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Purely Rickart Module

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Abstract

A module M over a ring R is said to be purely Rickart if the right annihilator in M of each endomorphism ring of a module M is a pure submodule of M. Purely Rickart module is a proper generalization of Rickart module. Some properties of the purely Rickart module are investigated. Also, we prove that the ring nxn matrix over R is a purely Rickart ring if and only if R is a weakly n-semiherditary ring. Every n-generated projective module is purely Rickart if and only if the free R-module R $^{(n)}$ is a purely Rickart. Others results are provided in this paper.

Keywords: purely Rickart module, Rickart module, weakly n-semihereditary ring

1. Introduction

Throughout this paper, all rings are associative with identity and all modules are unitary right R-modules. A ring R is a right PP (resp.PF) ring if every principal right ideal is projective (resp. flat). Left PP (resp. PF) is defined in a similar way. These concepts are introduced by A. Hattori [2]. A ring R is a right PP (resp. PF) ring if and only if the right annihilator of each element in R is a direct summand (resp. pure) in R [5]. An ideal I of a ring R is right (left) pure in R if for each element a∈I there is b∈I such that a=ab (ba) [12]. P.M. Cohn generalized the definition of purity for abelian group to a pure submodule. A submodule P of a module M is pure if and only if the sequence $0\rightarrow P\otimes E \rightarrow M\otimes E$ is exact for all left modules E [5]. Also, Fieldhouse [5] generalized the Von Neumann regular ring to a Von Neumann regular module. A module M is said to be Von Neumann module if and only if every submodule of M is pure. A ring R is

said to be (quasi-, right p.q.-) Baer if the right annihilator of every (right ideal, principal right) nonempty subset of R is generated (as a right ideal) by an idempotent of R [13], [7] and [9] respectively. Left p.q-Baer is defined in a similar way. In [11], L. Zhongkui and Z. Renyu, introduced a left APP-ring as a generalization of left p.q.-Baer rings and right PP-rings. A ring R is a left APP-ring if the left annihilator of every principle left ideal is right s-unital (an ideal I in R is s-unital if for each element a∈I there is an element x∈I such that ax=a) as an ideal of R. A. H. Al- saadi [3] introduced purely (quasi-)Baer modules as a generalization of (quasi-)Baer module. An R-module M is purely (quasi-) Baer if the right annihilator of every (two sided) left ideal I of $S=End_R(M)$ is pure in M.
In this paper, we introduce purely Rickart modules as a generalization to purely (quasi-) Baer modules and Rickart modules. A module M is purely Rickart if the right annihilator in M of each endomorphism of M is a pure submodule of M. This class contains the class of Von Neumann regular, semisimple and Rickart modules (the right annihilator in M of each endomorphism of M is a direct summand of M[8]). We will refer to the endomorphism ring of a right R-module M by $S=End_R(M)$, the right (left) annihilator of each element α∈S in M by $r_M(\alpha) =\{m\in M \mid \alpha(m)=0\}$($\ell_M(\alpha) =\{m \in M \mid m\alpha = 0\}$) and the right annihilator of each element a in a ring R is $r_R(a) =\{ r\in R \mid ar =0\}$. Also, we will refer to a (not) pure submodule by ($\not\leq^p$) $\leq^p$.

2. Purely Rickart modules

We introduce the following definition

Definition 2.1.
A module M is said to be *purely Rickart* if $\ker\alpha = r_M(\alpha)$ is a pure submodule of M for each $\alpha \in S = End_R(M)$. A ring R is right purely Rickart if R_R is purely Rickart module.

Remarks and examples 2.2.
1. A ring R is right purely Rickart if and only if R is a right PF ring.

