Block Transitive $2-(v,13,1)$ Designs and Classical Simple Groups

Shaojun Dai

Department of Mathematics, Tianjin Polytechnic University
No.399 Binshuixi Road, Xiqing District
Tianjin, 300387, P. R. China

Luozhong Gong

Department of Mathematics
Hunan University of Science and Engineering
Yongzhou, Hunan, 425199, P. R. China

Junqing Wang

Department of Mathematics, Tianjin Polytechnic University
No.399 Binshuixi Road, Xiqing District
Tianjin, 300387, P. R. China

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Abstract

This article is a contribution to the study of the automorphism groups of $2-(v,k,1)$ designs. Let $D$ be $2-(v,13,1)$ design, $G \leq Aut(D)$ be block transitive, point primitive but not flag transitive. Then $Soc(G)$, the socle of $G$, is not classical simple group over finite field $GF(q)(q$ odd).

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1 Introduction

A $2-(v, k, 1)$ design $D = (\mathcal{P}, \mathcal{B})$ is a pair consisting of a finite set $\mathcal{P}$ of $v$ points and a collection $\mathcal{B}$ of $k-$subsets of $\mathcal{P}$, called blocks, such that any 2-subsets of $\mathcal{P}$ is contained in exactly one block. We will always assume that $2 < k < v$.

Let $G \leq Aut(D)$ be a group of automorphisms of a $2-(v, k, 1)$ design $D$. Then $G$ is said to be block transitive on $D$ if $G$ is transitive on $\mathcal{B}$ and is said to be point transitive(point primitive on $D$ if $G$ is transitive (primitive) on $\mathcal{P}$). A flag of $D$ is a pair consisting of a point and a block through that point. Then $G$ is flag transitive on $D$ if $G$ is transitive on the set of flags.

The classification of block transitive $2-(v, 3, 1)$ designs was completed about thirty years ago (see [2]). In [3], Camina and Siemons classified $2-(v, 4, 1)$ designs with a block transitive, solvable group of automorphisms. Li classified $2-(v, 4, 1)$ designs admitting a block transitive, unsolvable group of automorphisms (see [8]). Tong and Li [11] classified $2-(v, 5, 1)$ designs with a block transitive, solvable group of automorphisms. Han and Li [5] classified $2-(v, 5, 1)$ designs with a block transitive, unsolvable group of automorphisms. Liu [10] classified $2-(v, k, 1)$ (where $k = 6, 7, 8, 9, 10$) designs with a block transitive, solvable group of automorphisms. In [6], Han and Ma classified $2-(v, 11, 1)$ designs with a block transitive classical simple groups of automorphisms.

This article is a contribution to the study of the automorphism groups of $2-(v, k, 1)$ designs. We prove that following theorem.

**Main Theorem** Let $D$ be $2-(v, 13, 1)$ design, $G \leq Aut(D)$ be block transitive, point primitive but not flag transitive. Then $Soc(G)$, the socle of $G$, is not classical simple group over finite filed $GF(q)$ ($q$ odd).

2 Preliminary Results

Let $D$ be a $2-(v, k, 1)$ design defined on the point set $\mathcal{P}$ and suppose that $G$ is an automorphism group of $D$ that acts transitively on blocks. For a $2-(v, k, 1)$ design, as usual, $b$ denotes the number of blocks and $r$ denotes the number of blocks through a given point. If $B$ is a block, $G_B$ denotes the setwise stabilizer of $B$ in $G$ and $G_{(B)}$ is the pointwise stabilizer of $B$ in $G$. Also, $G^B$ denotes the permutation group induced by the action of $G_B$ on the points of $B$, and so $G^B \cong G_B/G_{(B)}$.

If $n$ is a positive integer and $p$ is a prime number, then $|n|_p$ denotes the $p-$part of $n$ and $|n|_{p'}$ the $p'$-part of $n$. In other words, $|n|_p = p^t$ where $p^t | n$ but $p^{t+1} \not| n$, and $|n|_{p'} = n/|n|_p$.

**Lemma 2.1** ([6]) Let $G$, $D = (\mathcal{P}, \mathcal{B})$ be as in the Main Theorem and
$Soc(G) = T$. Then

$$|T| \leq \left\lceil \frac{v}{\lambda} \right\rceil \cdot |T_\alpha|^2 \cdot |G:T|,$$

where $\alpha \in \mathcal{P}$, $\lambda$ is the size of longest orbit of $G_\alpha$, and $[v/\lambda]$ is the smallest positive integer not less than $v/\lambda$.

**Lemma 2.2** ([5]) Let $G$ be a transitive group on the point set $\mathcal{P}$ and $T = Soc(G)$. Let $\alpha \in \mathcal{P}$ and let $\Gamma$ be a $G_\alpha$ orbit in $\mathcal{P}\setminus\{\alpha\}$. Then $\Gamma$ is a union of orbits of $T_\alpha$, all having the same size.

In this article, the classical simple group is one of the following groups:

$$PSL_n(q), \quad PSU_n(q), \quad PSp_n(q) \ (n \text{ even}),$$

$$PO_{n}^{\pm}(q) \ (n \text{ even}), \quad PO_n(q) \ (n \text{ even and } q \text{ odd}).$$

The following lemmas are very useful in our proof of the Main Theorem.

**Lemma 2.3** (Liebeck and Saxl [9]) Let $G$ be a primitive permutation group of odd degree $v$ on a set $\Sigma$ and let $H = G_\alpha$, where $\alpha \in \Sigma$.

(a) Either $(Z_p)^d \leq G \leq AGL(d, p)$ for some odd prime $p$, or $T^n \leq G \leq G_0 \wr S_m$, where $G_0$ is a primitive group of odd degree $n_0$ with simple socle $T$ and the wreath product has the product action of degree $n = n_0m$.

