Semilattices of $D^{**}$-Simple Semigroups

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Abstract

A right congruence $D^{**}$ on a semigroup is introduced, and semigroups which are semilattices of $D^{**}$-simple monoids are characterized.

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1. Introduction

For standard terminology and notation in semigroup theory see Howie [3]. As usual, $E(S)$ is the set of idempotents of a semigroup $S$.

Green [2] introduced a $\mathcal{L}$-relation on any semigroup $S$ : $a, b \in S$ are $\mathcal{L}$-related if and only if $a$ and $b$ generate the same principal left ideal. For $a, b \in S$, let $a\mathcal{L}\ast b$ if and only if $a$ and $b$ are $\mathcal{L}$-related in some oversemigroup of $S$. It was shown that $a\mathcal{L}\ast b$ if and only if for all $x, y \in S^1$,

$$ax = ay \text{ if and only if } bx = by$$

(see Fountain [1]).

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For any equivalence relation \( \sigma \) on a semigroup \( S \), let \( \sigma^r \) be the relation on \( S \) defined by: for any \( a, b \in S \), \( a \sigma^r b \) if and only if for all \( x, y \in S^1 \),

\[
ax \sigma ay \text{ if and only if } bx \sigma by.
\]

It is easy to see that \( \sigma^r \) is a right congruence on \( S \).

In particular when \( \sigma = 1_S \), the equality on \( S \), then \( L^* = 1^r_S \). Another particular case is the case where \( \sigma = R \), which was discussed in [5]. The notation \( L^{**} = R^* \) was introduced in [5]. In [5] semigroups which are semilattices of \( L^{**} \)-simple monoids was characterized. The purpose of this paper is to introduce right congruence \( D^r \) in which we are interested, denoted by \( D^{**} \), and to give a structure theorem for the class of all semigroups where all idempotents are central and each \( D^{**} \)-class contains an idempotent.

2. Semilattices of \( D^{**} \)-simple Semigroups

A particular case is the case where \( \sigma = D \), which is discussed in this section. The notation \( D^{**} = D^* \) is introduced. The \( D^{**} \)-class containing the element \( a \) is denoted by \( D^{**}_a \).

**Lemma 1.** Let \( S \) be a semigroup. If the \( D^{**} \)-class \( D^{**}_a \) contains an idempotent \( e \), then \( aDa \).

**Proof.** Assume that the \( D^{**} \)-class \( D^{**}_a \) contains an idempotent \( e \). From \( aD^{**}e \) we have that for all \( x, y \in S^1 \), \( axD^ay \) if and only if \( exDey \). Let \( x = e, y = 1 \). Since \( eeDe1, aDe1 \).

**Lemma 2.** ([5]) Let \( S \) be a semigroup in which the idempotents are central. If \( \sigma \) is a right congruence on \( S \) such that each \( \sigma \)-class contains a unique idempotent, then \( \sigma \) is a semilattice congruence on \( S \), and \( S/\sigma \cong E(S) \).

**Lemma 3.** Let \( S \) be a semigroup in which the idempotents are central. If each \( D^{**} \)-class contains an idempotent, then \( D^{**} \) is a semilattice congruence on \( S \), \( S/D^{**} \cong E(S) \), and each \( D^{**} \)-class is a monoid.

**Proof.** By Lemma 1, if the \( D^{**} \)-class \( D^{**}_a \) contains an idempotent \( e \), then \( aDa \). Hence there exists \( b \in S \) such that \( aLaRae \), and so \( a = xb, b = aey \) for some \( x, y \in S^1 \). Thus \( a = xae \). Since \( e \) is in the central of \( S \) we have that \( xae = exa \), hence \( a = exa \). Therefore \( ea = a = ae \). Let \( b, c \in D^{**}_a = D^{**}_e \). Then \( bcD^{**}ec \) since \( D^{**} \) is a right congruence on \( S \). Now \( ec = c \) we have \( bcD^{**}cD^{**}a \). We obtain that \( D^{**}_a \) is a monoid with identity \( e \). Since the identity of a monoid is unique, using the centrality of the idempotent, we have that each \( D^{**} \)-class contains a unique idempotent.

