Finite Groups with Nine Non-subnormal Subgroups

Aifang Feng
Department of Mathematics, Kunming University
Kunming, 650214, China

Zuhua Liu
Department of Mathematics, Kunming University
Kunming, 650214, China

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Abstract
In this paper, finite groups with nine non-subnormal subgroups are completely classified.

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1 Introduction
In group theory, there are various results about how the structure of a finite group related to its special subgroups. The structure of groups whose subgroups are all normal (the Dedekind groups) has been completely described in [3]. And the finite groups with one conjugate class of non-normal subgroups are classified in [4]. Moreover, finite groups with one conjugate class of non-subnormal subgroups are classified in [1]. And the authors have classified all groups with at most eight non-subnormal subgroups (see [8]). In this paper, finite groups with nine non-subnormal subgroups are completely classified.
Let $G$ be a finite group. $\mu(G)$ denotes the number of conjugate classes of non-subnormal subgroups of $G$. If $H \leq G$, we denote by $\mu_G(H)$ the number of $G$-conjugate classes of proper subgroups of $H$ that are non-subnormal in $G$. $N(G)$ denotes the number of non-subnormal subgroups of $G$. $A \rtimes B$ denotes the semidirect product of $A$ and $B$. The rest of notations are referred to [3].

## 2 Preliminary Notes

**Lemma 2.1 ([1]).** Let $G$ be a finite group. Then $\mu(G) = 1$ if and only if $G$ is a finite non-nilpotent inner-abelian group, that is

$$G \cong P \times Q = \langle a, b_1, b_2, \ldots, b_\beta \mid a^{p^n} = 1 = b_1^q = b_2^q = \cdots = b_\beta^q; [b_i, b_j] = 1, i, j = 1, 2, \ldots, \beta;$$

$$b_i^q = b_{i+1}, i = 1, 2, \ldots, \beta - 1; \quad b_\beta^q = b_1^{d_1}b_2^{d_2}\cdots b_\beta^{d_\beta} \rangle,$$

where $f(x) = x^q - d_3x^{q-1} - \cdots - d_2x - d_1$ is an irreducible polynomial over the field $\mathbb{F}_q$, which divides $x^p - 1$, and $q^\beta \equiv 1 \pmod{p}$.

**Lemma 2.2 ([5]).** Let $G$ be a finite $p$-group with cyclic maximal subgroup. Then one of the following holds

1. $G = \langle a | a^{p^n} = 1 \rangle, n \geq 1$.
2. $G = \langle a, b | a^{p^{n-1}} = b^p = 1, [a, b] = 1 \rangle, n \geq 2$.
3. $G = \langle a, b | a^{p^{n-1}} = b^p = 1, a^b = a^{1+p^{n-2}} \rangle, p \neq 2, n \geq 3$.
4. $G = \langle a, b | a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, a^b = a^{-1} \rangle, n \geq 3$.
5. $G = \langle a, b | a^{2^{n-1}} = 1, b^2 = 1, a^b = a^{-1} \rangle, n \geq 3$.
6. $G = \langle a, b | a^{2^{n-1}} = 1, b^2 = 1, a^b = a^{1+2^{n-2}} \rangle, n \geq 4$.
7. $G = \langle a, b | a^{2^{n-1}} = 1, b^2 = 1, a^b = a^{-1+2^{n-2}} \rangle, n \geq 4$.

**Lemma 2.3 ([7]).** Let $G$ be a finite $p$-group with cyclic maximal subgroup. Then the maximal subgroups of $G$ with the seven types are following respectively

1. $\langle a^p \rangle$.
2. $\langle b^a \rangle(i = 0, 1, \cdots, p-1), \langle a^p, b \rangle$.
3. $\langle b^a \rangle(i = 0, 1, \cdots, p-1), \langle a^p, b \rangle$.
4. $n > 3: \langle a \rangle, \langle a^2, b \rangle, \langle a^2, ba \rangle; n = 3: \langle a \rangle, \langle b \rangle, \langle ba \rangle$.
5. $\langle a \rangle, \langle a^2, b \rangle, \langle a^2, ba \rangle$.
6. $\langle a \rangle, \langle ba \rangle, \langle a^2, b \rangle$.
7. $\langle a \rangle, \langle a^2, b \rangle, \langle a^2, ba \rangle$.

**Lemma 2.4 ([6]).** Let $G$ be a finite group with $\mu(G) = 2$, and $H, K$ be non-subnormal and not conjugate in $G$. Then

1. $H < K$, and $H$ and $K$ are maximal in $K$ and $G$ respectively.
2. $H$ is cyclic, and $K = N_G(K)$. 
Lemma 2.5 ([8]). Let $G$ be a finite group, and $C$ be a conjugate class of non-subnormal subgroups of $G$. Then $|C| \geq 3$.

