Some Properties of $\gamma$-Sets in a Ring

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Abstract

Let $R$ be a ring with identity $1_R$. A subset $J$ of $R$ is called a $\gamma$-set if for every $a \in R \setminus J$, there exists $b, c \in J$ such that $a + b = 0$ and $ac = 1_R = ca$. A $\gamma$-set of minimum cardinality is called a minimum $\gamma$-set.

In this study, we introduced the concept of $\gamma$-sets in a ring and presented a few of its properties. Moreover, we gave a sharp upper bound of the number of minimum $\gamma$-sets in a finite division ring.

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1 Introduction

A dominating set of a graph $G = (V, E)$ is a subset $S$ of $V$ such that for each $v \in V \setminus S$ there exists $u \in S$ with $vu \in E$. Rosero et al. [3] attempted to extend

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the concept of dominating set to some mathematical structures. Thus, the concept of \( \mathcal{D} \)-sets in a group was introduced. In [3], they gave some properties of a \( \mathcal{D} \)-set and in [2], they give the number of \( \mathcal{D} \)-sets and the number of minimum \( \mathcal{D} \)-sets in finite groups. In this paper, we introduced and studied a parallel concept in rings. We call the new concept a \( \gamma \)-set in a ring.

A binary operation on a nonempty set \( G \) is a function \( G \times G \rightarrow G \). A semi-group is a nonempty set \( G \) together with a binary operation which is associative, that is, \( a(bc) = (ab)c \) for all \( a, b, c \in G \). A group is a semi-group \( G \) that satisfies the following properties: (1) There is an element \( e \in G \) with the property that \( ge = g = eg \) for all \( g \in G \); (2) For each \( g \in G \), there exists \( h \in G \) such that \( gh = e = hg \).

As found in [1], the Multiplication Principle states that if an event \( E \) can be decomposed into \( r \) ordered events \( E_1, E_2, \ldots, E_r \), and that there are \( n_1 \) ways for the event \( E_1 \) to occur, \( n_2 \) ways for the event \( E_2 \) to occur, \( \ldots, n_r \) ways for the event \( E_r \) to occur. Then the total number of ways for event \( E \) to occur is \( n_1n_2\cdots n_r \).

You may refer to [4] for the succeeding concepts and notations that were not discussed here.

2 Preliminary Results

We now give some properties of a \( \gamma \)-set of a ring \( R \). We denote by \( T_R \) the set of all \( \gamma \)-sets of \( R \). We note that \( T_R \neq \emptyset \) since \( R \) is a \( \gamma \)-set. The next result shows that the set of all \( \gamma \)-sets in a ring is a semi-group under the operation union.

**Theorem 2.1** Let \( R \) be a ring with identity \( 1_R \). Let \( T_R \) be the set of all \( \gamma \)-sets of \( R \) and \( T_R^C \) be the set of all \( J^C \) such that \( J \) is a \( \gamma \)-sets of \( R \). Then

1. The set \( T_R \) is a semi-group under the set operation union;
2. The set \( T_R^C \) is a semi-group under the set operation intersection.

**Proof:** (1) Let \( I, J \in T_R \) and consider \( I \cup J \). Let \( x \in R \setminus (I \cup J) = I^C \cap J^C \). Then \( x \in J^C \). Since \( J \) is a \( \gamma \)-sets, there exists \( y, z \in J \subseteq I \cup J \) such that \( x + y = 0 \) and \( xz = 1_R = zx \). This shows that \( I \cup J \) is a \( \gamma \)-sets of \( R \). Since union of sets is associative, \( T_R \) is a semi-group.

(2) Next, let \( H, K \in T_R^C \). Then \( H = I^C \) and \( K = J^C \) for some \( I, J \in T_R \). Consider \( H \cap K = (I \cup J)^C \). Since \( I, J \in T_R \), by (1) \( I \cup J \in T_R \). Hence, \( H \cap K = (I \cup J)^C \in T_R^C \). Since intersection of sets is associative, \( T_R^C \) is a semi-group. \( \blacksquare \)

What elements of a ring \( R \) are necessarily in every \( \gamma \)-set \( J \) of \( R \)? The next result tells us.
Theorem 2.2 Let \( R \) be a ring with identity \( 1_R \) and \( J \) be a \( \gamma \)-set of \( R \).