(b) If $G$ has simple socle $T$ then $G$ and $H$ are known, and one of (I), (II) and (III) below holds:

(I) $T$ is $A_c$, an alternating group; $H$ is $(S_k \times S_{c-k}) \cap G(1 \leq k < \frac{1}{2}c)$, or $H$ is $(S_a \wr S_b) \cap G(ab = c, a > 1, b > 1)$, or $G$ is $A_7$ of degree 15,

(II) $T$ is sporadic: all possibilities for $G, H$ are given by [1],

(III) $T = T(q)$, a simple group of Lie type over $GF(q)$, in which case

(A) if $q$ is even then $H \cap T$ is a parabolic subgroup of $T$,

(B) if $q$ is odd then one of (i), (ii), (iii) below holds:

(i) $H = N_{G}(T(q_0))$, where $q = q_0^c$ and $c$ is an odd prime;

(ii) $T$ is a classical group with natural projective module $V = V(n, q)$ and one of (1)-(7) below holds:

(1) $H$ is the stabilizer of a non-singular subspace (any subspace for $T = PSL_n(q)$),

(2) $T \cap H$ is the stabilizer of an orthogonal decomposition $V = \bigoplus V_i$ with all $V_i$ isometric (any decomposition $V = \bigoplus V_i$ with $\dim(V_i)$ constant for $T = PSL_n(q)$),
(3) $T = PSL_n(q)$, $H$ is the stabilizer of a pair $\{U, W\}$ of subspaces of complementary dimensions with $U \leq W$ or $U \oplus W = V$, and $G$ contains an automorphism of $T$ interchanging $U$ and $W$,

(4) $T \cap H = \Omega_7(2)$ or $\Omega_8^+(2)$ and $L$ is $P\Omega_7(q)$ or $P\Omega_8^+(q)$, respectively, $q$ is prime and $q \equiv \pm 3 \pmod{8}$,

(5) $T = P\Omega_8^+(q)$, $q$ is prime and $q \equiv \pm 3 \pmod{8}$, $G$ contains a triality automorphism of $T$ and $T \cap H$ is $2^3 \cdot 2^6 \cdot PSL_3(2)$,

(6) $T = PSL_2(q)$ and $T \cap H$ is dihedral $D_4$, $A_4$, $S_4$, $A_5$ or $PGL_2(q^2)$,

(7) $L = PSU_3(5)$ and $T \cap H = M_{10}$;

(iii) $T$ is an exceptional group: $T$, $H$ are as in [9](Table 1).

3 Proof of the Main Theorem

Let $G$, $D$ be as in the Main Theorem. Suppose that $T = Soc(G)$ is a classical simple group with natural projective module $V = V(n, q)$, where $q = p^f$ is odd and $p$ is prime.

Proposition 3.1 Let $G$, $D$ be as in the Main Theorem, $T = Soc(G)$ is a classical simple group and $T_\alpha = T \cap G_\alpha$, where $\alpha \in \mathcal{P}$. Then

(P1) $v = 156b + 1$;

(P2) $\frac{x}{v} < 157|G : T|$ or $\frac{v - 1}{x} \leq 156|G : T|$, where $x$ is the size of a $T_\alpha$-orbit in $\mathcal{P} \setminus \{\alpha\}$;

(P3) $\frac{y}{x} \leq 155|G : T|$, where $x$, $y$ are the sizes of two $T_\alpha$-orbit in $\mathcal{P} \setminus \{\alpha\}$;

(P4) $\frac{|T|}{|T_\alpha|} \leq 79|G : T|$;

(P5) If $(v - 1, q) = 1$, then there exists in $\mathcal{P} \setminus \{\alpha\}$ a $T_\alpha$-orbit of size $x$ such that $x \mid |T_\alpha|\nu'$.

Proof. (P1): Obviously, $13b = vr$ and $156b = v(v - 1)$. Since $G$ is not flag transitive, then $13 \mid v$ and $12 \mid (v - 1)$, and so $v \equiv 1 \pmod{156}$. Thus there exists a positive integer $b'$ such that $v = 156b' + 1$.

(P2): Let $\Gamma_1$ is a non-trivial suborbit of any $T_\alpha$-orbit, $\Gamma$ is a non-trivial suborbit of any $G_\alpha$-orbit and $\Gamma_1 \subseteq \Gamma$. $x$ and $\lambda$ are the sizes of $\Gamma_1$ and $\Gamma$. Then $\lambda \leq x|G : T|$, by Lemma 2.1. Again by Lemma 2.1 of [8], $b' \mid \lambda$. Thus $b' \leq \lambda \leq x|G : T|$, $\frac{v}{x|G : T|} \leq \frac{v}{\lambda} \leq \frac{v}{b'} < 157$. 
(P3): Let $B = \{1, 2, \cdots, 13\} \in \mathcal{B}$. Then the structure of $G^B$, the rank and subdegree of $G$ do not occur:

<table>
<thead>
<tr>
<th>Type of $G^B$</th>
<th>Rank of $G$</th>
<th>Subdegree of $G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1)$</td>
<td>157</td>
<td>$1, b', \cdots, b'$</td>
</tr>
</tbody>
</table>

Otherwise, $|G^B|$ is odd. We have $|G|$ is odd, a contradiction with $|T|$ is a classical simple group. Thus $\lambda \geq 2b'$, where $\lambda$ is the size of longest orbit of $G_{\alpha}$. Let $\Delta_1$ and $\Delta_2$ are two non-trivial suborbits of any $T_{\alpha}$-orbit, $\Gamma_1$ and $\Gamma_2$ are two non-trivial suborbits of any $G_{\alpha}$-orbit, $\Delta_1 \subseteq \Gamma_1$, $\Delta_2 \subseteq \Gamma_2$. Set $x, y, \lambda_1$ and $\lambda_2$ are the sizes of $\Delta_1, \Delta_2, \Gamma_1$ and $\Gamma_2$. By Lemma 2.2,

$$\lambda_1 = t_1x, \quad \lambda_2 = t_2y,$$

where $t_1$ and $t_2$ are the factors of $|G : T|$. By $\lambda \geq 2b'$, we have $\frac{\lambda_2}{\lambda_1} = \frac{t_2y}{t_1x} \leq 155$.

Thus $\frac{v}{\lambda} \leq 155 |G : T|$.