Proof. Follows from [5, Theorem 2.2, CH.6]. ◘
2. Every Rickart (and hence semisimple) module M is purely Rickart.
3. The converse of (2) is not true in general. As the example in[11, Example 2.5], let Z be the ring of integers and let $S = (\prod_{i=1}^{\infty} Z/2Z)$ / $(\oplus_{i=1}^{\infty} Z/2Z)$. Then in the power series R= S[[x]], every principal ideal flat and hence R is PF ring. So, by (1) R is purely Rickart ring. But by [11, Example 2.5.], R is not Rickart (PP) ring.
4. Every Von Neumann regular module is purely Rickart module. The converse is not true in general. For example, Q_Z is purely Rickart which is not Von Neumann regular module: in fact, $Z\not\leq^p Q$ where the sequence $0 \to Z \to Q \to \frac{Q}{Z} \to 0$ is not pure exact.

5. Every purely Baer module is a purely Rickart.
6. The purely Rickart property transforms under an isomorphism map.

Proof. Let M_1 and M_2 be two modules such that M_1 is a purely Rickart and there is an isomorphism β: $M_1 \to M_2$. Let $S_i = End_R(M_i)$ for i =1,2. Define: $S_1 \to S_2$ by (g)=$\beta g \beta^{-1}$ for each g $\in S_1$. Since βis an isomorphism, then so is ψ. Let α∈S_2 .We are finished if we can prove β $(r_{M_1}(\psi^{-1}(\alpha))) = r_{M_2}(\alpha)$. For that: put $\psi^{-1}(\alpha) = \varphi \in S_1$. Since M_1 is purely Rickart, then $r_{M_1}(\varphi) \leq^p M_1$. If b ∈ $r_{M_2}(\alpha)$, then β (a) = b for some a∈M_1 (β is onto). Now b ∈ $r_{M_2}(\alpha)$ and (φ) = α, mean 0 = ψ (φ) b = $\beta\varphi\beta^{-1}(b)$ = βφ (a). Since β is a homomorphism, φ (a) = 0. Hence a ∈ $r_{M_1}(\psi^{-1}(\alpha))$ such that b = β (a) ∈ β $(r_{M_1}(\psi^{-1}(\alpha))$. Thus $r_{M_2}(\alpha) \leq \beta\, (r_{M_1}(\psi^{-1}(\alpha))$. Now, let β (x) ∈ β $(r_{M_1}(\psi^{-1}(\alpha)))$ and $\psi^{-1}(\alpha) = \alpha$. So (g) β(x) = $\beta g \beta^{-1}(\beta(x)) = \beta g(x) = 0$. Hence β(x) ∈ $r_{M_2}(\psi\ (g)) = r_{M_2}(\alpha)$. Thus β $(r_{M_1}(\psi^{1}(\alpha))) \leq r_{M_2}(\ \alpha\ )$ and so $\beta(r_{M_1}(\psi^{-1}(\alpha)) = r_{M_2}(\alpha)$. Now, since $r_{M_1}(\psi^{-1}(\alpha)) \leq^p M$ where M_1 is purely Rickart, then $r_{M_2}(\alpha) \leq^p M$. Therefore M_2 is purely Rickart. ◘

7. Homomorphic image of a purely Rickart module may be not purely Rickart. In fact, the Z-module Z is purely Rickart while an epimorphic image Z_4 of Z is not purely Rickart as Z-module. For that: let f: $Z_4 \to Z_4$ defined by $f(\bar{1}) = \bar{2}$. Then $r_{Z_4}(f) = \{\bar{0}, \bar{2}\}$ is not pure submodule of Z_4.

A submodule of a purely Rickart module may be not purely Rickart in general. For example the Z-module $Q \oplus Z_2$ is purely Rickart while the submodule $Z \oplus Z_2$ is not. In fact, if one takes α: $Z \oplus Z_2 \to Z \oplus Z_2$ defined by $\alpha(a, \bar{b}) = (0, \bar{a})$, then kerα = $2Z \oplus Z_2$ is not pure submodule in $Z \oplus Z_2$ where 2Z is not pure in Z. Therefore $Z \oplus Z_2$ is not purely Rickart sumodule.

Proposition 2.3. Let M be a purely Rickart module and N ≤ M. If every α ∈$End_R(N)$ can be extended to $\bar{\alpha}$ ∈S = $End_R(M)$ then N is purely Rickart module .