1. If \( a \in R \) with \( 2a = 0 \), then \( a \in J \).

2. If \( a \in R \) with \( a^2 = 1 \), then \( a \in J \).

3. \( 0, 1_R \in J \).

Proof: (1) Let \( a \in R \) with \( 2a = 0 \) and \( a \notin J \). Since \( J \) is a \( \gamma \)-set, there exists \( b \in J \) such that \( a + b = 0 \). Hence, \( a + a = 2a = 0 = a + b \), i.e. \( a = b \). This is a contradiction.

(2) Let \( a \in R \) with \( a^2 = 1 \) and \( a \notin J \). Since \( J \) is a \( \gamma \)-set, there exists \( b \in J \) such that \( ab = 1_R = ba \). Hence, \( aa = a^2 = 1_R = ab \), i.e. \( a = b \). This is a contradiction.

(3) Since \( (1_R)^2 = 1_R \), by (2) \( 1_R \in J \). Similarly, since \( 2(0) = 0 \), we have by (1) \( 0 \in J \). \( \blacksquare \)

Theorems 2.3 and 2.4 give some of the conditions wherein a ring \( R \) has a unique \( \gamma \)-set, i.e., \( |T_R| = 1 \).

Theorem 2.3 Let \( R \) be a ring with identity \( 1_R \) and \( J \) be a \( \gamma \)-set of \( R \). If \( 2a = 0 \) for all \( a \in R \), then \( |T_R| = 1 \).

Proof: Recall that \( R \in T_R \). Hence, \( |T_R| \geq 1 \). Suppose \( |T_R| \neq 1 \), i.e. \( |T_R| > 1 \).

Let \( S \in T_R \) with \( S \neq R \). Let \( x \in R \setminus S \). Since \( S \) is a \( \gamma \)-set, there exists \( y \in S \) such that \( x + y = 0 \). This implies that there exists \( x \in R \) with \( x + y = 0 \) and \( x \neq y \). This is a contradiction. Hence, the assertion follows. \( \blacksquare \)

Theorem 2.4 Let \( R \) be a ring with identity \( 1_R \) and \( J \) be a \( \gamma \)-set of \( R \). If \( a^2 = 1_R \) for all \( a \in R \), then \( |T_R| = 1 \).

Proof: Proved similarly. \( \blacksquare \)

We denote by \( 1_R \) the multiplicative identity of all the ring structures mentioned in the succeeding discussions. Theorem 2.5 characterizes ring with unique \( \gamma \)-set.

Theorem 2.5 Let \( R \) be a division ring and \( J \) be a \( \gamma \)-set of \( R \). If \( |T_R| = 1 \), then \( 2a = 0 \) or \( a^2 = 1_R \) for all \( a \in R \).

Proof: Assume that there exists \( a \in R \) with \( 2a \neq 0 \) and \( a^2 \neq 1_R \). If \( a + a = 2a \neq 0 \), then there exists \( b \in R \) with \( b \neq a \) and \( a + b = 0 = b + a \). Note that \( a \neq 0 \) since \( 2a \neq 0 \). Since \( R \) is a division ring, there exists \( c \in R \) such that \( ac = 1_R = ca \). Since \( a^2 = 1_R \), \( c \neq a \). Now, consider \( J = R \setminus \{a\} \). Clearly, \( J \) is a \( \gamma \)-set. Hence, \( |T_R| > 1 \), i.e. \( |T_R| \neq 1 \). \( \blacksquare \)
Theorem 2.6 Let $J$ be a $\gamma$-set in $R$ and $u \in R$. Then $u \in R \setminus J$ if and only if there exists $v \in R$ such that $uv = 1_R = vu$ and $u \neq v$.

Proof: Suppose $u \in R \setminus J$, where $J$ is a $\gamma$-set. Then there exists $v \in J$ such that $uv = 1 = vu$. Suppose $u = v$. Then $uv = uu = u^2 = 1$ which implies that $u \in J$. This is a contradiction. Thus $u \neq v$. The converse is trivial. $\blacksquare$

Theorem 2.7 Let $R$ be a ring with identity $1_R$. Then $a \not\in R \setminus J$ for all $\gamma$-set $J$ in $R$ if $a \in R$ is a zero divisor.