(P4): By Lemma 2.1 and $\lambda \geq 2b'$, we have

$$\frac{|T|}{|T_{\alpha}|^2} \leq \frac{v}{\lambda} \cdot |G : T| \leq \frac{v}{2b'} \cdot |G : T| \leq \frac{156b' + 1}{2b'} \cdot |G : T| \leq 79 |G : T|.$$

(P5): Let $t$ be the size of any $T_{\alpha}$-orbit in $\mathcal{P} \setminus \{\alpha\}$. Suppose to the contrary that $t \nmid |T_{\alpha}|$. Since $t \mid |T_{\alpha}|$, we have $p \mid t$. Furthermore, since $\mathcal{P} \setminus \{\alpha\}$ is a union of $T_{\alpha}$-orbits, $p \mid (v-1)$. Thus $p \mid (v-1, q)$, which contradicts $(v-1, q) = 1$.

Because $G$ is point primitive and $v = 156b' + 1$ is odd, $G$ is a primitive group of odd degree. So we can use Lemma 2.3 to continue our proof. Thus $T$ and $H = G_{\alpha}$ are one of the following cases:

1. $H = N_G(T(q_0))$, where $q = q_0^c$ and $c$ is an odd prime;
2. $H$ is the stabilizer of a non-singular subspace (any subspace for $T = PSL_n(q)$);
3. $T \cap H$ is the stabilizer of an orthogonal decomposition $V = \bigoplus V_i$ with all $V_i$ isometric (any decomposition $V = \bigoplus V_i$ with $\text{dim}(V_i)$ constant for $T = PSL_n(q)$);
4. $T = PSL_n(q)$, $H$ is the stabilizer of a pair $\{U, W\}$ of subspaces of complementary dimensions with $U \bigoplus W = V$ or $U \subseteq W$, and $G$ contains an automorphism of $T$ interchanging $U$ and $W$;
5. $T \cap H$ is $\Omega_7(2)$ or $\Omega_8^+(2)$ and $L$ is $P\Omega_7(q)$ or $P\Omega_8^+(q)$, respectively, $q$ is prime and $q \equiv \pm 3 \pmod{8}$;
(6) $T = PΩ^+_8(q)$, $q$ is prime and $q \equiv \pm 3 \pmod{8}$, $G$ contains a triviality automorphism of $T$ and $T \cap H$ is $2^3 \cdot 2^6 \cdot PSL_3(2)$;

(7) $T = PSL_2(q)$ and $T \cap H$ is dihedral $D_4$, $A_4$, $S_4$, $A_5$ or $PGL_2(q^{1/2})$;

(8) $L = PSU_3(5)$ and $T \cap H = M_{10}$.

In order to prove the Main Theorem, we will rule out these cases one by one.

**Lemma 3.2** Case (1) cannot occur.

**Proof.** Suppose that $H = N_G(T(q_0))$, where $q = q_0^c$ and $c$ is an odd prime. Clearly $f \geq c \geq 3$. By Propositions 4.5.3, 4.5.4, 4.5.8, 4.5.10 and 7.5.1 of [7], we have $T(q_0)$ is a maximal subgroup of $T(q)$. Then $T_\alpha = T \cap G_\alpha = T(q_0)$. By [4], we also have

$$|T| = \frac{1}{d} q^N (q^{d_1} - \varepsilon_1) (q^{d_2} - \varepsilon_2) \cdots (q^{d_l} - \varepsilon_l),$$

where $d_1 > d_2 > \cdots > d_l$, $\varepsilon = \pm 1$, $i = 1, 2, \ldots, l$. And by Theorem 9.3.4 of [4], we have $d_1 + d_2 + \cdots + d_l = N + l$. So

$$|T| = \frac{1}{d} q^N (q^{d_1} - \varepsilon_1) (q^{d_2} - \varepsilon_2) \cdots (q^{d_l} - \varepsilon_l) \geq \frac{1}{d} q^N (q^{d_1} - 1) (q^{d_2} - 1) \cdots (q^{d_l} - l) \geq \frac{1}{d} q^{\frac{4N+1}{2}}.$$

(we have used the inequalities $q^i - 1 > q^{i-1}(q - 1)$ and $(q - 1)^i > q^i$).

For $T(q_0)$ we have

$$|T(q_0)| = \frac{1}{d_0} q_0^N (q_0^{d_1} - 1) (q_0^{d_2} + 1) \cdots (q_0^{d_l} - (-1)^{l+1}) < \frac{1}{d} q^{2N+l}.$$

(we have used the inequalities $q^i + 1 < q^{i+1}$ and $q^i - 1 < q^{i+1}$).

Thus

$$\frac{|T|}{|T_\alpha|^2} > \frac{d_0^2 q^{\frac{4N+1}{2}}}{d_0^{4N+2l}} > \frac{d_0^2 q^{\frac{4N+1}{2}}}{d q^{\frac{4N+1}{2}}} > 79|G : T| \quad \text{if } T \neq A_1(q).$$

If $T = A_1(q)$, i.e. $T = PSL_2(q)$, we have

$$|T| = \frac{1}{2} q(q^2 - 1), \quad |T_\alpha| = \frac{1}{2} q_0(q_0^2 - 1), \quad v = \frac{q_0^{c-1}(q_0^{2c} - 1)}{q_0^2 - 1}.$$

Since $(v - 1, q) = 1$, there exists in $\mathcal{P} \setminus \{\alpha\}$ a $T_\alpha$-orbit of size $x$ such that

$$x | |T_\alpha|_p \leq \frac{1}{2} (q_0^2 - 1).$$
Thus \( x \leq \frac{1}{2}(q_0^2 - 1) \). It follows that

\[
\frac{v}{x} > \frac{2q_0^{c-1}(q_0^{2c} - 1)}{(q_0^2 - 1)^2} > 2q_0^{3c-5} > 2q > 157|G : T| \quad (q \neq 27),
\]

contradicting \((P_2)\). If \( q = 27 = 3^3 \), then \( v = 819 \), contradicting \((P_1)\).

**Lemma 3.3** Case (2) cannot occur.

**Proof.** Suppose that \( H \) is the stabilizer of a non-singular subspace (any subspace for \( T = PSL_n(q) \)).

(1) \( T = PSL_n(q) \) \((n \geq 2)\).