Proof. Let α ∈End (N). So there is $\bar{\alpha}$ ∈S such that $\bar{\alpha}\,|_N = \alpha$. Since M is purely Rickart, hence Ker$\bar{\alpha} \leq^p$ M. So Kerα $\leq^p$ M but Kerα ≤ N. Hence Kerα $\leq^p$N [5, Proposition 1.2-2, CH.1]. Therefore N is purely Rickart module. ◘

Proposition 2.4. Every direct summand of a purely Rickart module is purely Rickart.

Proof. Let M be a purely Rickart module and S= $End_R(M)$, N $\leq^{\oplus}$M. Then N = eM for some e^2=e ∈S. Let α∈ $End_R(N)$= $End_R(eM)$. Put β= αe. Then kerβ= [eM∩kerα]⊕(1-e) M = kerα⊕(1-e)M. Since M is purely Rickart and β ∈S, kerβ $\leq^p$M. But kerα $\leq^{\oplus}$kerβ hence kerα $\leq^p$ kerβ $\leq^p$ M and so kerα $\leq^p$ M. But kerα ≤ N, then kerα $\leq^p$N[5, Proposition 1.2, CH.1].◘

Recall that a module M is pure split if every pure submodule of M is a direct summand [6]. Clear that every semisimple is pure split, the Z-module Z is pure split while the Z-module $\prod_{\alpha\in I} Z_\alpha$ is not where if M = $\prod_{\alpha\in I} Z_\alpha$ and N = $\oplus_{\alpha\in I} Z_\alpha \leq$ M then N is pure submodule which is not direct summand in M. A module M ≠ 0 is pure simple if the only pure submodule of M is the trivial submodules of M [5].

Proposition 2.5. Let M be a pure split module. Then M is purely Rickart if and only if M is Rickart module.

Proposition 2.6. Let M be a pure simple, then the following conditions are equivalent
1. M is a purely Rickart
2. Every nonzero endomorphism of M is a monomorphism.
3. M is a Rickart

Proof. (1⇒2) Let α be a nonzero endomomorphism of M. Since M is purely Rickart, then Kerα $\leq^p$ M. But M is pure simple and α ≠0. Hence kerα = 0. Thus α is a monomorphism.
(2⇒3) and (3⇒1) Obvious. ◘

Remark 2.7. Consider the Z-module M = $Z_2 \oplus Z_2$, then M is purely Rickart module (where M is semisimple) which is not pure simple. One can easily show the projection map α: M→Z_2 is not monomorphism.

Recall that an exact sequence of right modules 0→N→M →F→0 is purely exact if and only if for every left module D, the sequence 0 → N⊗D → M⊗D → F⊗D → 0 is exact [5].

Proposition 2.8. A module M is purely Rickart if and only if the short exact sequence 0 → Kerα →M→ $\frac{M}{\operatorname{Ker}\alpha}$ → 0 is purely exact for each α ∈ S = $End_R(M)$.

Corollary 2.9. Let M be a module and S=$End_R(M)$. Then M is a purely Rickart if and only if Imα is a flat submodule of M for each α∈ S.

Proof. Follows $\frac{M}{\operatorname{Ker}\alpha} \cong$ Imα and [5, Theorem 1.7, CH.1] and Proposition 2.8 ◘
Recall that a module M is CF if every cyclic submodule of M is flat [1].

Corollary 2.10. Every cyclic CF module is purely Rickart.

Proof. Suppose that M is a cyclic C.F. module and α ∈ S= End_R (M). Then Imα is cyclic submodule of M. So, Imα is a flat submodule of M. Since α is an arbitrary endomorphism of S, then from Corollary (2.9), M is a purely Rickart module. ◘

A module P is said to be pure projective if and only if for any pure exact sequence 0→N→M → F→ 0, the induced sequence 0→ Hom(N,P) →Hom(M,P) → Hom(F,P) → 0 is exact[5].