Since \( D^{**} \) is a right congruence on \( S \), it follows from Lemma 2 that \( D^{**} \) is a semilattice congruence on \( S \), and \( S/D^{**} \cong E(S) \).
Remark By the proof of Lemma 3 we have that the $D^{**}$-class which contains a central idempotent $e$ is a monoid with identity $e$, and the $D^{**}$-class contains a unique idempotent.

To characterize the semigroups which are semilattices of $D^{**}$-simple monoids, we need the following definitions.

A semigroup $S$ is called $D^{**}$-simple if $D^{**} = S \times S$. A semigroup $S$ is called $D$-left cancellative if for all $a, x, y \in S$,

$$axDy \text{ if and only if } xDy.$$  

**Lemma 4.** A monoid is $D^{**}$-simple if and only if it is $D$-left cancellative.

**Proof.** Suppose that $S$ is a monoid with identity $e$. Let $S$ be $D^{**}$-simple. Since $aD^{**}e$ for all $a \in S$, we have that for all $x, y \in S, axDy$ if and only if $x = exDey = y$. Thus $S$ is $D$-left cancellative.

Conversely, let $S$ be $D$-left cancellative, and $a \in S$. Let $x, y \in S^1$. When at least one of $x$ and $y$ is 1, without loss of generality assume that $x = 1$. Then $a = aeDy$ if and only if $eDy$ if and only if $eDey$. When $x$ and $y$ are not 1, that is, $x, y \in S$. Then $axDy$ if and only if $xDy$ if and only if $exDey$. Therefore $aD^{**}e$, and so $S$ is $D^{**}$-simple.

Before we give a structure theorem for the semigroups in which we are interested, we recall the following notion.

Let $Y$ be a semilattice and for each $\alpha \in Y$, let $S_\alpha$ be a monoid with identity $e_\alpha$. Assume that $S_\alpha$ are pairwise disjoint. For each $\alpha \geq \beta$ in $Y$ let $\varphi_{\alpha,\beta}$ be a monoid homomorphism of $S_\alpha$ into $S_\beta$ such that

(i) $\varphi_{\alpha,\alpha}$ is the identity mapping on $S_\alpha$ for each $\alpha \in Y$,

(ii) $\varphi_{\alpha,\beta} \varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$ if $\alpha \geq \beta \geq \gamma$ in $Y$.

On $S = \bigcup_{\alpha \in Y} S_\alpha$ define a multiplication $*$ by

$$a * b = (a\varphi_{\alpha,\alpha})(b\varphi_{\beta,\beta}) \quad (a \in S_\alpha, b \in S_\beta).$$

Then $(S, *)$ is a semigroup which is called a strong semilattice of the monoids $S_\alpha, \alpha \in Y$, and is denoted by $[Y; S_\alpha, \varphi_{\alpha,\beta}]$ (see Section III.7 of [4]).

**Lemma 5.** Let $S = [Y; S_\alpha, \varphi_{\alpha,\beta}]$ is a strong semilattice of the monoids $S_\alpha, \alpha \in Y$. If $a \in S_\alpha, b \in S_\beta$, and $aDb$, then $\alpha = \beta$.

**Proof.** Since $aDb$, there exists $c \in S$ such that $aLcRb$. Suppose that $c \in S_\gamma$. Let $x, y \in S^1$ be such that $a = xc, c = ya$. When at least one of $x$ and $y$ is 1, then $a = c$, and so $\alpha = \gamma$. When $x$ and $y$ are not 1, that is, $x, y \in S$, assume that $x \in S_{\alpha_1}, y \in S_{\gamma_1}$. Then $a \in S_{\alpha_1\gamma_1}$, hence $\alpha = \alpha_1 \gamma$. Thus

$$\alpha \gamma = \alpha_1 \gamma \cdot \gamma = \alpha_1 \gamma = \alpha.$$

Similarly, $\gamma = \gamma_1 \alpha$, and so $\gamma \alpha = \gamma$. But $\alpha \gamma = \gamma \alpha$. Thus $\alpha = \gamma$. From $cRb$, we can similarly obtain that $\gamma = \beta$. Theorfore $\alpha = \beta$.
The following theorem gives a structure theorem for the class of all semiroups where all idempotents are central and each $D^{**}$-class contains an idempotent.