Let \( W \) be a subspace of \( V \), \( \{e_1, e_2, \ldots, e_n\} \) and \( \{e_1, e_2, \ldots, e_m\} \) be bases of \( V \) and \( W \), respectively, where \( m \leq \frac{n}{2} \), and \( G_\alpha \) denotes the stabilizer of the subspace \( W \), then

\[
\hat{T}_\alpha = \left\{ A = \begin{pmatrix} A_1 & A_2 & A_3 \\ A_2 & A_3 \end{pmatrix} \in SL_n(q) | A_1 \in GL_m(q), A_2 \in GL_{n-m}(q) \right\},
\]

where \( \hat{T}_\alpha \) is the inverse image of \( T_\alpha \) under the homomorphism \( SL_n(q) \to PSL_n(q) \).

If \( m \geq 2 \), let \( \{e_1, \ldots, e_{m-1}, e_{m+1}\} \) generate a subspace \( W' \), and \( G_\beta \) denotes the stabilizer of \( W' \), then there exists an element \( g \) in \( SL_n(q) \) such that \( \hat{T}_\alpha g = \hat{T}_\beta \). Thus

\[
\hat{T}_{\alpha \beta} = \hat{T}_\alpha \cap \hat{T}_\beta = \left\{ B = \begin{pmatrix} B_1 & * & a \\ * & 0 & b \\ * & * & B_2 \end{pmatrix} \in SL_n(q) \right\}.
\]

Now let \( \{e_{m+1}, \ldots, e_{2m-1}, e_{2m}\} \) generate a subspace \( \tilde{W} \), and \( G_\gamma \) denotes the stabilizer of \( \tilde{W} \), then there exists an element \( g_1 \in SL_n(q) \) such that \( \hat{T}_\alpha g_1 = \hat{T}_\beta \). Thus

\[
\hat{T}_{\alpha \gamma} = \hat{T}_\alpha \cap \hat{T}_\gamma = \left\{ B = \begin{pmatrix} X & Y & Z \\ 0 & X & * \\ * & * & Z \end{pmatrix} \in SL_n(q) \right\}.
\]

Hence \( T_\alpha \) has two orbits with sizes \( x \) and \( y \), respectively, such that

\[
\frac{x}{y} \geq \frac{(g-1)^2q_{m+n-m^2-3}|GL_{m-1}(q)||GL_{n-m-1}(q)|}{q^{2m(n-2m)}|GL_m(q)||GL_{n-2m}(q)|} > q^{\frac{2m-n-2m^2-3n+2m+2}{2}} > 155|G : T| \quad (m \geq 4 \text{ or } m = 3 \text{ and } q \neq 3),
\]
contradicting \((P_3)\).

If \(m = 3\), \(q = 3\) and \(n > 6\), then
\[
\frac{x}{y} > q^{\frac{3n-10}{2}} > 155|G : T|,
\]
contradicting \((P_3)\). If \(m = 3\), \(q = 3\) and \(n = 6\), then \(v = 10 \cdot 28 \cdot 121\) is even, contradicting \((P_1)\).

If \(m = 2\) and \(n \geq 10\), then
\[
\frac{x}{y} \geq \frac{(q-1)3^{m-7}||GL_n(q)||}{q^{3n-10}||GL_2(q)||^2||GL_{n-4}(q)||} > q^{n-4} > 155|G : T|,
\]
contradicting \((P_3)\). If \(m = 2\) and \(4 \leq n \leq 9\), we can easily prove that \(v\) is even, contradicting \((P_1)\).

If \(m = 1\), then \(T_{\alpha}\) is transitive on the subspaces of dimension 1. Hence \(T\) is 2-transitive, a contradiction.

(2) \(T = \text{PSp}_n(q), \text{PSU}_n(q), \text{PO}_{2n+1}(q), \text{PO}^+_{2n}(q), \text{PO}_n^-(q)\).

We deal with these cases by two steps, first we show that \(q^2\) is a divisor of \(v\), and next show that \(T\) has a suborbit with size \(x\) such that \(q^2 \mid x\). By \((P_2)\) this is a contradiction. We only prove the case of \(\text{PO}^+_{2n}(q)\) in detail.

By Theorem 4.1.6 of [7], the stabilizer of a non-singular subspace of \(\text{PO}^+_{2n}(q)\) \((n \geq 4)\) is of type \(O_m^\epsilon(q) \perp O_{2n-m}^\epsilon(q)\), where \(1 \leq m < n, \epsilon = \{+, -, \circ\}\).

Let \(\epsilon = \circ\). Then the stabilizer of a non-singular subspace of \(\text{PO}^+_{2n}(q)\) is of type \(O_m^\epsilon(q) \perp O_{2n-m}^\epsilon(q)\).

If \(m = 1\), then the stabilizer \(T_{\alpha}\) of the subspace \(W\) with dimension 1 is of type \(O_1(q) \perp O_{2n-1}(q)\). Hence \(v = \frac{1}{2}q^{n-1}(q^{n-1} - 1)\). Let \(\{x\}\) and \(\{x_1; e_1, f_1; e_2, f_2; \ldots; e_{n-1}, f_{n-1}\}\) \((\{e_i, f_j\} = \delta_{i,j}\) and \((e_i, e_j) = (f_i, f_j) = 0\) for all \(i, j\) \) be the standard basis of \(W\) and \(W^\perp\), respectively. Then there exists an element \(t\) in \((e_{n-1}, f_{n-1})\) such that \(Q(t) = Q(x)\). Let \(\tilde{W} = \langle t \rangle\), and \(T_{\beta}\) denotes the stabilizer of \(\tilde{W}\). We have an element \(g_1\) in \(I\) such that \(\tilde{T}_{\alpha}g_1 = \tilde{T}_{\beta}\).