Proposition 2.11. If Imα is a pure projective (and hence projective) submodule of M for any α∈ S, then M is a purely Rickart module if and only if M is a Rickart module.

Proof. Suppose that M is a purely Rickart module. Since Imα is a pure projective submodule of M and $r_M(\alpha) \leq^p M$ for all α ∈ S, then the short exact sequence 0 → $r_M(\alpha)$ → M→ Imα →0 is a pure exact and hence splits[5, Theorem 2.4, CH.2]. So $r_M(\alpha) \leq^{\oplus}$ M. Therefore M is a Rickart module. The converse is clear. ◘

Proposition 2.12. If a module M is a purely Rickart module then for each N ≤ M with $\frac{M}{N} \cong L \leq^{\oplus}$ M we have $N \leq^p$ M.

Proof. Suppose M is purely Rickart module. Let N ≤ M with $\frac{M}{N} \cong L \leq^{\oplus}$ M. Then, there is an isomorphism α: $\frac{M}{N} \to L$, ρ: M → $\frac{M}{N}$ the projection map and j: L→ M the inclusion map. Put β = jαρ, so clearly that kerβ = $\rho^{-1}(0)$ = N. Since M is a purely Rickart, hence $N \leq^{\rho}$ M. ◘

The following proposition shows the converse of the proposition (2.12) is true if Imα is an isomorphic to a direct summand of M.

Proposition 2.13. For a module M, if Imα is an isomorphic to a direct summand of M for each α ∈ S= $End_R(M)$ and for each N ≤ M with $\frac{M}{N} \cong L \leq^{\oplus}$ M implies N $\leq^{\rho}$ M, then M is purely Rickart .

Proof. Let α ∈S. Since $\frac{M}{Ker\alpha} \cong$ Imα ≅ A $\leq^{\oplus}$ M. Then by hypothesis, kerα $\leq^{\rho}$M. ◘

Recall that from Remarks and examples (2.2-7), a quotient of purely Rickart not necessary purely Rickart. Consider the following proposition.

Proposition 2.14. If M is a purely Rickart module, then for any $N \leq^{\oplus}$ M, $\frac{M}{N}$ is purely Rickart module.

Proof. Let $N \leq^{\oplus} M = N \oplus L$ for some L ≤ M, so $\frac{M}{N} = \frac{N\oplus L}{N} \cong \frac{L}{N\cap L} \cong L$. Then by Proposition (2.4) L is purely Rickart and hence $\frac{M}{A}$ is purely Rickart (Remarks and examples (2.2-6)) ◘

We needed to the following proposition which appears in[5,Proposition 1.5,CH.1]

Proposition 2.15. Let 0 → P_i→ M_i→ N_i→ 0 be exact for all i∈ I, any index set, and let P= ⊕P_i (i in I) and M = ⊕M_i (i in I). Then P is pure in M if and only if P_i is pure in M_i for all i in I.

The Z-module $Z\oplus Z_2$ is not purely Rickart even though each of Z and Z_2 are purely Rickart Z-module. Recall that a submodule N of a module M is fully invariant if f (N) ≤ N for each f ∈ S =End (M). It's well known that if a module $M=M_1\oplus M_2$, then M_1 is a fully invariant submodule of M if and only if $Hom_R(M_1,M_2) = 0$. The following result proves: the purely Rickart property is closed under the direct sum if every direct summand is fully invariant purely Rickart.

Proposition 2.16. Let $M = \oplus_{i\in I} M_i$ where each of M_i is fully invariant submodule, then M is purely Rickart if and only if M_i is purely Rickart.

Proof. Suppose that each of M_i (i∈I) is purely Rickart module. Let $M = \oplus_{i\in I} M_i$, $S=End_R(M)$ and $\alpha = (\alpha_{ij}) \in S$ be arbitrary where $\alpha_{ij} \in Hom_R(M_j,M_i)$. Since M_i is fully invariant for each i∈ I then $Hom_R (M_i, M_j) = 0$ for each i ≠ j and $\alpha (M_i) \leq M_i$ for each i∈ I. Since $r_M (\alpha) = \oplus_{i\in I} r_{M_i}(\alpha_{ii})$, then by Proposition(2.15) M is purely Rickart module. The converse follows from Proposition (2.4). ◘

Let R be a Commutative ring and M be an R-module. Then the idealization of a module M (R (+) M for short) is a commutative ring with multiplication $(r_1,m_1)(r_2,m_2) = (r_1r_2, r_1m_2+r_2m_1)$ [4].