Lemma 2.6 ([9]). Let $G$ be a finite group and $H \leq G$. If $H$ is non-subnormal and $\mu_G(H) = 0$, then $H$ is a cyclic $p$-group.

Lemma 2.7 ([8]). Let $G$ be a finite group with $N(G) = 6$. Then

(i) $G \cong \langle a, b, c \mid a^{2^m} = 1 = b^3 = c^3; [b, a] = [b, c] = 1, c^a = c^{-1} \rangle$.

(ii) $G \cong \langle a, b, c \mid a^{2^m} = 1 = b^3 = c^3, [b, a] = [b, c] = 1, c^a = c^{-1} \rangle$, where $q \neq 2, 3$.

3 Main Results

Theorem 3.1. Let $G$ be a finite group with $N(G) = 9$. Then one of the following holds:

(i) $G \cong \langle a, b, c \mid a^{2^m} = b^3 = c^3 = 1, [a, c] = [b, c] = 1, b^a = b^{-1} \rangle$, where $r \neq 2, 3$.

(ii) $G \cong \langle a, b \mid a^{2^m} = b^2 = 1, b^a = b^r \rangle$, where $r \equiv 1 \mod 9$.

(iii) $G \cong \langle a, b, c \mid a^{2^{m-1}} = b^3 = c^3 = 1, [a, b] = [b, c] = 1, c^a = c^{-1} \rangle, m \geq 2$.

(iv) $G \cong \langle a, b, c \mid a^4 = c^3 = 1, b^2 = a^2, a^b = a^{-1}, [b, c] = 1, c^a = c^{-1} \rangle$.

(v) $G \cong \langle a, b, c \mid a^{2^{m-1}} = b^2 = c^3 = 1, [b, c] = 1, a^b = a^{1+2^{m-2}}, c^a = c^{-1} \rangle, m \geq 4$.

Proof. Let $G$ be a finite group with $N(G) = 9$. Then $\mu(G) \leq 3$ by lemma 2.5, and hence $G$ is solvable (see [9]). Clearly $G$ is not nilpotent, thus there is a Sylow subgroup of $G$ is not normal in $G$ since the group with all Sylow subgroups are normal is nilpotent. Let $P \in Syl_p(G)$, and $P$ be not normal in $G$.

Case 1 $\mu(G) = 1$. Now, $|G| = p^m q^n$ and $P$ is maximal in $G$ by lemma 2.1. Hence $P = N_G(P)$, and $9 = |G : N_G(P)| = |G : P|$, and $q = 3, n = 2$. By Sylow Theorem 9 $\equiv 1 \mod p$, and $p = 2$. So $G \cong \langle a, b, c \mid a^{2^m} = 1 = b^3 = c^3, [b, c] = 1, b^a = c, c^a = b^d \rangle$, where $f(x) = x^3 - d_2 x - d_1$ is an irreducible polynomial over the field $\mathbb{F}_3$. Since $\langle a^2 \rangle Q < G$, $\langle a^2 \rangle Q < G$ is abelian group by lemma 2.1, and hence $[a^2, b] = 1$. Thus $b = b^a^2 = c^a = b^{d_1} c^{d_2}$, hence $d_1 = 1, d_2 = 0$, and $f(x) = x^2 - 1$ is a reducible polynomial over the field $\mathbb{F}_3$, which is a contradiction.

Case 2 $\mu(G) = 2$. Let $C_1, C_2$ be the two conjugate classes of non-subnormal subgroups of $G$. Then $|C_1| \geq 3, |C_2| \geq 3$ according to Lemma 2.5. Let $H \in C_1, K \in C_2$. Then $H < K$ and $K = N_G(K)$ by lemma 2.4. We assert that $N_G(H) = H$ or $N_G(H) = K$. Otherwise, $H < N_G(H) \triangleleft G$, a contradiction. Thus $|C_2| = |C_1|$, it follows that $|C_1| = 6, |C_2| = 3$ since $|C_1| + |C_2| = 9$. Hence $N_G(H) = H$, and $|K : H| = 2$. However $H < K$ by $|K : H| = 2$, and $H < K \leq N_G(H)$, a contradiction.
Case 3 $\mu(G) = 3$. Let $C_1, C_2, C_3$ be the three conjugate classes of non-subnormal subgroups of $G$. Then $|C_1| + |C_2| + |C_3| = 9$. Thus $|C_1| = |C_2| = |C_3| = 3$ since $|C_i| \geq 3, i = 1, 2, 3$, and hence $3 | |G|$. Let $P \in C_1, H \in C_2, K \in C_3$. Then $3 \equiv 1(\text{mod} p)$ by Sylow Theorem, and hence $p = 2$. So $P \in \text{Syl}_2(G)$, and all Sylow subgroups except Sylow 2-subgroups are normal in $G$. Let $Q \in \text{Syl}_3(G)$. Then $Q \triangleleft G$.