Now we will prove there exists an element \(g\) in \(\Omega\) such that \(\tilde{T}_{\alpha}g = \tilde{T}_{\beta}\). If \(g_1 \in \Omega\), then \(g_1\) is the element we need. If \(g_1 \not\in \Omega\), we have two cases: (a) \(g_1 \in S\), (b) \(g_1 \not\in S\). If (a) holds, it follows from Proposition 2.5.6 of [7] that \(g_1\) can be written as a product of an even number of reflections \(g_1 = r_{v_1} \cdots r_{v_k}\), for some non-singular vectors \(v_i\). Since \(g_1 \not\in \Omega\), then \(\theta(g_1) = \prod_{i=1}^k(v_i, v_i) \in F^* \setminus (F^*)^2\) (i.e. \(\theta(g_1)\) is non-square). Let \(a\) be non-square and \(g = r_{e_1} + f_{1}r_{e_1} + af_1g_1\), then we have
\[
\theta(g) = (e_1 + f_1, e_1 + f_1)(e_1 + af_1, e_1 + af_1)\theta(g_1) = 4a\theta(g_1) \in (F^*)^2.
\]
Hence \(g \in \Omega\) and \(\tilde{T}_{\alpha}g = \tilde{T}_{\beta}\). If (b) holds, it follows from Proposition 2.5.6 of [7] that \(g_1\) can be written as a product of an odd number of reflections \(g_1 = r_{v_1} \cdots r_{v_k}\), for some non-singular vectors \(v_i\). Let \(g_2 = r_{e_1} + f_1g_1\), then
If \( g_2 \in S \), then \( g_2 \) is the element we need. If \( g_2 \not\in \Omega \), then by case (a) we have an element \( g_3 \in \Omega \) such that \( T^g_\alpha = T_\beta \). Since \( T_\alpha \beta \) stabilizes \( W, W^\perp, \bar{W} \) and \( \bar{W}^\perp \), then also stabilizes \( W, \bar{W} \) and \( W^\perp \cap \bar{W}^\perp \). Thus

\[
\hat{T}_{\alpha\beta} \leq M = \left\{ A = \begin{pmatrix} a & A \\ B & \end{pmatrix} \in \Omega^+_2(q) \mid \right\},
\]

where \( A \in O_{2n-3}(q), B \in O^+_2(q) \).

Therefore \( T \) has a non-trivial suborbit of size \( x \) such that

\[
x = \frac{|T_\alpha|}{|T_{\alpha\beta}|}, \quad y = \frac{|T_\alpha|}{|M|} = q^{3n-4} \cdot Y, \quad y \mid x,
\]

where \( Y \) is a positive integer. It follows that \( q^2 \mid (v, x) \), contradicting \((P_2)\).

If \( m \geq 3 \), let

\[
\{ x; e_1, f_1; \cdots; e_{\frac{m-3}{2}}; f_{\frac{m-3}{2}}; e_{\frac{m+1}{2}}; f_{\frac{m+1}{2}} \}
\]

and

\[
\{ x; e_1, f_1; \cdots; e_{\frac{m-3}{2}}; f_{\frac{m-3}{2}}; e_{\frac{m+1}{2}}; f_{\frac{m+1}{2}} \}
\]

be the standard basis, and \( T_\alpha \) and \( T_\beta \) be the stabilizers of \( W \) and \( \bar{W} \). Similarly there exists an element \( g \in \Omega \) such that \( \hat{T}_\alpha^g = \hat{T}_\beta^g \). Then

\[
\hat{T}_{\alpha\beta} = \left\{ A = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix} \in \Omega^+_2(q) \mid \right\},
\]

\[
B_1 \in O_{m-2}(q), B_2, B_3 \in O^+_2(q), B_4 \in O_{2n-m-2}(q)
\]

Thus we have

\[
v = \frac{q^{2mn-m^2-1} \prod_{i=1}^{n-1} (q^{2i} - 1)}{\prod_{i=1}^{m-1} (q^{2i} - 1) \prod_{i=1}^{2n-m-1} (q^{2i} - 1)},
\]

\[
x = \frac{q^{2n-4} \prod_{i=1}^{m-1} (q^{2i} - 1) \prod_{i=1}^{m-1} (q^{2i} - 1)}{(q + 1)^2 \prod_{i=1}^{\frac{m}{2}} (q^{2i} - 1) \prod_{i=1}^{\frac{m}{2}} (q^{2i} - 1)}.
\]

Therefore

\[
\frac{v}{x} > q^{2mn-m^2-5n+5}.
\]

If \( m \geq 4 \) or \( m = 3 \) and \( n \geq 9 \), then

\[
\frac{v}{x} > q^5 > 157 |G : T|,
\]
contradicting \((P_2)\).

If \(m = 3, n = 8, q \geq 5\), then
\[
\frac{v}{x} > q^4 > 157|G : T|,
\]
contradicting \((P_2)\). If \(m = 3, n = 8, q = 3\), then \(v = 3^{19}(3^{12} + \cdots + 3^2 + 1)\), contradicting \((P_1)\). It is not difficult to exclude the exceptional cases with \(m = 3\) and \(n \leq 7\) by direct calculation.

Let \(\varepsilon = +\). Then the stabilizer \(T_\alpha\) of the subspace \(W\) is of type \(O_{2m}^+(q) \perp O_{2n-2m}^+(q)\). Let
\[
\begin{align*}
&\{e_1, f_1; \cdots; e_{m-1}, f_{m-1}; e_m, f_m\}, \\
&\{e_1, f_1; \cdots; e_{m-1}, f_{m-1}; e_{m+1}, f_{m+1}\},
\end{align*}
\]
be the standard basis of \(W\) and \(\tilde{W}\), and \(T_\alpha\) and \(T_\beta\) be the stabilizers of \(W\) and \(\tilde{W}\). Similarly there exists an element \(g\) in \(\Omega\) such that \(\hat{T}_\alpha^g = \hat{T}_\beta\). Then
\[
\hat{T}_{\alpha\beta} = \left\{ \begin{array}{c}
A = \begin{pmatrix}
B_1 & B_2 \\
B_3 & B_4
\end{pmatrix} \\
B_1 \in O_{2m-2}^+(q), B_2, B_3 \in O_2^+(q), B_4 \in O_{2n-2m-2}^+(q)
\end{array} \right\}.
\]

Similarly we can get \(v\) and \(x\) and exclude them by \((P_1)\) and \((P_2)\).

Let \(\varepsilon = -\). Then the stabilizer \(T_\alpha\) of the subspace \(W\) is of type \(O_{2m}^-(q) \perp O_{2n-2m}^-(q)\). Let
\[
\begin{align*}
&\{x, y; e_2, f_2; \cdots; e_m, f_m\}, \\
&\{x_1, y_1; e_{m+2}, f_{m+2}; \cdots; e_n, f_n\}, \\
&\{x_1, y_1; e_2, f_2; \cdots; e_m, f_m\},
\end{align*}
\]
be the standard basis of \(W\), \(W^\perp\) and \(\tilde{W}\), and \(T_\alpha\) and \(T_\beta\) be the stabilizers of \(W\) and \(\tilde{W}\). Similarly there exists an element \(g\) in \(\Omega\) such that \(\hat{T}_\alpha^g = \hat{T}_\beta\). Then
\[
\hat{T}_{\alpha\beta} = \left\{ \begin{array}{c}
A = \begin{pmatrix}
B_1 & B_2 \\
B_3 & B_4
\end{pmatrix} \\
B_2 \in O_{2m-2}^+(q), B_1, B_3 \in O_2^{-}(q), B_4 \in O_{2n-2m-2}^+(q)
\end{array} \right\}.
\]

Similarly we can get \(v\) and \(x\) and exclude them by \((P_1)\) and \((P_2)\).