Proposition 2.17. Let B= R (+) M be the idealization of a module M. If B is purely Rickart, then so is R. The converse is true when M = 0.

Proof. Let a ∈ R and $b\in r_R (a)$. Put B = R (+) M, then (b,0) ∈ B. Since $r_B((a,0)) = r_R(a) \times r_M(a)$ and (a,0)(b,0)=(ab,0) = 0, then $(b,0)\in r_B((a,0))$. So, there exist (c, d) $\in r_B ((a,0))$ such that (b,0)(c, d) = (b, 0) where B is a purely Rickart ring. Since (b, 0) (c, d) = (bc, bd+0) = (b, 0). Then bc = b and hence $r_R(a) \leq^P R$. Therefore R is a purely Rickart ring. Conversely, suppose that R is a purely Rickart and M = 0. Then R (+) M = R(+){0}. Hence, for each (a, c) ∈ R (+) M, (a, c) = (a, 0). If (n, m) $\in r_B((a,0)) = r_R(a)\times\{0\}$, then (n, m) = (n, 0) and so there is a $c\in r_R(a)$ such that nc=n. That means, there is (c, 0) $\in r_B$ (a, 0) such that (n, 0) (c, 0) = (nc, 0) = (n, 0) $\in r_B$ (a, 0). So, $r_B (a, 0) \leq^P B$ = R (+) M. Therefore R (+) M is a purely Rickart ring. ◘

Proposition 2.18. Let R and S be rings and $_RV_S$ R-S bimodule and $C = \begin{pmatrix} R & V \\ 0 & S \end{pmatrix}$. If C is right purely Rickart ring. Then both R and S are right purely Rickart.

Proof . Suppose that C is a left purely Rickart. Let $\alpha = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in C$. Since C is purely Rickart ring, then for each element $\gamma = \begin{pmatrix} r & v \\ 0 & s \end{pmatrix} \in r_C(\alpha)$ there is $\beta = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in r_C(\alpha)$ such that γ = γβ. Now, $\alpha\gamma = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r & v \\ 0 & s \end{pmatrix} = \begin{pmatrix} ar & av \\ 0 & 0 \end{pmatrix} =$

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. So $r_C(\alpha) = \begin{pmatrix} r_R(a) & r_v(a) \\ 0 & S \end{pmatrix}$. Since $\gamma\beta = \begin{pmatrix} r & v \\ 0 & s \end{pmatrix}\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} rx & ry+vz \\ 0 & sz \end{pmatrix} = \begin{pmatrix} r & v \\ 0 & s \end{pmatrix} = \gamma$. So r = rx. Since $\alpha\beta = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} ax & ay \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ then $x \in r_R(a)$. Now, for all $r \in r_R(a)$, there is $x \in r_R(a)$ such that r = rx . Therefore R is right purely Rickart ring. Let $\mu = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \in C$, by the some way we can show that S is right purely Rickart. ◘

Proposition 2.19. Let R be a Noetherian ring. Then R is a right purely Rickart if and only if R is a right Rickart ring.

Proof. Follows the fact, in a Noetherian ring, a pure ideal and a direct summand ideal are equivalent [12, proposition 1.1.14]. ◘

Freser and Nicholson in [10] proved that a ring R is a left PP ring if and only if for any nonempty $X \leq R$ then $a \in r(X)a$ for all $a \in r(X)$. We will analogous this resulted in the following.

