $l$, then the number of elements of order $l$ (denoted by $n_l(G)$) is $n_l(G) = n\phi(l)$, where $\phi(l)$ is the Euler function of $l$. In particular, if $n$ denotes the number of cyclic subgroups of $G$ of maximal order $k$, then $M(G) = n\phi(k)$.

By above lemma, we have:

**Lemma 2.2** If $M(G) = 22$ and $k$ is maximal element order of $G$, then possible values of $n$, $k$ and $\phi(k)$ are given in following table:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\phi(k)$</th>
<th>$k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>22</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>11</td>
<td>2</td>
<td>3,4,6</td>
</tr>
<tr>
<td>2</td>
<td>11</td>
<td>null</td>
</tr>
<tr>
<td>1</td>
<td>22</td>
<td>23,46</td>
</tr>
</tbody>
</table>

**Lemma 2.3** [1, Lemma 8] If the number of elements of maximal order $k$ is $m$, then there exists a positive integer $\alpha$ such that $|G|$ divides $mk^\alpha$.

**Lemma 2.4** Let $G$ be a finite 2-group. If $\exp(G) = 4$, then there is no group $G$ with $M(G) = 22$. 
Finite groups having exactly 22 elements

proof. If $G$ is a nonabelian 2-group with $\text{exp}(G) = 4$ and every $x$ in $G$ of order 2 is contained in $Z(G)$. We prove that $|G| < 64$. Suppose that $|G| \geq 64$. Then $G$ has a subgroup $H \cong C_2 \times C_2 \times C_2 \times C_2 \times C_4$. Since every element of order 2 is contained in $Z(G)$ and $\text{exp}(G) = 4$. Obviously, $n_4(H) = 32$, a contradiction. If $G$ is abelian, let $|G| = 2^t$. Then $2^{t-1} \leq 22$ by [3, Lemma 2.5]. Hence $t \leq 5$ and $|G| \leq 32$. Therefore $|G| = 32$. Now the lemma holds by [1, Lemma 9].

3 Proof of Main Theorem

By the hypothesis $M(G) = 22$, then $k \neq 2, 3$ by [1, Lemma 6]. In the following we prove our theorem case by case for the remaining possible values of $k$.

**case 1** $k = 4$. By Lemma 2.3, in this case $G$ is a 2-group. By Lemma 2.4, such $G$ does not exist.

**case 2** $k = 6$. In this case $|G| = 2^a3^b$, where $a > 0$ and $b > 0$ by Lemma 2.3. Let $x$ be an element of order 6. Then $|C_G(\langle x \rangle)| = 2^a \cdot 3^b$. Since there exists no element of order 9 or 4 in $C_G(x)$, we have $v \leq 2$ and $u \leq 2$ by $M(G) = 22$. Since $G$ has exactly 11 cyclic subgroups of order 6, we have $|G : N_G(\langle x \rangle)| = 1, 2, 3, 4, 6, 8$ or 9. If there is an element $y$ of order 6 in $G$ such that $|G : N_G(\langle x \rangle)| = 8$ or 9, then there exists another element $z$ of order 6 in $G$ such that $|G : N_G(\langle x \rangle)| = 1, 2, 3, 4$ or 6. That is to say, $G$ always has an element $x$ of order 6 such that $|G : N_G(\langle x \rangle)| = 1, 2, 3, 4$ or 6. Therefore $|G| |2^5 \cdot 3^3$ since $|G| = |G : N_G(\langle x \rangle)| \cdot |N_G(\langle x \rangle) : C_G(\langle x \rangle)| \cdot |C_G(\langle x \rangle)|$. Thus (1) follows.

**case 3** $k \in \{23, 46\}$. Let $x$ be an element of order $k$. Then $C_G(x) = \langle x \rangle \trianglelefteq G$. Therefore, $G/C_G(x) \cong Aut(C_k)$ and $C_G(x) \cong C_k$. Thus (2) holds.

References


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