(1) Assume that $P < N_G(P)$. Then $N_G(P) = H \triangleleft K$. Without loss of generality let $N_G(P) = K$. Then $K = N_G(K)$ since $|C_1| = |C_3|$, and hence $|C_1| = |G : K| = 3$, which follows that $K$ is maximal in $G$.

(i) Suppose that $P$ is maximal in $K$. Then $|K : P| = r$, where $r$ is an odd prime, since $K$ is solvable. That is $|K| = 2^m r, |G| = 2^m 3r$. Let $K = P \times K_r$, where $K_r$ is the Sylow $r$-subgroup of $K$. It is easily to verified that $K_r \leq Z(G)$, and of course, $K_r \not\leq N_G(H)$. If $P < H$, then $K \leq N_G(H)$. It follows that $N_G(H) = K$ since $|C_2| = |C_3| = 3$, thus $H < K$, which contradicts the maximality of $P$ in $K$. So $\mu_G(H) = 0$, and $H$ is a cyclic 2-group by lemma 2.6. That is $H < P < K$. If $r \neq 3$, then $|Q| = 3$. Let $L = P \times Q$. Then $N(L) = 6$, and $|L| = 2^m 3$, which is a contradiction according to lemma 2.7. If $r = 3$, then $|Q| = 3^2$, and $|G| = 2^m 3^2$. Let $H = L \times Q$. Then $\mu(L) = 1$, and $H$ is self-normalizing in $L$ by lemma 2.1, which contradicts $N_G(H) = K$.

(ii) Suppose that $P$ is not maximal in $K$. Then we must have $P < H < K$. In fact, if $P < L < K$, and $L \neq H$, then $L \triangleleft G$, and hence $P \triangleleft G$, a contradiction. Thus $\mu_G(P) = 0$, and $P = \langle a | a^{2m} = 1 \rangle$ is cyclic by lemma 2.6. Of course, $P$ and $H$ are maximal in $H$ and $K$ respectively, and $N_G(P) = N_G(H) = N_G(K) = K$ by $|C_1| = |C_2| = |C_3| = 3$. Since $H$ and $K$ are solvable, $|H : P| = r$, and $|K : H| = s$, where $r, s$ are odd primes. We assert that $r = s$. Otherwise $r \neq s$, let $K = P \times K_r \times K_s$, and $H = P \times K_r$, where $K_r$ and $K_s$ be the Sylow $r$-subgroup and the Sylow $s$-subgroup of $K$ respectively. Then $P \times K_s$ is also non-subnormal, and $P \times K_s$ is not conjugate to $P, H$ and $K$, hence $\mu(G) \geq 4$, a contradiction. That is $|H| = 2^m r, |K| = 2^m r^2$. If $K_r = \langle c_1 \rangle \times \langle c_2 \rangle$ is elementary abelian, then $P, H = P \times \langle c_1 \rangle, P \times \langle c_2 \rangle$, and $K$ are all non-subnormal in $G$, and not conjugate to each other. It follows that $\mu(G) \geq 4$, a contradiction. So $K_r = \langle c | c^2 = 1 \rangle$ is cyclic.

If $r \neq 3$, then $Q = \langle b | b^3 = 1 \rangle$, and hence $G \cong \langle a, b, c | a^{2m} = b^3 = c^2 = 1, [a, c] = [b, c] = 1, b^a = b^{-1} \rangle$, where $r \neq 2, 3$. Conversely, it is easily verified that the non-subnormal subgroups of $G$ are $\langle a \rangle, \langle a \rangle^b, \langle a \rangle^{b^2}, \langle a, c \rangle, \langle a, c \rangle^b, \langle a, c \rangle^{b^2}$, and $\langle a, c \rangle^b, \langle a, c \rangle^{b^2}$. Hence $N(G) = 9$, and Case (i) in Theorem 3.1 holds.