**Lemma 3.4** Case (3) cannot occur.
Proof. Suppose that $T \cap H$ is the stabilizer of an orthogonal decomposition $V = \bigoplus V_i$ with all $V_i$ isometric (any decomposition $V = \bigoplus V_i$ with $\dim(V_i)$ constant for $T = PSL_n(q)$).

(1) $T = PSL_n(q)$.

Let $T = T \cap G_\alpha$ be the stabilizer of $V = \bigoplus V_i$ with $\dim(V_i)$ constant,

$$\{e_1, \cdots, e_m, e_{m+1}, \cdots, e_{2m}, \cdots, e_{m(n-1)+1}, \cdots, e_{tm}\}$$

and be bases of $V$, and correspondingly $\{e_{m(i-1)+1}, \cdots, e_{im}\}$ be a base of $V_i (i = 1, 2, \cdots, t)$. By Theorem 4.2.9 of [7], $T_\alpha$ is of type $GL_m(q) \wr S_t$.

If $m = 1$, Then

$$\frac{|T|}{(n, q - 1)|T_\alpha|^2} = \frac{q^{n(n-1)}}{(q-1)^{2n}(n!)^2} > \frac{q^{2n^2 - 3n}}{(n!)^2} > 79|G : T| \quad (n \geq 6).$$

If $n = 2$, let

$$V = \langle e \rangle \perp \langle f \rangle = \langle e + f \rangle \perp \langle f \rangle,$$

then $v = \frac{1}{2}q(q+1)$ and $x = 2(q-1)$, where $x$ is the size of a $T_\alpha$-orbit in $\mathcal{P} \setminus \{\alpha\}$. Thus

$$\frac{v-1}{x} > \frac{q}{4} > 157|G : T| \quad (f \geq 8),$$

contradicting $(P_2)$. If $f = 1$ and $p > 626$, then $\frac{v-1}{x} > 157|G : T|$, contradicting $(P_2)$. If $f = 1$ and $p < 626$, we can get $v$, contradicting $(P_1)$. It is not difficult to exclude the exceptional cases with $n = 2$ by direct calculation. The proof of $n = 3, 4, 5$ is similar to $n = 2$ and omitted here.

If $m \geq 2$, let

$$V_1' = \langle e_1, \cdots, e_{m-1}, e_{2m} \rangle,$$

$$V_2' = \langle e_{m+1}, \cdots, e_{2m-1}, e_{2m} \rangle,$$

$$V_i' = V_i \quad (i = 3, 4, \cdots, t),$$

and $T_\beta$ denotes the stabilizer of $V = \bigoplus_{i=1}^t V_i'$, then there exists an element $g$ in $SL_n(q)$ such that $T_\alpha = T_\beta$. Thus the element in $T_{\alpha \beta}$ has the forms

$$\begin{pmatrix} A & a \\ B & b \\ C \end{pmatrix}$$

or

$$\begin{pmatrix} A & a \\ B & b \\ C \end{pmatrix},$$

where $A, B \in GL_{m-1}(q), C \in GL_{n-2m}(q), a, b \in GF(q)^*$ and $C$ is the stabilizer of the decomposition $V = \bigoplus_{i=3}^t V_i$. Thus we have

$$v = \frac{|GL_n(q)|}{|GL_m(q)|^t!} \geq \frac{1}{t!} q^{2n^2 - 2m - n},$$
\[
x = \frac{|GL_m(q)|^t t!}{2(q-1)^2 |GL_{m-1}(q)|^t |GL_m(q)|^{t-2}(t-2)!} < \frac{t(t-1)}{2} q^{4m-3},
\]
where \( x \) is the size of a \( T_\alpha \)-orbit in \( \mathcal{P} \setminus \{\alpha\} \). Therefore
\[
\frac{v}{x} > \frac{2}{t(t-1)t!} q^{2n^2-2mn-n^2m+6} > 157|G : T| \quad (n \geq 6),
\]
contradicting \((P_2)\). When \( n = 4 \), we can give the proof easily by \((P_1)\) and \((P_2)\) and omit here.

2) \( T = PSp_n(q), PSU_n(q), P\Omega_{2n+1}(q), P\Omega^\pm_{2n}(q) \).

These cases are separated into two steps to discuss. First when \( m \leq 2 \) we consider \(|T|/|T_\alpha|^2\) and arrive at a contradiction by Lemma 2.1, second when \( m > 2 \) we consider \( v/x \), where \( x \) is the size of a suitable non-trivial orbit of \( T_\alpha \), and have a contradiction by \((P_2)\). We only show the case of \( P\Omega^\pm_{2n}(q) \) in detail.

From Theorems 4.2.11, 4.2.14 and 4.2.15 of [7], we know that the stabilizer of an orthogonal decomposition \( V = \bigoplus_{i=1}^t \) of \( P\Omega^\pm_{2n}(q) \) \((n \geq 4)\) is of type \( O_m^\varepsilon(q) \left< S_t \right> \), where \( \varepsilon = \{+, -, \circ\} \).

Let \( \varepsilon = \circ \). Then the stabilizer of an orthogonal decomposition \( V = \bigoplus_{i=1}^t \) of \( P\Omega^\pm_{2n}(q) \) \((n \geq 4)\) is of type \( O_m(q) \left< S_{2n} \right> \), where \( 2n = mt \).

If \( m = 1 \), then \( T_\alpha \) is of type \( O_1(q) \left< S_{2n} \right> \), where \( q = p \).

\[
\frac{|T|}{|T_\alpha|^2} = \frac{dq^{n(n-1)}(q^n - 1) \prod_{i=1}^{n-1} (q^{2i} - 1)}{2^{2n-1}((2n)!)^2} > \frac{dq^{4n^2-3n}}{2^{2n-1}((2n)!)^2} > 79|G : T|,
\]
contradicting \((P_1)\).