Proof: Suppose $a \in R$ is a zero divisor (which implies that $a \neq 0$). Then there exists $b \neq 0$ in $R$ such that $ab = 0$. Suppose there exists $c \in R$ such that $ac = 1 = ca$. Then $0 = c0 = c(ab) = (ca)b = 1b = b$, which is a contradiction. Thus, by the previous theorem, $a \not\in R \setminus J$. $\blacksquare$

3 Number of Minimum $\gamma$-Sets in a Finite Division Ring

Theorem 3.1 Let $R$ be a ring with identity $1_R$ and $a \in R$. If there exists $b \in R$ with $b \neq a$ such that $ab = 1_R = ba$, then there exists $c \in R$ with $c \neq a$ and $a + c = 0$. Hence, $R \setminus \{a\}$ is a $\gamma$-set.

Proof: Let $R$ be a ring with identity $1_R$ and $a \in R$. Suppose that there exists $b \in R$ with $b \neq a$ such that $ab = 1_R = ba$ (this would imply that $a \neq 0$). Assume further that for all $c \in R$ with $c \neq a$, $a + c \neq 0$. If for all $c \in R$ with $c \neq a$, $a + c \neq 0$, then we must have $a + a = 0$. This is a contradiction. Hence, the theorem follows. $\blacksquare$

We call the pair $a$ and $b$ of Theorem 3.1 as super-couple. In the succeeding discussions, we are motivated by the problem of knowing how many minimum $\gamma$-sets and how many $\gamma$-sets a finite division ring have.

Theorem 3.2 Let $R$ be a ring with identity $1_R$ and suppose that $R$ has no zero divisors. Let $J$ be a $\gamma$-set of $R$, and $a \in J$. If there exists $b \in J$ with $b \neq a$ such that $ab = 1_R$, then either, there exists $c \in J$ with $c \neq a$ and $a + c = 0$, or there exists $e \in J$ with $b + e = 0$. Hence, $J \setminus \{a\}$ or $J \setminus \{b\}$ is also a $\gamma$-set of $R$.

Proof: Let $R$ be with $1_R$ and suppose that $R$ has no zero divisors and $J$ be a $\gamma$-set of $R$. Let $a \in J$ and assume that there exists $b \in R$ with $b \neq a$ such that $ab = 1_R$. (This implies that $a \neq 0$.) If there exists $c \in J$ with $c \neq a$ such
that \( a + c = 0 \), then we are done. So we assume that for all \( c \in J \) with \( c \neq a \), \( a + c \neq 0 \). If for all \( c \in J \) with \( c \neq a \), \( a + c \neq 0 \), then there exists \( d \in R \setminus J \) with \( a + d = 0 \). Since \( d \in R \setminus J \) and \( J \) is a \( \gamma \)-set, there exists \( e \in J \) such that \( de = 1_R \). Thus, \( 0 = 0e = (a + d)e = ae + de = ae + 1_R \), i.e., \( ae = -1_R \). Hence, \( a(b + e) = ab + ae = 1_R + (-1_R) = 0 \). Since \( a \neq 0 \) and \( R \) has no zero divisors, \( b + e = 0 \). Hence, the theorem follows.

Theorem 3.2 suggests that every super-couple determines a \( \gamma \)-set, in the same way as in [2] that every non-involution determines a \( \mathcal{S} \)-set.

Let \( R \) be a division ring and \( T = \{0\} \cup \{t \in R : t^2 = 1_R\} \). We note that if \( x \in R \setminus T \), then \( x \neq x^{-1} \). Since \( x \) and \( x^{-1} \) must be in \( R \setminus T \) for all \( x \in R \setminus T \), \( |R \setminus T| \) must be even. We denote by \( c \) the number \( |R \setminus T|/2 \) and we call it the \( c \)-number of \( R \).

**Lemma 3.3** Let \( R \) be a division ring and \( T = \{0\} \cup \{t \in R : t^2 = 1_R\} \). Define a relation \( \sim \) on \( R \setminus T \) by \( x \sim y \) if and only if \( x = y \) or \( xy = 1_R \). Then \( R \) is an equivalence relation on \( R \setminus T \).

