Construction of the $\alpha_2$–Automorphism

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Abstract
In this paper, let $\lambda$ a monomorphism from $A$ to $A'$ where $A, A' \in \Gamma$, we consider $B'$ a basic subgroup of $A'$:

$$B' = \bigoplus_{i \geq k} B_k$$

with

$$B_k = \bigoplus_{i \in I_k} <x_{k,i}>,$$

we suppose there exists $n_0 \in \mathbb{N}^*$ such that the restriction of $\lambda$ to $p^{n_0}A$ is an isomorphism from $p^{n_0}A$ to $p^{n_0}A'$ and we pose: $\lambda(A) = A_1$.

We show that $B'_1 \cap A_1$ is a direct factor of $B'_1$ and if $B'_1 = (B'_1 \cap A_1) \bigoplus B^x_1$ and $\alpha \in Aut(A)$ is written in the form: $\alpha = \pi id_A + \rho$, where $\pi$ is an invertible $p$–adic number and $\rho \in Hom(A, A^1)$ with $A^1$ is the first Ulm subgroup of $A$ then, there exists an automorphism $a_2$ of $A_2 = A_1 \bigoplus B^x_1$ such that for all $a_2 \in A_2$:

$$\alpha(a_2) = \pi a_2 + p^{n_0}a'_1$$

where $a'_1 \in A_1$.

Throughout this paper, $\Gamma$ is a class of abelian $p$–groups, $o(x)$ will denote the order of the element $x$ and $p$ is a prime number.

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1 Introduction

In 1987, P. Schupp showed, in [3], that the extension property in the category of groups, characterizes the inner automorphisms. M. R. Pettet gives in, [4], a simpler proof of Schupp’s result and shows that the inner automorphisms of a group are also characterized by the lifting property in the category of groups. The automorphisms of abelian $p$-groups having the extension property in the category of abelian $p$-groups are characterized in [1].

Let $\lambda$ be a monomorphism from $A$ to $A'$ where $A, A' \in \Gamma$, we suppose there exists $n_0 \in \mathbb{N}^*$ such that the restriction of $\lambda$ to $p^{n_0}A$ is an isomorphism from $p^{n_0}A$ to $p^{n_0}A'$.

We pose: $\lambda(A) = A_1$ hence $\lambda$ is an isomorphism from $A$ to $A_1$

so $\lambda(p^{n_0}A) = p^{n_0}A_1$ and since $\lambda(p^{n_0}A) = p^{n_0}A'$ then $p^{n_0}A_1 = p^{n_0}A'$.

By (1) there exists $t \in I_k$ such that $p^{n_0}t_k = p^{n_0}x$.

We pose: $B_k' = \bigoplus_{i \in I_k} < t_{k,i} >$

By (2) there exists $k, i \in A_1$ such that $p^{n_0}t_{k,i} = p^{n_0}x$.

Lemma 1.1 For all $k \geq n_0 + 1$, we have: $B_k'[p] = B_k^0[p]$.

Proof Let $x \in B_k'[p]$ then $x \in \bigoplus_{i \in I_k} < x_{k,i} >$ and $px = 0$

i.e. $\begin{cases} x = \sum_{i=1}^{r} m_j x_{k,i} & \text{where } r \in \mathbb{N}^* \quad (3) \\ px = 0 \end{cases}$

hence $p\sum_{j=1}^{r} m_j x_{k,i} = \sum_{j=1}^{r} pm_j x_{k,i} = 0$

then $\forall j = 1, ..., r : pm_j x_{k,i} = 0$

which implies that $\forall j = 1, ..., r : p^k | pm_j$

i.e. $\forall j = 1, ..., r : p^k - 1 | m_j$

i.e. $\forall j = 1, ..., r : p^{n_0} | m_j$ because $k \geq n_0 + 1$

i.e. $\forall j = 1, ..., r : \exists m_j' \in \mathbb{N}^*, m_j = p^{n_0}m'_j \quad (4)$

then by (3),(4) and (2) we have:

$x = \sum_{j=1}^{r} m_j x_{k,i} = \sum_{j=1}^{r} p^{n_0} m'_j x_{k,i} = \sum_{j=1}^{r} m'_j p^{n_0} t_{k,i}$

hence $x \in \bigoplus_{i \in I_k} < t_{k,i} >$ and $px = 0$ i.e. $x \in B_k^0[p]$

then for all $k \geq n_0 + 1 : B_k'[p] \subset B_k^0[p]$
by the same procedure we prove for all \( k \geq n_0 + 1 : B_k^0[p] \subset B_k'[p] \)
consequently for all \( k \geq n_0 + 1 : B_k^0[p] = B_k'[p] \).