If \( m = 2k + 1 \geq 3 \), let
\[
\{x_i; e_{i1}, f_{i1}; \cdots; e_{ik}, f_{ik}\} \quad (i = 1, 2, \cdots, t)
\]
be a standard basis of \( V_i \) \((i = 1, 2, \cdots, t)\), and let
\[
V'_1 = \langle x_1; e_{i1}, f_{i1}; \cdots; e_{1(k-1)}, f_{1(k-1)}; e_{2k}, f_{2k} \rangle,
V'_2 = \langle x_2; e_{21}, f_{21}; \cdots; e_{2(k-1)}, f_{2(k-1)}; e_{1k}, f_{1k} \rangle,
V'_i = V_i \quad (i = 3, 4, \cdots, t),
\]
then \( V = \bigoplus_{i=1}^t V'_i \). Denote by \( T_\beta \) the stabilizer of an orthogonal decomposition \( V = \bigoplus_{i=1}^t V'_i \), then there exists an element \( g \) in \( \Omega \) such that \( \hat{T}_\alpha^g = \hat{T}_\beta \). Obviously the element in \( \hat{T}_\alpha \hat{T}_\beta \) has the forms
\[
\begin{pmatrix}
A & a \\
B & b \\
C & 
\end{pmatrix}
\quad \text{or} \quad
\begin{pmatrix}
A & a \\
B & b \\
C & 
\end{pmatrix},
\]
where $A, B \in O_{m-2}(q), C \in O_{2n-2m}^+(q), a, b \in O_2^+(q)$ and $C$ is the stabilizer of the decomposition $V = \bigoplus_{i=3}^t V_i$. Thus

$$v = \frac{|O_{2n}^+(q)|}{|O_m(q)|^t|t!|} > \frac{1}{2^{t-1}t!} q^{\frac{4n^2-3m-2n}{2}},$$

$$x = \frac{|O_{m}(q)|^t|t!|}{2|O_{m-2}(q)|^2|O_m(q)|^t-2(t-2)!|O_2^+(q)|^2} < \frac{t(t-1)}{8} q^{4m-5},$$

where $x$ is the size of a $T_\alpha$-orbit in $P \setminus \{\alpha\}$. Therefore

$$\frac{v}{x} > \frac{1}{2^{t-4}t(t-1)} q^{\frac{4n^2-3m-2n-8m+10}{2}} > 157|G : T|.$$

This is a contradiction.

When $\varepsilon = +$ or $-$, the proof is similar to $\varepsilon = \circ$ and omitted here.

**Lemma 3.5** Case (4) cannot occur.

**Proof.** Suppose that $T = PSL_n(q), H$ is the stabilizer of a pair $\{U, W\}$ of subspaces of complementary dimensions with $U \bigoplus W = V$ or $U \leq W$, and $G$ contains an automorphism of $T$ interchanging $U$ and $W$.

(1) If $U \leq W$ and $\dim(U) + \dim(W) = n$, let $\dim(U) = m$, and

$$\{e_1, e_2, \ldots, e_m\}, \quad \{e_1, e_2, \ldots, e_m, \ldots, e_{n-m}\}, \quad \{e_1, e_2, \ldots, e_n\}$$

be bases of $U, W$ and $V$, respectively. Then the element in $T_\alpha$ has the form

$$\begin{pmatrix}
A_m \\
* \\
* \\
B_{n-2m} \\
* \\
* \\
C_m
\end{pmatrix}.$$

If $m = 1$, then

$$v = \frac{(q^n - 1)(q^{n-1} - 1)}{(q - 1)^2}$$

is even, which is impossible.

If $m = 2$, then

$$v = \frac{(q^n - 1)(q^{n-1} - 1)(q^{n-2} - 1)(q^{n-3} - 1)}{(q - 1)^2(q^2 - 1)^2}$$

is even. A contradiction.
If $m \geq 3$, let $\{e_2, e_3, \ldots, e_{m+1}\}$ generate a subspace $\tilde{U}$, $\tilde{W} = W$, and $T_\beta$ denote the stabilizer of the pair $\{\tilde{U}, \tilde{W}\}$. Then there exists an element $g$ in $SL_n(q)$ such that $\tilde{T}_\alpha^g = \tilde{T}_\beta$. So the element in $\tilde{T}_{\alpha\beta}$ has the form

$$
\begin{pmatrix}
a & * & 0 & 0 & 0 \\
0 & A_{m-1} & 0 & 0 & 0 \\
0 & * & b & 0 & 0 \\
* & * & * & B_{n-2m-1} & 0 \\
* & * & * & * & C_m
\end{pmatrix}.
$$

Therefore

$$
v = \frac{|GL_n(q)|}{|GL_m(q)||GL_{n-2m}(q)|q^{m(2n-3m)} > q^{\frac{4mn-n-6m^2}{2}}},
$$

$$
x = \frac{|GL_m(q)||GL_{n-2m}(q)|q^{m(2n-3m)}}{(q-1)^2q^{2mn-3m^2-3m+3n}|GL_{m-1}(q)||GL_m(q)||GL_{n-2m-1}(q)|} < q^{n-m+1}.
$$

where $x$ is the size of a $T_\alpha$-orbit in $P \setminus \{\alpha\}$. Using above inequalities, we have

$$
v > q^{\frac{4mn-n-6m^2}{2} - 3n + 2m - 2} > q^6 > 157,
$$

a contradiction.