**Proof:** Let \( R \) be a division ring and \( T = \{0\} \cup \{t \in R : t^2 = 1_R\} \). Define a relation \( \sim \) on \( R \setminus T \) by \( x \sim y \) if and only if \( x = y \) or \( xy = 1_R \). Since \( x = x \) for all \( x \in R \setminus T \), \( x \sim x \) for all \( x \in R \setminus T \), i.e., \( \sim \) is reflexive. If \( x \sim y \), then \( xy = 1_R = yx \), i.e., \( y \sim x \). Hence, \( \sim \) is symmetric. Finally, if \( x \sim y \) and \( y \sim z \), then \( xy = 1_R \) and \( yz = 1_R = zy \). Thus, \( xy = zy \), i.e., \( x = z \). If \( x = z \), then \( x \sim z \). This shows that \( \sim \) is transitive. Accordingly, \( R \) is an equivalence relation on \( R \setminus T \).

**Remark 3.4** The equivalence relation \( \sim \) in \( R \setminus T \) of Lemma 3.3 partitions \( R \setminus T \) into equivalence classes \( \mathcal{C} = \{\{x, x^{-1}\} : x \in R \setminus T\} \). If \( R \) is finite, then we denote this partition by \( \mathcal{C} = \{A_1, A_2, \ldots, A_c\} \).

**Theorem 3.5** Let \( R \) be a division ring. \( E \) is a minimum \( \gamma \)-set of \( R \) if and only if \( E = T \cup \{a_1, a_2, \ldots\} \) where \( a_i \in A_i \) for all \( i \). In particular, if \( R \) is finite, then \( E \) is a minimum \( \gamma \)-set of \( R \) if and only if \( E = T \cup \{a_1, a_2, \ldots, a_c\} \) where \( a_i \in A_i \) for all \( i = 1, 2, \ldots, c \) (where \( \{A_1, A_2, \ldots, A_c\} \) is the partition of \( R \setminus T \) in the sense of Remark 3.4).

**Corollary 3.6** Let \( R \) be a finite division ring and \( J \) be a minimum \( \gamma \)-set of \( R \). Then \( |J| = |T| + (|R \setminus T|)/2 \).

**Proof:** Let \( J \) be a minimum \( \gamma \)-set and \( T = \{0\} \cup \{t \in R : t^2 = 1_R\} \). Then by Theorem 3.5, \( J = T \cup \{a_1, a_2, \ldots, a_c\} \) for some \( a_1 \in A_1, a_2 \in A_2, \ldots, a_c \in A_c \). Therefore, \( |J| = |T| + (|R \setminus T|)/2 \).
The index minimum of a finite ring $R$ is the number of minimum $\gamma$-sets of $R$ and is denoted by $\text{ind}(R)$. The next result gives an upper bound on the number of minimum $\gamma$-set of a finite division ring.

**Corollary 3.7** Let $R$ be a finite division ring. Then $\text{ind}(R) \leq 2^c$ where $c$ is the $c$-number of $R$.

**Proof:** Let $J$ be a minimum $\gamma$-set and $T = \{0\} \cup \{t \in R : t^2 = 1_R\}$. Then by Theorem 3.5, $J = T \cup \{a_1, a_2, \ldots, a_c\}$ for some $a_1 \in A_1, a_2 \in A_2, \ldots, a_c \in A_c$. By Theorem 3.2, the number of ways to choose $a_i$ is either 1 or 2 for all $i = 1, 2, \ldots, c$. Therefore, the number of ways to choose a minimum $\gamma$-set, by Multiplication Principle, is less than or equal $|A_1| \cdot |A_2| \cdots |A_c| = 2 \cdot 2 \cdot 2 \cdots 2 = 2^c$. ■

The bound given in Corollary 3.7 is sharp. Equality is attained if for all $x \in R \setminus T$, $x^{-1} = -x$, for example $\text{ind}(\mathbb{Z}_5) = 2^1$, note that $c$ here is equal to 1.

**Conjecture 3.8** Let $R$ be a finite division ring. Then $|T_R| \leq 3^c$ where $c$ is the $c$-number of $R$. ■

**References**


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