**Theorem 6** For a semigroup $S$ satisfying $\mathcal{D}$ is a left congruence on $S$ the following are equivalent:

1. the idempotents of $S$ are central, each $D^{**}$-class of $S$ contains an idempotent, and for any $a, b$ in each $D^{**}$-class of $S$, $a$ and $b$ have $\mathcal{D}$-relation in the $D^{**}$-class whenever $a$ and $b$ have $\mathcal{D}$-relation in $S$;

2. $S$ is a strong semilattice of $D^{**}$-simple monoids with a unique idempotent, and for any $a, b$ in each $D^{**}$-simple monoids, $a$ and $b$ have $\mathcal{D}$-relation in the $D^{**}$-simple monoids whenever $a$ and $b$ have $\mathcal{D}$-relation in $S$.

**Proof.** (1) implies (2). By the remark after Lemma 3, each $D^{**}$-class is a monoid. Let $a, x, y \in D^{**}_e$, where $e$ is the identity of $D^{**}_e$. If $ax \mathcal{D} ay$ in $D^{**}_e$, then $ax \mathcal{D} ay$ in $S$ and since $aD^{**}e$ we have $x = ex \mathcal{D} ey = y$ in $S$. By hypothesis, $xDy$ in $D^{**}_e$. Conversely, if $xDy$ in $D^{**}_e$, then $xDy$ in $S$. Notice the condition that $\mathcal{D}$ is a left congruence on $S$. Hence $ax \mathcal{D} ay$ in $S$. By hypothesis again, $ax \mathcal{D} ay$ in $D^{**}_e$. Therefore, $D^{**}_e$ is $\mathcal{D}$-left cancellative, hence is $D^{**}$-simple by Lemma 4.

By the remark after Lemma 3, we have that each $D^{**}$-class contains a unique idempotent. It follows from Lemma 3 that $S$ is a semilattice of its $D^{**}$-classes $D^{**}_e, e \in E(S)$. For $e \geq f$ in $E(S)$, define

$$\varphi_{e,f} : D^{**}_e \rightarrow D^{**}_f, a \mapsto af.$$ 

Using that $\mathcal{D}$ is a left congruence on $S$ and the centrality of idempotents of $S$, we can deduce that $S = [E(S); D^{**}_e, \varphi_{e,f}]$, and (2) is satisfied.

(2) implies (1). Assume that $S$ is a strong semilattice of $D^{**}$-simple monoids, let $S = [\mathcal{Y}; S_\alpha, \varphi_{\alpha,\beta}]$, where each $S_\alpha$ is a $D^{**}$-simple monoid with a unique idempotent $e_\alpha$. Then $E(S) = \{e_\alpha : \alpha \in Y\}$. For $e_\alpha \in E(S), b \in S_\beta, \alpha, \beta \in Y$ we have

$$e_\alpha b = (e_\alpha \varphi_{\alpha,\alpha \beta})(b \varphi_{\beta,\alpha \beta}) = e_{\alpha \beta}(b \varphi_{\beta,\alpha \beta}) = (b \varphi_{\beta,\alpha \beta})e_{\alpha \beta} = (b \varphi_{\beta,\alpha \beta})(e_\alpha \varphi_{\alpha,\alpha \beta}) = b e_\alpha.$$ 

Therefore, the idempotents of $S$ are central.

We show that if $a \in S_\alpha$, then $aD^{**}e_\alpha$ in $S$ as follows. Let $x, y \in S^1$.

