Subgroup Lattices and Tables of Marks

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Abstract
We define isomorphisms of subgroup lattices and tables of marks, and study some of the invariants preserved by them.

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1 Introduction
The table of marks of a group is a useful invariant, and even though it provides a considerable amount of data about the group, it is not enough to identify it up to isomorphism (as proved by Thévenaz in [4]). Another important group invariant, but from a different perspective, is the subgroup lattice of the group (see [3] for an introduction to this topic). In this paper we combine these

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two concepts in a new definition (an isomorphism of subgroup lattices and tables of marks). We prove that groups sharing these two combined invariants must have many more properties in common than one would expect. In fact, the question remains whether these two properties may be enough to prove that the groups are isomorphic as groups: the only known examples of non-isomorphic groups with isomorphic tables of marks fail when the definition is extended in this manner.

2 Tables of marks

Let \( G \) be a finite group. Let \( \mathcal{C}(G) \) be the family of all conjugacy classes of subgroups of \( G \). We usually assume that the elements of \( \mathcal{C}(G) \) are ordered non-decreasingly according to the size of the subgroups. The matrix whose \( H, K \)-entry is \( \#(G/K)^H \) (that is, the number of fixed points of the set \( G/K \) under the action of \( H \)) is called the table of marks of \( G \) (where \( H, K \) run through all the elements in \( \mathcal{C}(G) \)).

The Burnside ring of \( G \), denoted \( \mathbb{Z}[\mathcal{C}(G)] \), is the subring of \( \mathbb{Z}[\mathcal{C}(G)] \) spanned by the columns of the table of marks of \( G \).

**Definition 1.** Let \( G \) and \( Q \) be finite groups. Let \( \psi \) be a function from \( \mathcal{C}(G) \) to \( \mathcal{C}(Q) \). Given a subgroup \( H \) of \( G \), we denote by \( H' \) any representative of \( \psi([H]) \). We say that \( \psi \) is an isomorphism between the tables of marks of \( G \) and \( Q \) if \( \psi \) is a bijection and if \( \#(Q/K')^{H'} = \#(G/K)^H \) for all subgroups \( H, K \) of \( G \). We usually refer to \( H' \) as the image of \( H \) under the isomorphism of table of marks.

We list some properties of isomorphisms of tables of marks, whose proofs can be found in [?]. An isomorphism between tables of marks preserves the order of the subgroups, the order of their normalizers, the number of elements of a given order, the number of (conjugacy classes of) subgroups of a given order, the number of normal subgroups of a given order, it sends cyclic groups to cyclic groups and elementary abelian groups to elementary abelian groups. It also sends the derived subgroup of \( G \) to the derived subgroup of \( Q \), maximal subgroups of \( G \) to maximal subgroups of \( Q \), Sylow \( p \)-subgroups to Sylow \( p \)-subgroups (same \( p \)), and the Frattini subgroup of \( G \) to the Frattini subgroup of \( Q \).

Assume now that \( G \) and \( Q \) are finite groups with isomorphic tables of marks. As we mentioned, \( G \) and \( Q \) must have the same order. It is also easy to check that if \( G \) is abelian or simple, then \( G \) and \( Q \) must be isomorphic groups. If \( G \) is a direct product, so is \( Q \), and their corresponding factors have isomorphic tables of marks. If \( G \) is a semidirect product \( N \rtimes H \) then \( Q \) is a semidirect product \( N' \rtimes H' \) where \( H \) and \( H' \) have isomorphic tables of marks.
(although we cannot say much about $N$ and $N'$, other than they correspond under the isomorphism of tables of marks).

However, an isomorphism of tables of marks may not preserve abelian subgroups, and it may not send the centre of $G$ to the centre of $Q$. This can be seen in two nonisomorphic groups of order 96 which have isomorphic tables of marks (see [2]). This will change when we have isomorphisms of subgroup lattices and tables of marks.

3 Isomorphisms of subgroup lattices and tables of marks

We now combine the notion of isomorphism of tables of marks with that of an isomorphism of subgroup lattices. Even though subgroup lattices have been studied for a long time (see [1]), many open problems in the area remain. However, combining subgroup lattices with tables of marks, more can be said about the structure of a group.

Definition 2. Let $G$ and $Q$ be finite groups with subgroup lattices $S(G)$ and $S(Q)$ respectively. Let $\varphi : S(G) \rightarrow S(Q)$ be a map. We say that $\varphi$ is an isomorphism of subgroup lattices and tables of marks if $\varphi$ is an isomorphism of lattices such that:

- (Order Axiom) For every $H \in S(G)$, $|H| = |\varphi(H)|$.

- (Subconjugation Axiom) For every $H, K, L \in S(G)$ with $H$ and $K$ subgroups of $L$, we have that $H$ is conjugate to $K$ by an element of $L$ if and only if $\varphi(H)$ is conjugate to $\varphi(K)$ by an element of $\varphi(L)$.

Remark 3. Note that if we omit the Order Axiom, then the cyclic groups $C_{p^n}$ and $C_{q^n}$ for $p, q$ primes have isomorphic subgroup lattices that satisfy the Subconjugation Axiom.

4 Properties of isomorphisms of subgroup lattices and tables of marks

We proceed to derive properties preserved by isomorphisms of subgroup lattices and tables of marks. Throughout this discussion, let $G$ and $Q$ be finite groups and $\varphi : S(G) \rightarrow S(Q)$ an isomorphism of subgroup lattices and tables of marks. Let $L$ denote an arbitrary subgroup of $G$ (sometimes we shall take $L = G$) and $H, K$ subgroups of $L$. 
Property 1. \( \varphi \) induces an isomorphism of subgroup lattices and tables of marks between \( L \) and \( \varphi(L) \).