**Lemma 1.2** Let \( B'' = B_1'' \oplus \ldots \oplus B_{n_0}'' \oplus B_{n_0+1}' \oplus B_{n_0+2}'' \oplus \ldots \)
with \( B_k'' = \begin{cases} B_k' & \text{if } k \leq n_0 \\ B_k^0 & \text{if } k > n_0 \end{cases} \)
then \( B'' \) is a basic subgroup of group \( A' \).

**Proof** - For \( n \leq n_0 - 1 \)
Since \( B' \) is a basic subgroup of group \( A' \) while by theorem 33.1 [2] :
\[
(p^n A')[p] = B_{n+1}'[p] \oplus (p^{n+1} A')[p] = B_{n+1}''[p] \oplus (p^{n+1} A')[p]
\]
- For \( n \geq n_0 \)
Since \( B' \) is a basic subgroup of group \( A' \) while by theorem 33.1 [2] and lemma 2.1:
\[
(p^n A')[p] = B_{n+1}'[p] \oplus (p^{n+1} A')[p] = B_{n+1}^0[p] \oplus (p^{n+1} A')[p]
\]
then \( B'' \) is a basic subgroup of group \( A' \).

**Proposition 1.1** (i) \( B_1' \cap A_1 \) is a direct factor of \( B_1' \)

(ii) Let \( B_1^\times \) such that \( B_1' = (B_1' \cap A_1) \oplus B_1^\times \) if \( A_2 = A_1 \oplus B_1^\times \)
then \( A_2 = A_1 + B_1' \)

**Proof** (i) By Lemma 2.2 and theorem 32.4 [2] we have:
\[
A' = B'' \oplus \ldots \oplus B_{n_0}'' \oplus (B_{n_0}^\times + p^{n_0} A')
\]
with \( B_{n_0}^\times = B_{n_0+1}' \oplus B_{n_0+2}'' \oplus \ldots \)
but since \( A_1 \subset A' \) and \( B_{n_0}^\times + p^{n_0} A' \subset A_1 \)
then \( A_1 = [A_1 \cap (B_1' \oplus \ldots \oplus B_{n_0}'')] \oplus (B_{n_0}^\times + p^{n_0} A') \).
since we have \( B_1' = \bigoplus_{i \in I_1} < x_{1,i} > \) and \( o(x_{1,i}) = p, \forall i \in I_1 \)
then \( pB_1' = 0 \) (i.e. \( B_1' \) is a bounded group )
thus \( B_1' \) is a \( \mathbb{Z}/p\mathbb{Z} \)-vector space
then \( B_1' \cap A_1 \) is a direct factor of \( B_1' \)

(ii) We have \( A_2 = A_1 \oplus B_1^\times \) hence \( A_2 = A_1 + B_1^\times \)
and since \( B_1^\times \subset B_1' \) because \( B_1' = (B_1' \cap A_1) \oplus B_1^\times \)
then \( A_2 \subset A_1 + B_1' \).
Conversely let \( x \in A_1 + B_1' \) then \( x = a_1 + b_1' \) where \( a_1 \in A_1 \) and \( b_1' \in B_1' \)
we have \( b_1' \in B_1' \) hence \( b_1' = y + b_1'' \) where \( y \in B_1' \cap A_1 \subset A_1 \) and \( b_1'' \in B_1'' \)
then \( x = a_1 + y + b_1'' \in A_2 \) because \( A_2 = A_1 \oplus B_1^\times \)
so \( A_1 + B_1' \subset A_2 \) consequently \( A_2 = A_1 + B_1' \).
Definition 1.3 We define the mapping $\alpha_2$ of $A_2$ by:

$$
\begin{align*}
\alpha_2|_{A_1} &= \lambda\alpha\lambda^{-1} \quad \text{where} \quad \alpha \in \text{Aut}(A) \\
\alpha_2|_{B_1^\times} &= \pi id_{B_1^\times}
\end{align*}
$$

Remark 1.1 $\alpha_2$ is well defined because:

$A_2 = A_1 \oplus B_1^\times$ and $\alpha_1 := \lambda\alpha\lambda^{-1} \in \text{Aut}(A_1)$ and $\pi id_{B_1^\times} \in \text{Aut}(B_1^\times)$

and we have $\forall a_2 \in A_2$, $\exists!(a_1, b_1^\times) \in A_1 \times B_1^\times$:

$$a_2 = a_1 + b_1^\times \text{ and } \alpha_2(a_2) = \alpha_1(a_1) + \pi b_1^\times$$

Proposition 1.2 $\alpha_2$ is an automorphism of $A_2$

Proof

Let $a_2 \in kero_2$ then $\alpha_2(a_2) = 0$

and by remark 2.1 we have: $\alpha_1(a_1) + \pi b_1^\times = 0$

i.e.

$$
\begin{align*}
\alpha_1(a_1) &= 0 \\
\pi b_1^\times &= 0
\end{align*}
$$

this is equivalent to $a_1 = b_1^\times = 0$ i.e. $kero_2 = 0$ then $\alpha_2$ is injective.

On the other hand let $(a_1', b_1^\times') \in A_1 \times B_1^\times$

we have $\alpha_1 \in \text{Aut}(A_1)$ hence $\exists! a_1 \in A_1$ such that $\alpha_1(a_1) = a_1'$

and since $b_1^\times = \pi(-1)b_1^\times = \pi b_1^\times$ where $b_1^\times = \pi^{-1}b_1^\times$

we then obtain:

$$a_1' + b_1^\times = \alpha_1(a_1) + \pi b_1^\times = \alpha_2(a_1) + \alpha_2(b_1^\times) = \alpha_2(a_1 + b_1^\times)$$

which means that $\alpha_2$ is surjective.

It follows that $\alpha_2$ is an automorphism of $A_2$.

Proposition 1.3 The following diagram is commutative..:

$$
\begin{array}{ccc}
A & \xrightarrow{\lambda} & A_2 \\
\alpha \downarrow & & \downarrow \alpha_2 \\
A & \xrightarrow{\lambda} & A_2
\end{array}
$$

Proof

We have $\forall x \in A$, $\lambda(x) \in A_1 \subset A_2$

then

$$
\begin{align*}
\alpha_2\lambda(x) &= \alpha_1\lambda(x) \quad \text{because} \quad \alpha_2|_{A_1} = \alpha_1 \\
&= \lambda\alpha\lambda^{-1}\lambda(x) \quad \text{because} \quad \alpha_1 = \lambda\alpha\lambda^{-1} \\
&= \lambda\alpha(x)
\end{align*}
$$

therefore $\alpha_2\lambda = \lambda\alpha$
**Theorem 1.4** Let \( \alpha \in \text{Aut}(A) \) such that \( \alpha = \pi \text{id}_A + \rho \) where \( \pi \) is an invertible \( p \)-adic number and \( \rho \in \text{Hom}(A, A^1) \) with \( A^1 \) is the first Ulm subgroup of \( A \). Then \( \forall a_2 \in A_2, \exists a'_1 \in A_1, \exists n_0 \in \mathbb{Z} \) such that \( \alpha_2(a_2) = \pi a_2 + p^{n_0}a'_1 \).

**Proof**

Let \( a_2 \in A_2 \) since \( A_2 = A_1 \oplus B_1^\times \) hence \( a_2 = a_1 + b_1^\times \) where \( a_1 \in A_1 \), \( b_1^\times \in B_1^\times \) then

\[
\alpha_2(a_2) = \alpha_1(a_1) + \pi b_1^\times \\
= \lambda \alpha \lambda^{-1}(a_1) + \pi b_1^\times \\
= \lambda \alpha(a) + \pi b_1^\times \text{ where } \lambda^{-1}(a_1) = a \in A \\
= \lambda(\pi a + \rho(a)) + \pi b_1^\times \\
= \pi a_1 + \lambda \rho \lambda^{-1}(a_1) + \pi b_1^\times \\
= \pi a_2 + \lambda \rho \lambda^{-1}(a_1)
\]

since \( a_1 \in A_1 \), then \( \rho \lambda^{-1}(a_1) \in A^1 \subset p^{n_0}A \)

hence \( \exists a \in A, \rho \lambda^{-1}(a_1) = p^{n_0}a \)

then \( \lambda \rho \lambda^{-1}(a_1) = p^{n_0}a' = p^{n_0}a'_1 \) where \( a'_1 \in A_1 \)

it follows that \( \alpha_2(a_2) = \pi a_2 + p^{n_0}a'_1 \) where \( a'_1 \in A_1 \).

**References**


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