(i) When $x, y \in S$, where $x \in S_\beta, y \in S_\gamma$. Assume that $ax \mathcal{D} ay$, Since $ax \mathcal{D} ay$, and $ay \mathcal{D} ay$, it follows from Lemma 5 that $\alpha \beta = \alpha \gamma$. Then $(ae_\beta)(xe_\delta) = ax \mathcal{D} ay = (ae_\beta)(ye_\delta)$ in $S$. Notice that $(ae_\delta)(xe_\delta), (ae_\delta)(ye_\delta)$ are in $S_\delta$. By hypothesis, $(ae_\delta)(xe_\delta) \mathcal{D} (ae_\delta)(ye_\delta)$ in $S_\delta$. Thus $S_\delta$ is $D^{**}$-simple, i.e., is $D$-left cancellative, we have that $e_\alpha x = e_\delta(xe_\delta)e_\delta(ye_\delta) = e_\alpha y$. Conversely, if $e_\alpha x \mathcal{D} e_\alpha y$, then $ax = a(e_\alpha x) \mathcal{D} a(e_\alpha y) = ay$.

(ii) When at least one of $x$ and $y$ is 1, without loss of generality assume that $x = 1$. Let $y \in S_\gamma$. Assume that $a \mathcal{D} y$, Since $a \in S_\alpha$ and $ay \in S_\alpha \gamma$, it follows
from Lemma 5 that \( \alpha = \alpha \gamma \). Then \((ae_\alpha)e_\alpha = aD\alpha y = (ae_\alpha)(ye_\alpha)\) in \( S \). Notice that \((ae_\alpha)e_\alpha, (ae_\alpha)(ye_\alpha)\) are in \( S_\alpha \). By hypothesis, \((ae_\alpha)e_\alpha D(ae_\alpha)(ye_\alpha)\) in \( S_\alpha \). Since \( S_\alpha \) is \( \mathcal{D}^{**}\)-simple, i.e., is \( \mathcal{D}\)-left cancellative, we have that \( e_\alpha D e_\alpha(ye_\alpha) = e_\alpha y \). Conversely, if \( e_\alpha D e_\alpha y \), then \( a = ae_\alpha D a(ye_\alpha) = ay \). This proved that \( a \mathcal{D}^{**} e_\alpha \) in \( S \).

For each \( \mathcal{D}^{**}\)-class \( D^{**}_a \) of \( S \), suppose \( a \in S_\alpha \). Then \( a \mathcal{D}^{**} e_\alpha \) in \( S \). Notice \( e_\alpha \in D^{**}_a \). Hence \( D^{**}_a \) contains an idempotent.

Now we show that each \( \mathcal{D}^{**}\)-class \( D^{**}_a \) of \( S \) contains some \( S_\alpha \). Let \( a \in S_\alpha \). To prove that \( S_\alpha \subseteq D^{**}_a \), let any element \( b \) be in \( S_\alpha \). Then \( b \mathcal{D}^{**} e_\alpha \) in \( S \). Hence \( b \in D^{**}_e \). Thus \( S_\alpha \subseteq D^{**}_e \). But \( D^{**}_e = D^{**}_a \). Therefore \( S_\alpha \subseteq D^{**}_a \). It follows that \( D^{**}_a \) is a union of some of the \( S_\alpha \):

\[
D^{**}_a = \bigcup_{b \in S_\beta} S_\beta.
\]

By the remark after Lemma 3, \( D^{**}_a \) contains a unique idempotent. Hence \( D^{**}_a \) can not contain more than one of the \( S_\alpha \). Thus \( D^{**}_a = S_\alpha \).

On the other hand, for each \( S_\alpha \), similar with the proof of the above part, we may prove that \( S_\alpha \) is contained in a \( \mathcal{D}^{**}\)-class \( D^{**}_a \) of \( S \), and \( S_\alpha = D^{**}_a \). Consequently the \( S_\alpha, \alpha \in Y \), are precisely the \( \mathcal{D}^{**}\)-classse of \( S \) in the strong semilattice decomposition \( S = [Y; S_\alpha, \varphi_{\alpha, \beta}] \), and (1) holds. \( \square \)

References


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