(2) If $U \oplus W = V$ and $G$ contains an automorphism of $T$ interchanging $U$ and $W$, let $\{e_1, e_2, \ldots, e_m\}$, $\{e_{m+1}, \ldots, e_n\}$, $\{e_1, \ldots, e_{m-1}, e_{m+1}\}$ and $\{e_m, e_{m+2}, \ldots, e_n\}$ generate a subspace $U$, $W$, $\tilde{U}$ and $\tilde{W}$, respectively. Denote by $T_\alpha$ and $T_\beta$ the stabilizer of the pair $\{U, W\}$ and $\{\tilde{U}, \tilde{W}\}$, respectively. Then there exists an element $g$ in $SL_n(q)$ such that $\tilde{T}_\alpha^g = \tilde{T}_\beta$. So the element in $\tilde{T}_{\alpha\beta}$ have the forms

$$
\begin{pmatrix}
A_m \\
B_{n-m}
\end{pmatrix} \quad \text{and} \quad
\begin{pmatrix}
A_{m-1} \\
a & b \\
B_{n-m-1}
\end{pmatrix},
$$

respectively. Therefore

$$
v = \frac{|GL_n(q)|}{|GL_m(q)||GL_{n-m}(q)|},
$$

$$
x = \frac{|GL_m(q)||GL_{n-m}(q)|}{(q-1)^2|GL_{m-1}(q)||GL_{n-m-1}(q)|},
$$

where $x$ is the size of a $T_\alpha$-orbit in $P \setminus \{\alpha\}$. This is a contradiction by $(P_2)$.

**Lemma 3.6** Cases (5) cannot occur.
Proof. Suppose that $T \cap H$ is $\Omega_7(2)$ or $\Omega_8^+(2)$ and $L$ is $P\Omega_7(q)$ or $P\Omega_8^+(q)$, respectively, $q$ is prime and $q \equiv \pm 3 \pmod{8}$.

If $T = P\Omega_7(q)$ and $T_\alpha = P\Omega_7(2)$. Then

$$\frac{|T|}{|T_\alpha|^2} = \frac{\frac{1}{2}q^9(q^2 - 1)(q^4 - 1)(q^6 - 1)}{2^{18}(2^2 - 1)^2(2^4 - 1)(2^6 - 1)^2} > 79|G : T| \quad (q \geq 5),$$

contradicting $(P_4)$. If $q = 3$, then $v = 3^5 \cdot 13$, contradicting $(P_1)$.

If $T = P\Omega_8^+(q)$ and $T_\alpha = P\Omega_8^+(2)$. Then

$$\frac{|T|}{|T_\alpha|^2} = \frac{\frac{1}{2}q^{12}(q^2 - 1)(q^4 - 1)(q^6 - 1)}{2^{24}(2^2 - 1)^2(2^4 - 1)(2^6 - 1)^2} > 79|G : T| \quad (q \geq 5),$$

contradicting $(P_4)$. If $q = 3$, then $v = 3^7 \cdot 13$, contradicting $(P_1)$.

Lemma 3.7 Case (6) cannot occur.

Proof. Suppose that $T = P\Omega_8^+(q)$, $q$ is prime and $q \equiv \pm 3 \pmod{8}$, $G$ contains a triviality automorphism of $T$ and $T \cap H$ is $2^3 \cdot 2^6 \cdot PSL_3(2)$.

Obviously, $T = P\Omega_8^+(q)$ and $T_\alpha = 2^3 \cdot 2^6 \cdot PSL_3(2)$. Thus

$$\frac{|T|}{|T_\alpha|^2} = \frac{\frac{1}{2}q^{12}(q^2 - 1)(q^4 - 1)(q^6 - 1)}{2^{21} \cdot 3^3 \cdot 7^2} > 79|G : T|,$$

contradicting $(P_4)$.

Lemma 3.8 Case (7) cannot occur.

Proof. Suppose that $T = PSL_2(q)$ and $T \cap H$ is $D_4$, $A_4$, $S_4$, $A_5$ or $PGL_2(q^\frac{1}{2})$.

If $T_\alpha = D_4$, then $|T| = \frac{q(q^2 - 1)}{2}$ and $|T_\alpha| = 4$. Thus

$$\frac{|T|}{|T_\alpha|^2} = \frac{q(q^2 - 1)}{32} > 79|G : T| \quad (f \geq 3),$$

contradicting $(P_4)$. If $f = 2$ and $p \geq 3$, then $\frac{|T|}{|T_\alpha|^2} > 79|G : T|$, contradicting $(P_4)$. If $f = 2$ and $p = 3$, then $v = 90$, contradicting $(P_1)$. If $f = 1$ and $p \geq 17$, then $\frac{|T|}{|T_\alpha|^2} > 79|G : T|$, contradicting $(P_4)$. If $f = 1$ and $p = 3$, then $v = 3 < k = 13$, which is impossible. If $f = 1$ and $p = 5, 7, 11$ or $13$, then $v = 15, 42, 165$ and $272$, contradicting $(P_1)$. When $T_\alpha = A_4$, $S_4$ or $A_5$, the proof is similar to $T_\alpha = D_4$ and omitted here.

If $T_\alpha = PGL_2(q^\frac{1}{2})$, then

$$v = \frac{1}{2}q^\frac{1}{2}(q + 1), \quad |T_\alpha| = q^\frac{1}{2}(q - 1).$$
By \((v-1, q) = 1\) and \((P_3)\), then \(x \leq q-1\), where \(x\) is the size of a \(T_\alpha\)-orbit in \(P \setminus \{\alpha\}\). Therefore

\[
\frac{v-1}{x} > \frac{1}{2}q^\frac{1}{2}(q+1) - 1 > \frac{1}{2}q^\frac{1}{2}.
\]

When \(f \geq 18\), then \(\frac{v-1}{x} > 155|G : T|\), contradicting \((P_2)\). If \(f = 2\) and \(p > 620\), then \(\frac{v-1}{x} > 155|G : T|\), contradicting \((P_2)\). If \(f = 2\) and \(p < 620\), we can easily prove that \(v\) is even, contradicting \((P_1)\). The proof of \(4 \leq f \leq 16\) is similar to \(f = 2\) and omitted here.

**Lemma 3.9** Case (8) cannot occur.

**Proof.** Suppose that \(L = PSU_3(5)\) and \(T \cap H = M_{10}\). Obviously, \(T = PSU_3(5)\) and \(T_\alpha = M_{10}\). Thus

\[
v = \frac{|PSU_3(5)|}{|M_{10}|} = \frac{\frac{5^3 \cdot 24 \cdot 126}{2^3 \cdot 3^2 \cdot 5}} = 1050
\]

is even, contradicting \((P_1)\).

By Lemmas 3.2-3.9, it follows that \(T\) is not a classical simple group.

Thus the proof of the Main Theorem is complete.

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**References**


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