If $r = 3$, then $Q$ is a 3-group with cyclic maximal subgroup, and $|Q| = 3^3$. Suppose that $Q$ is of type (2), (3) of lemma 2.2. Let $Q = \langle b, c | b^2 = b^3 = 1 \rangle$. Then $[a, b] = 1$, and $H = P \times \langle b^2 \rangle$, $K = P \times \langle b \rangle$. Let $M = \langle b^3, c \rangle$, and $N = P \times M$. Then $\mu(N) = 1$, and $P$ is self-normalizing in $N$, which is a contradiction. Suppose that $Q = \langle b | b^6 = 1 \rangle$. Then $K = P \times \langle b^3 \rangle$, and $G \cong \langle a, b | a^{2m} = b^{27} = 1, b^a = b^r \rangle$. Since $[a, b^3] = 1$, $b^3 = (b^3)^a = (b^3)^3 = (b^r)^3 = b^{3r}$,
hence $3r \equiv 3 \pmod{27}$, that is $r \equiv 1 \pmod{9}$. Conversely, it is easily verified that the non-subnormal subgroups of $G$ are $\langle a \rangle, \langle a \rangle^b, \langle a \rangle^{b^2}, \langle a, b^3 \rangle, \langle a, b^3 \rangle^b, \langle a, b^3 \rangle^{b^2}$, and $\langle a, b^9 \rangle, \langle a, b^9 \rangle^b, \langle a, b^9 \rangle^{b^2}$. Hence $N(G) = 9$, and Case (ii) in Theorem 3.1 holds. 

(2) Assume that $P = N_G(P)$, then $G = P \rtimes Q$, where $|Q| = 3$, since $|C_1| = |G : N_G(P)| = |G : P| = 3$. Hence $N_G(H) = N_G(K) = P$ by $|C_2| = |C_3| = 3$, it follows that $H, K < P$. If $H < K < P$, then $N(K \rtimes Q) = 6$, which contradicts lemma 2.7. So $H$ and $K$ are all cyclic maximal subgroups of $P$, which implies that $P$ has at least two cyclic maximal subgroups. According to lemma 2.3, $P$ is of type (2), or (4), or (6) of lemma 2.2. Let $P = \langle a, b \rangle$, where $a^{2^{m-1}} = 1$. Now, $H = \langle a \rangle$, and $K = \langle ba \rangle$ by lemma 2.3, and clearly $\mu(H \rtimes Q) = 1, \mu(K \rtimes Q) = 1$. Let $Q = \langle c | c^3 = 1 \rangle$, $B = \langle b \rangle$. Then $B \triangleleft G$ by $\mu(G) = 3$, hence $B \triangleleft BQ = M$ since $B$ is the Sylow 2-subgroup of $M$. Thus $M = B \times Q$, and $[b, c] = 1$.

Suppose that $P$ is of type (2) of lemma 2.2. That is $P = \langle a, b | a^{2^{m-1}} = b^2 = 1, [a, b] = 1 \rangle$, $n \geq 2$. Now, $G \cong \langle a, b, c | a^{2^{m-1}} = b^2 = c^3 = 1, [a, b] = [b, c] = 1, c^a = c^{-1} \rangle, m \geq 2$. Conversely, it is easily verified that the non-subnormal subgroups of $G$ are $\langle a, b \rangle, \langle a, b \rangle^c, \langle a, b \rangle^{c^2}, \langle a \rangle, \langle a \rangle^c, \langle a \rangle^{c^2}$, and $\langle ba \rangle, \langle ba \rangle^c, \langle ba \rangle^{c^2}$. Hence $N(G) = 9$, and Case (iii) in Theorem 3.1 holds.

Suppose that $P$ is of type (4) of lemma 2.2. Since $P$ has at least two cyclic maximal subgroups, $m = 3$ by lemma 2.3. That is $P = \langle a, b | a^4 = 1, b^2 = a^2, a^b = a^{-1} \rangle$, and $G \cong \langle a, b, c | a^4 = c^3 = 1, b^2 = a^2, a^b = a^{-1}, [b, c] = 1, c^a = c^{-1} \rangle$. Conversely, it is easily verified that the non-subnormal subgroups of $G$ are $\langle a, b \rangle, \langle a, b \rangle^c, \langle a, b \rangle^{c^2}, \langle a \rangle, \langle a \rangle^c, \langle a \rangle^{c^2}$, and $\langle ba \rangle, \langle ba \rangle^c, \langle ba \rangle^{c^2}$. Hence $N(G) = 9$, and Case (iv) in Theorem 3.1 holds.

Suppose that $P$ is of type (6) of lemma 2.2. That is $P = \langle a, b | a^{2^{m-1}} = 1, b^2 = 1, a^b = a^{1+2^{m-2}}, m \geq 4$. Now, $G \cong \langle a, b, c | a^{2^{m-1}} = b^2 = c^3 = 1, [b, c] = 1, a^b = a^{1+2^{m-2}}, c^a = c^{-1} \rangle, m \geq 4$. Conversely, it is easily verified that the non-subnormal subgroups of $G$ are $\langle a, b \rangle, \langle a, b \rangle^c, \langle a, b \rangle^{c^2}, \langle a \rangle, \langle a \rangle^c, \langle a \rangle^{c^2}$, and $\langle ba \rangle, \langle ba \rangle^c, \langle ba \rangle^{c^2}$. Hence $N(G) = 9$, and Case (v) in Theorem 3.1 holds.

\[ \square \]

References


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