Proposition 2.20. For a ring R the following conditions are equivalent

1. R is a right purely Rickart
2. For all finitely generated left ideal J of R, the right annihilator of J in R is a pure in R as a right ideal.
3. For all finitely generated ideal J and a finite set $\{x_1,\dots, x_n\} \leq r_R(J)$, there is $b \in r_R(J)$ such that $x_i = x_i b$ for all i= 1,…n.

Proof. Clearly that 3⇒2⇒1.

1⇒2) Suppose that R is a purely Rickart ring and J is a left ideal in R generated by $\{x_1, x_2, \dots, x_n\}$. Then $r_R(J) = \bigcap_{i=1}^{n} r_R(Rx_i) = \bigcap_{i=1}^{n} r_R(x_i)$. Since R is a purely Rickart, then for each $1 \leq i \leq n$, $r_R(x_i) \leq^{P} R$. Now, if $b \in r_R(J)$ then there exists $c_i \in r_R(x_i)$ for each $1 \leq i \leq n$ such that $b = bc_i$. Put $c = c_1c_2\dots c_n \in \bigcap_{i=1}^{n} r_R(x_i) = r_R(J)$. Then $b = bc \in br_R(J)$.

(2⇒3) If n=1, then by (2), the proof is complete. When n >1 one can choose $y_n \in r_R(J)$ such that $x_n = x_n y_n$ and $z_i = x_i - x_i y_n$ for i= 1,…, n-1. Proceeding inductively, we can find $\acute{y} \in r_R(J)$ such that $z_i = z_i\acute{y}$ for each i = 1,…,n-1. Now, put $y = \acute{y} + y_n - \acute{y}y_n \in r_R(J)$. It's clear that for each i = 1,…,n, $x_i = x_i y$. ◘

Recall that a ring R is weakly n-semihereditray if each n-generated left (and/or right) ideal is flat [14].

Theorem 2.20. For a ring R and a fixed positive integer n, the following conditions are equivalent

1. Every n-generated projective right R-module is a purely Rickart.
2. The free R-module $R^{(n)}$ is a purely Rickart.

Proof. (1⇒2) Since every free module is projective, then it's clear that the free R-module $R^{(n)}$ is n-generated projective right R-module. So, by hypothesis, $R^{(n)}$ is a purely Rickart R-module.

(2⇒1) Let M be n-generated projective right R-module. So M is an image of free R-module A. So there's an epimorphism α: A→ M. But A is a free and M is n-generated projective, then $\alpha: R^{(n)} \rightarrow M$ splits, where $A \cong R^{(n)}$. Hence M is a direct summand of $R^{(n)}$. By hypothesis, $R^{(n)}$ is a purely Rickart, then so is M (Proposition 2.4). ◘

Proposition 2.21. For a ring R and a fixed positive integer n, then R is a right weakly n-semihereditary ring if and only if $Mat_n(R)$ is a right purely Rickart ring.
Proof. From [13, Theorem 2.10] R is a weakly n-semihereditary ring if and only if R^{nxn} is a PF [13, Theorem 2.10]. But $R^{nxn} \cong Mat_n(R)$ then $Mat_n(R)$ is a PF if and only if for each element $A \in Mat_n(R)$, $r_{Mat_n}(A) \leq^p Mat_n(R)$ [5, Theorem 2.2, CH.6] if and only if $Mat_n(R)$ is a purely Rickart ring. ◘

Proposition 2.22. If $R^{(n)}$ is purely Rickart R-module , where n is a fixed positive integer, then R is weakly n-semiherditary ring.

Proof. Let I be n-generated right ideal of R. i.e $I = \sum_{i=1}^{n} a_i R$ for some $a_i \in R$, i=1,…,n. Let $\alpha: R^{(n)} \rightarrow R$ defined by $\alpha((r_i)_{i=1}^{n})) = \sum_{i=1}^{n} a_i r_i$ clearly that α is a module homomorphism. Put $\psi = i \circ \alpha$ where $i: R \rightarrow R^{(n)}$ is the canonical injection map. Since $R^{(n)}$ is purely Rickart, then Im ψ = Imα = I is flat (Corollary 2.9). Hence R is weakly n- semiherditary ring. ◘

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