Property 2. \( \varphi \) induces a bijection between the conjugacy classes of subgroups of \( G \) and \( Q \).

Property 3. \( H \) is normal in \( L \) if and only if \( \varphi(H) \) is normal in \( \varphi(L) \).

Property 4. \( \varphi(N_G(H)) = N_Q(\varphi(H)) \). More generally, \( \varphi(N_L(H)) = N_{\varphi(L)}(\varphi(H)) \).

Property 5. Let \( \alpha_L(H,K) \) denote the number of subgroups of \( L \) that are conjugate to \( K \) in \( L \) and that contain \( H \), and let \( \beta_L(H,K) \) denote the number of subgroups of \( L \) that are conjugate to \( H \) in \( L \) and that are contained in \( K \). Then \( \alpha_L(H,K) = \alpha_{\varphi(L)}(\varphi(H),\varphi(K)) \) and \( \beta_L(H,K) = \beta_{\varphi(L)}(\varphi(H),\varphi(K)) \).

Property 6. For any subgroup \( L \) of \( G \), the map \( \varphi \) induces an isomorphism between the tables of marks of \( L \) and \( \varphi(L) \).

In view of the last properties, we have the following result.

Theorem 4. Let \( G \) and \( Q \) be finite groups with subgroup lattices \( S(G) \) and \( S(Q) \) respectively. Let \( \varphi : S(G) \to S(Q) \) be an isomorphism of subgroup lattices. The following are equivalent:

1. \( \varphi \) is an isomorphism of subgroup lattices and tables of marks.

2. for every subgroup \( L \) of \( G \), and for every subgroups \( H, K \) of \( L \), we have that \( |H| = |\varphi(H)| \) and \( \alpha_L(H,K) = \alpha_{\varphi(L)}(\varphi(H),\varphi(K)) \) (or equivalently, with \( \beta \) instead of \( \alpha \): \( \beta_L(H,K) = \beta_{\varphi(L)}(\varphi(H),\varphi(K)) \)). As a consequence, we have \( \varphi(N_L(H)) = N_{\varphi(L)}(\varphi(H)) \).

3. for every subgroup \( L \) of \( G \), \( \varphi \) induces an isomorphism between the tables of marks of \( L \) and \( \varphi(L) \).

Proof. (of Theorem): We have that 1 implies 2 and 2 implies 3 immediately. To prove that 3 implies 1, simply note that \( H \) is conjugate to \( K \) in \( L \) if and only if \( |H| = |K| \) and \( \alpha_L(H,K) = 1 \). \( \square \)

Property 7. If \( L \) is a normal subgroup of \( G \), then \( \varphi(L) \) is a normal subgroup of \( Q \) and \( \varphi \) induces an isomorphism of subgroup lattices and tables of marks between \( G/L \) and \( Q/\varphi(L) \). More generally, if \( H \) is a normal subgroup of \( L \), then \( \varphi(H) \) is a normal subgroup of \( \varphi(L) \) and \( \varphi \) induces an isomorphism of subgroup lattices and tables of marks between \( L/H \) and \( \varphi(L)/\varphi(H) \).
Property 8. The following properties are derived from the isomorphisms of tables of marks: \( L \) is abelian if and only if \( \varphi(L) \) is abelian, and in this case, \( L \) is isomorphic to \( \varphi(L) \); in particular, \( L \) is cyclic if and only if \( \varphi(L) \) is cyclic (this will help us deal with elements of \( G \)); \( L \) is simple if and only if \( \varphi(L) \) is simple, and in this case, \( L \) is isomorphic to \( \varphi(L) \); \( L \) is soluble if and only if \( \varphi(L) \) is soluble; \( \varphi \) preserves (that is, gives a correspondence between) derived subgroups, Frattini subgroups, Sylow \( p \)-subgroups, maximal subgroups, elementary abelian subgroups.

Property 9. For a subgroup \( L \), denote its center by \( Z(L) \). Then \( \varphi(Z(L)) = Z(\varphi(L)) \).

Proof. (of Properties 1 to 9)

1. Both axioms hold for subgroups of \( L \).

2. This follows immediately from the Subconjugation Axiom.

3. We have that \( H \) is normal in \( L \) if and only if for every \( K \) in \( L \) conjugate to \( H \) in \( L \), we must have \( K = H \).

4. The normalizer in \( G \) of \( H \) is the largest subgroup \( N \) of \( G \) such that \( H \) is a normal subgroup of \( N \), and these properties are preserved by \( \varphi \). Repeat the argument for \( L \) instead of \( G \).

5. This follows from the Subconjugation Axiom.

6. Note that \( \varphi \) preserves the orders of the subgroups, the orders of their normalizers and the alphas, both with respect to \( G \) and to any subgroup \( L \).

7. This follows from the Correspondence Theorem.

8. All these properties hold for isomorphisms of tables of marks.

9. The center of a group is the largest abelian normal subgroup \( Z \) such that for any cyclic subgroup \( C \), \( ZC \) is abelian.

Epilogue

We conjecture that finite \( p \)-groups with isomorphic subgroup lattices and tables of marks must be isomorphic groups, but we believe that for arbitrary finite groups this is not the case. However, we do not have a counterexample yet.
References


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